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Stability Results for a Class of Nonlinear Caputo Volterra-Fredholm System: Physics and Engineering Application

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ABSTRACT

This study intends to provide and prove a novel stability theorem for the non-linear Volterra-Fredholm integro-differential equation with Caputo fractional derivative using the weighted space method and fixed-point technique. Specifically, the study investigates the H-U-R stability and semi-U-H-R stability results. Eventually, the investigation discusses an example of the capability of this method.

1. INTRODUCTION

As a result of their frequent appearance in a wide range of engineering and scientific disciplines, systems of fractional differential and integral equations are currently the focus of active research [1]. A system of integral-differential equations must therefore have approximate solutions. Besides, fractional derivatives provide a powerful tool for many types of physical modeling, such as stochastic dynamical systems, electrodynamics of complex medium, plasma physics, signal processing, economics, and so on researches [2, 3].

Budak et al. [4] reported that the stability issue of differential equations solutions presented. One of the most essential topics in differential equation theory is Ulam-Hyers stability. Because of the broad scope of fractional calculus, many authors focused on the study of stability for fractional differential equations [5-8]. In the same regard, fractional integro-differential equations also drew the attention of several authors [9-16].

Chalishajar and Kumar [5] enhanced a new direction of research via studied the existence and uniqueness of the solutions as well as discussed two types of stability. In same regard, Khan et al. [7] used Perov's fixed point theorem and generalized metric space to derive some relaxed requirements for the uniqueness of positive solutions to the aforementioned problem. Dong et al. [9] investigated the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the random fractional integro-differential equation using the fixed point theorem.

The stability theory of fractional integro-differential equations is a significant branch of fractional calculus. The Ulam-type stability of an integro-differential equation implies that we can find the exact solution to the problem near an approximate solution. Several varieties of Ulam-type stability for nonlinear fractional integro-differential equations have been studied in recent decades [5, 7, 15-17].

Recently, Sevgin and Sevli [12] examined the U-H stability and the U-H-R stability in formulations of fixed-point techniques for the nonlinear Volterra equation:

$$\Delta'(\upsilon) = A(\upsilon,\Delta(\upsilon)) + \int_0^{\upsilon} \Phi(\upsilon,\varsigma,\Delta(\varsigma))d\varsigma \tag{1}$$

Vu and Van Hoa [15] addressed the nonlinear IVP of the Volterra equations, and they used the successive approximation approach to explain the U-H and U-H-R stability of the following equations.

$$\Delta'(\upsilon) = A\big(\upsilon,\Delta(\upsilon)\big) + \int_a^{\upsilon} \Phi\big(\upsilon,\varsigma,\Delta(\varsigma)\big)d\varsigma,\ \upsilon \in [a,b] \tag{2}$$

$$\Delta(a) = \Delta_0 \tag{3}$$

Sousa and De Oliveira [18, 19] introduced U-H stability for the Volterra -Hilfer fractional problem using the Banach fixed-point approach.

$${}^{H}D_{0+}^{\alpha,\beta;\psi\psi}\Delta(\upsilon) = A\big(\upsilon,\Delta(\upsilon)\big) + \int_0^{\upsilon} \Phi\big(\upsilon,\varsigma,\Delta(\varsigma)\big)d\varsigma \tag{4}$$

where, ${}^{H}D_{0+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative.

Herein, the current study is interested in the following Caputo fractional nonlinear Volterra-Fredholm integro-differential problem:

$$\Theta'(\upsilon) + {}^{c}D_{0+}^{\alpha}\Theta(\upsilon) = g\big(\upsilon,\Theta(\upsilon)\big) + \int_0^{\upsilon} \Upsilon\big(\upsilon,\varsigma,\Theta(\varsigma)\big)d\varsigma + \int_0^{1} \Psi\big(\upsilon,\varsigma,\Theta(\varsigma)\big)d\varsigma,\ \upsilon \in [0,1] \tag{5}$$

$$\Theta(0) = \eta \tag{6}$$

where, $\Theta \in C^1[0,1], 0 < \alpha < 1, \Upsilon, \Psi: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g: [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Motivated by the above studies, the current study will investigate another problem of stability theorem for the non-linear Volterra-Fredholm integro-differential equation with Caputo fractional derivative using the weighted space method and fixed-point technique.

Therefore, the aim of this work is to investigate the H-U, H-U-R, and semi-U-H-R stability for the system (5) under some new standards.

2. PRELIMINARIES

In this segment, the study introduces some useful preliminaries for fractional derivatives [20, 21]. Moreover, we recall concepts of stability for Eq. (5).

Let:

$$\rho(\Theta,\omega)=\sup_{v\in[0,1]}\frac{|\Theta(v)-\omega(v)|}{\xi(v)},\Theta,\omega\in C^1[0,1] \quad (7)$$

The weighted metric, where function ξ is a continuous non-decreasing defined as $\zeta:[0,1]\to(0,+\infty)$ then there is $\xi\in[0,1)$, satisfies:

$$\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\varsigma(\varsigma)d\varsigma\le\ \xi\zeta(v) \quad (8)$$

Obviously, $(C^1[0,1],\rho)$ is a complete metric space.

Definition 2.1 [20, 21] Let $f:(0,+\infty)\to\mathbb{R}$ be integrable function, the R-L fractional integral is given by:

$$I_{0+}^{\alpha}f(\Lambda)=\frac{1}{\Gamma(\alpha)}\int_0^{\Lambda}(\Lambda-v)^{\alpha-1}f(v)dv,\Lambda>0,0<\alpha<1. \quad (9)$$

Definition 2.2 [20, 21] The left Caputo fractional derivative of differentiable function f(υ) is given by:

$${}^{c}D_{0+}^{\alpha}f(\Lambda)=I_{0+}^{1-\alpha}f'(\Lambda)=\frac{1}{\Gamma(1-\alpha)}\int_0^{\Lambda}(\Lambda-v)^{-\alpha}f'(v)dv,0<\alpha<1. \quad (10)$$

Definition 2.3 [20, 21] The function of Mittag-Leffler is given by:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\beta,\alpha,z\in\mathbb{C},Re(\beta)>0,Re(\alpha)>0 \quad (11)$$

The Laplace transform of the Mittag-Leffler and Caputo derivative given by:

$$\mathcal{L}\{v^{\beta-1}E_{\alpha,\beta}(\pm av^{\alpha})\}(\varsigma)=\frac{\varsigma^{\alpha-\beta}}{(\varsigma^{\alpha}\mp a)},Re(\varsigma)>|a|^{\frac{1}{\alpha}},Re(\beta)>0,Re(\alpha)>0,\beta,\alpha\in\mathbb{C} \quad (12)$$

$$\mathcal{L}\left\{v^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm av^{\alpha})\right\}(\varsigma)=\frac{k!\varsigma^{\alpha-\beta}}{(\varsigma^{\alpha}\mp a)^{k+1}},Re(\varsigma)>|a|^{\frac{1}{\alpha}},\alpha,\beta\in\mathbb{C}, \quad (13)$$

where:

$$E_{\alpha,\beta}^{(k)}(y)=\frac{d^k}{dy^k}E_{\alpha,\beta}(y)=\sum_{j=0}^{\infty}\frac{(j+k)!y^j}{j!\Gamma(\alpha j+\alpha k+\beta)},k=0,1,2,..., \quad (14)$$

and

$$\mathcal{L}^{c}D_{0+}^{\alpha}f(v)\}(\varsigma)=\varsigma^{\alpha}\tilde{f}(\varsigma)-\varsigma^{\alpha-1}f(0),0<\alpha<1, \quad (15)$$

respectively [20, 21].

Definition 2.4 [20, 21] If $\Lambda(v)$ is a given differential function, satisfying:

$$\Big|\Lambda'(v)+{}^{c}D_{0+}^{\alpha}\Lambda(v)-g(v,\Lambda(v))-\int_0^v\Upsilon(v,\varsigma,\Lambda(\varsigma))d\varsigma-\int_0^1\Psi(v,\varsigma,\Lambda(\varsigma))d\varsigma\Big|\le\theta,v\in[0,1],\theta>0, \quad (16)$$

There is C>0 and $\Theta(v)$ is a solution of the system in Eq. (5), where:

$$|\Lambda(v)-\Theta(v)|\le C\theta,v\in[0,1] \quad (17)$$

Then, the system in Eq. (5) has the U-H stability.

If $\Lambda(v)$ satisfies (16) there is a solution $\Theta(v)$ of the system (5) and C>0, such that:

$$|\Lambda(v)-\Theta(v)|\le C\phi(v),v\in[0,1] \quad (18)$$

where, the function φ is continuous nonnegative defined as $\phi:[0,1]\to(0,+\infty)$, then the system (1) has the semi-U-H-R stability.

If $\phi:[0,1]\to(0,+\infty)$ is a continuous and $\Lambda(v)$ satisfying:

$$\Big|\Lambda'(v)+{}^{c}D_{0+}^{\alpha}\Lambda(v)-g(v,\Lambda(v))-\int_0^v\Upsilon(v,\varsigma,\Lambda(\varsigma))d\varsigma-\int_0^1\Psi(v,\varsigma,\Lambda(\varsigma))\Big|\le\varphi(v) \quad (19)$$

There is C>0 and $\Theta(v)$ is a solution of the system (5), where:

$$|\Lambda(v)-\Theta(v)|<-C\varphi(v),v\in[0,1] \quad (20)$$

Then, the system in Eq. (1) has the U-H-R stability.

3. STABILITY RESULTS

The study will investigate in this segment the stabilities of U-H-R, semi-U-H-R and U-H for the system (1) in $C^1[0,1]$.

3.1 U-H-R stability for the system in Eq. (5)

Here, the study will investigate the equivalent integral equation of the system (5) and study the U-H-R stability for the system (5) in $(C^1[0,1],\rho)$.

Lemma 3.1 Assume that $f:[0,1]\to\mathbb{R}$ is a continuous function, and $0<\alpha<1,\Theta(v)\in C^1[0,1]$, the unique solution of the following equation.

$$\Theta'(v)+{}^{c}D_{0+}^{\alpha}\Theta(v)=f(v),\Theta(0)=\eta \quad (21)$$

is given by:

$$\Theta(v)=\eta+\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})f(\varsigma)d\varsigma. \quad (22)$$

Proof: The Laplace transforms of both $\Theta'(v)$ and ${}^{c}D_{0+}^{\alpha}\Theta(v)$ exist for $\Theta(v)\in C^1[0,1]$, applying the Laplace transform on two sides of Eq. (22). Then,

$$s\widetilde{\Theta}(\varsigma)-\eta+\varsigma^{\alpha}\widetilde{\Theta}(\varsigma)-\varsigma^{\alpha-1}\Theta(0)=\tilde{f}(\varsigma) \quad (23)$$

$$\widetilde{\Theta}(\varsigma)=\frac{1}{\varsigma}\eta+\frac{1}{\varsigma^{\alpha}+\varsigma}\tilde{f}(\varsigma) \quad (24)$$

It can take the inverse Laplace transform on the both sides of Eq. (23), then get:

$$\Theta(v)=\eta+\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)f(\varsigma)d\varsigma \quad (25)$$

Then, $\Theta(v)$ satisfies Eq. (21)⇔ $\Theta(v)$ satisfies Eq. (23). As a result, Eq. (23) is the equivalent integral equation of Eq. (21).

Theorem 3.2 Assume that a function ζ is continuous non-decreasing defined as $\zeta:[0,1]\to(0,\infty)$, and there exists $\xi\in[0,1)$, satisfying:

$$\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\zeta(\varsigma)d\varsigma\le\xi\zeta(v) \quad (26)$$

The following hypotheses are introduced:

[D_1] Assume that a continuous function g defined as $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, such that:

$$|g(v,h_1)-g(v,h_2)|\le\epsilon_1|h_1-h_2|,v\in[0,1],h_1,h_2\in\mathbb{R} \quad (27)$$

with $\epsilon_1>0$.

[D2] Suppose that the kernels $\Upsilon,\Psi:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying:

$$|\Upsilon(v,\varsigma,h_1)-\Upsilon(v,\varsigma,h_2)|\le\epsilon_2^k|h_1-h_2|,v,\varsigma\in[0,1],h_1,h_2\in\mathbb{R} \quad (28)$$

$$|\Psi(v,\varsigma,h_1)-\Psi(v,\varsigma,h_2)|\le\epsilon_2^h|h_1-h_2| \quad (29)$$

with $\epsilon_2^k,\epsilon_2^h>0$. If $\Lambda\in C^1[0,1]$ satisfies:

$$\Big|\Lambda'(v)+{}^cD_{0+}^{\alpha}\Lambda(v)-g\big(v,\Lambda(v)\big)-\int_0^v\Upsilon\big(v,\varsigma,\Lambda(\varsigma)\big)d\varsigma-\int_0^1\Psi\big(v,\varsigma,\Lambda(\varsigma)\big)\,d\varsigma\Big|\le\zeta(v),v\in[0,1], \quad (30)$$

and if:

$$(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi<1 \quad (31)$$

Then we have $\Theta(v)$ is a solution of the system (5) satisfies:

$$|\Lambda(v)-\Theta(v)|\le\frac{\xi\zeta(v)}{1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi},v\in[0,1] \quad (32)$$

Proof: Applying Lemma 3.1, the equivalent equation of Eq. (5) is given by:

$$\Theta(v)=\eta+\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\Big[g\big(\varsigma,\Theta(\varsigma)\big)+\int_0^\varsigma\Upsilon\big(\varsigma,\tau,\Theta(\tau)\big)d\tau+\int_0^1\Psi\big(\varsigma,\tau,\Theta(\tau)\big)\,d\tau\Big]\,d\varsigma \quad (33)$$

Define the operator $\Omega: C^1[0,1]\to C^1[0,1]$ by:

$$\begin{aligned}(\Omega\omega)(v)&=\eta+\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)g\big(\varsigma,\omega(\varsigma)\big)d\varsigma\\&+\int_0^v\int_0^\varsigma E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\Upsilon\big(\varsigma,\tau,\omega(\tau)\big)d\tau d\varsigma\\&+\int_0^v\int_0^1 E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\Psi\big(\varsigma,\tau,\omega(\tau)\big)d\tau d\varsigma,v\in[0,1],\omega\in C^1[0,1]\end{aligned} \quad (34)$$

From the hypotheses [D1] and [D2], so **Ω** is continuous.

Now, we will show that **Ω** is strictly contractive in ($C^1[0,1]$, ρ). By weighted metric ρ definition and Eqns. (26)-(28), for any $\omega,w\in C^1[0,1]$, it can obtain:

$$\begin{aligned}\rho(\Omega\omega,\Omega w)&\le\sup_{v\in[0,1]}\frac{\left|\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\left[g\big(\varsigma,w(\varsigma)\big)-g\big(s,w(\varsigma)\big)\right]d\varsigma\right|}{\zeta(v)}\\&+\sup_{v\in[0,1]}\frac{\left|\int_0^v\int_0^\varsigma E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\left[\Upsilon\big(\varsigma,\tau,\omega(\tau)\big)-\Upsilon\big(\varsigma,\tau,w(\tau)\big)\right]d\tau d\varsigma\right|}{\varsigma(v)}\\&+\sup_{v\in[0,1]}\frac{\left|\int_0^v\int_0^1 E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\left[\Psi\big(\varsigma,\tau,\omega(\tau)\big)-\Psi\big(\varsigma,\tau,w(\tau)\big)\right]d\tau d\varsigma\right|}{\varsigma(v)}\\&\le\epsilon_1\sup_{v\in[0,1]}\frac{\left|\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)|\omega(\varsigma)-w(\varsigma)|d\varsigma\right|}{\varsigma(v)}\\&+\epsilon_2^k\sup_{v\in[0,1]}\frac{\left|\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\int_0^\varsigma|\omega(\tau)-w(\tau)|d\tau d\varsigma\right|}{\varsigma(v)}\\&+\epsilon_2^h\sup_{v\in[0,1]}\frac{\left|\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\int_0^1|\omega(\tau)-w(\tau)|d\tau d\varsigma\right|}{\varsigma(v)}\\&\le c_1\xi\rho(\omega,w)+c_2^k\xi\rho(\omega,w)+c_2^h\xi(\omega,w)=(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi\rho(\omega,w)\end{aligned} \quad (35)$$

From the hypothesis $(\epsilon_1+\epsilon_2^k+\epsilon_h)\xi<1$, so Ω is strictly contractive.

Here, it can assume that $\Lambda(v)\in C^1[0,1]$ satisfies Eq (30). By Eqns. (30), (26) and Lemma 3.1, then:

$$\begin{aligned}&\Big|\Lambda(v)-\eta-\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\Big[g\big(s,\Lambda(\varsigma)\big)+\int_0^\varsigma\Upsilon\big(\varsigma,\tau,\Lambda(\tau)\big)d\tau+\int_0^1\Psi\big(\varsigma,\tau,\Lambda(\tau)\big)d\tau\Big]d\varsigma\Big|\le\\&\left|\int_0^v E_{1-\alpha,1}\left(-(v-\varsigma)^{1-\alpha}\right)\zeta(\varsigma)d\varsigma\right|\le\xi\zeta(v),\end{aligned} \quad (36)$$

The procedures for proving Eq (36) are the same as for proving Lemma 3.1. From Ω definition and Eq. (36), it can conclude:

$$|(\Omega\Lambda)(v)-\Lambda(v)|\le\xi\zeta(v) \quad (37)$$

Then, by ρ definition, it get:

$$\rho(\Omega\Lambda,\Lambda)\le\xi<1<\infty \quad (38)$$

Let $C^*[0,1]=\{y\in C^1[0,1]:\rho(\Omega\Lambda,y)<\infty\}$. By Banach fixed-point theorem, there is a unique solution $\Theta\in C^*[0,1]$ such that ΩΘ=Θ, that means Θ is a solution of the system (33). Therefore, Θ is the solution of the system (1). Then:

$$\rho(\Lambda,\Theta) \le \frac{1}{1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi}\rho(\Omega\Lambda,\Lambda) \le \frac{\xi}{1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi} \quad (39)$$

From ρ definition, the Eq. (33) holds. This completes the proof.

3.2 Semi-U-H-R and U-H stabilities for the system (5)

The study will investigate in this segment the stabilities of U-H and semi-U-H-R in $C^1[0,1]$ for the system in Eq. (5).

Theorem 3.3 Assume that a function ζ is continuous non-decreasing defined as $\zeta: [0, 1] \to (0, \infty)$ and there is $\xi \in [0, 1)$ satisfying:

$$\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\zeta(\varsigma)d\varsigma \le \xi\zeta(v) \quad (40)$$

Let $\epsilon_1, \epsilon_2^k, \epsilon_2^h > 0$ for $(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi < 1$. Assume that $h: [0,1]\times\mathbb{R}\to\mathbb{R}$ and $\Upsilon, \Psi: [0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying:

$$\begin{cases} |g(v,h_1)-g(v,h_2)| \le \epsilon_1|h_1-h_2|, v\in[0,1], h_1,h_2\in\mathbb{R} \\ |\Upsilon(v,\varsigma,h_1)-\Upsilon(v,\varsigma,h_2)| \le \epsilon_2^k|h_1-h_2|, v,\varsigma\in[0,1], h_1,h_2\in\mathbb{R} \\ |\Psi(v,\varsigma,h_1)-\Psi(v,\varsigma,h_2)| \le \epsilon_2^h|h_1-h_2|, v,\varsigma\in[0,1], h_1,h_2\in\mathbb{R} \end{cases} \quad (41)$$

If $\Lambda \in C^1[0,1]$ satisfies:

$$\left|\Lambda'(v) + {}^cD_{0+}^\alpha\Lambda(v) - g(v,\Lambda(v)) - \int_0^v \Upsilon(v,\varsigma,\Lambda(\varsigma))d\varsigma - \int_0^1 \Psi(v,\varsigma,\Lambda(\varsigma))d\varsigma\right| \le \theta, v\in[0,1] \quad (42)$$

with $\theta>0$, thus there is $\Theta(v)$ solution of the system (5) satisfies:

$$|\Lambda(v)-\Theta(v)| \le \frac{\theta\zeta(v)}{[1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi]\zeta(0)}\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right|, v\in[0,1] \quad (43)$$

This means that under above conditions, the system (5) has the semi-U-H-R stability.

Proof. Consider $\Omega: C^1[0,1] \to C^1[0,1]$, defined by:

$$(\Omega\omega)(v) = \eta + \int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\left[g(s,\omega(\varsigma)) + \int_0^\varsigma \Upsilon(\varsigma,\tau,\omega(\tau))d\tau + \int_0^1 \Psi(\varsigma,\tau,\omega(\tau))d\tau\right]d\varsigma \quad (44)$$

where, $v\in[0,1], \omega\in C^1[0,1]$.

For any $\omega, w \in C^1[0,1]$,then it has:

$$\rho(\Omega\omega,\Omega\omega) \le (\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi\rho(\omega,w) \quad (45)$$

From $(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\,\xi < 1$, then Ω is strictly contractive in $(C^1[0,1],\rho)$.

Next, suppose that $\Lambda(v)\in C^1[0,1]$ satisfies Eq. (42). By Eq. (42) and Lemma 3.1 it can get:

$$\left|\Lambda(v) - \eta - \int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\left[g(s,\Lambda(\varsigma)) + \int_0^\varsigma \Upsilon(\varsigma,\tau,\Lambda(\tau))d\tau + \int_0^1 \Psi(\varsigma,\tau,\Lambda(\tau))d\tau\right]d\varsigma\right| \le \theta\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right|, v\in[0,1] \quad (46)$$

From the continuity of Mittag-Leffler function, we have $\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right|$ is a continuous nonnegative function. From the Eqns. (44) and (46), then:

$$|(\Omega\Lambda)(v)-\Lambda(v)| \le \theta\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right| \quad (47)$$

Since ζ is a continuous function, it can get:

$$\rho(\Omega\Lambda,\Lambda) = \sup_{v\in[0,1]}\frac{|(\Omega\Lambda)(v)-\Lambda(v)|}{\zeta(v)} \le \sup_{v\in[0,1]}\frac{\theta\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right|}{\zeta(0)} < \infty \quad (48)$$

Let $C^*[0,1] = \{y\in C^1[0,1]: \rho(\Omega\Lambda,y)<\infty\}$.

Applying the Banach fixed-point theorem, thus there is a solution $\Theta\in C^*[0,1]$ such that ΩΘ=Θ. That means $\Theta(v)$ is a unique solution of the system (5).

From the Banach fixed-point theorem and Eq. (48), then:

$$\rho(\Lambda,\Theta) \le \frac{1}{1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi}\rho(\Omega\Lambda,\Lambda) \le \frac{\theta\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right|}{[1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi]\zeta(0)} \quad (49)$$

Thus, by the definition of ρ, then:

$$|\Lambda(v)-\Theta(v)| \le \frac{\theta}{[1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi]\zeta(0)}\zeta(v)\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right| \quad (50)$$

where, $\zeta(v)\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\,d\varsigma\right|$ is a continuous non-negative function. This completes the proof.

Remark 3.4 For any $v\in[0,1]$, $\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma$ is real number convergent series. Then, there exists $N>0$, such that:

$$\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\,d\varsigma\right| < N \quad (51)$$

Theorem 3.5 Assume that $\epsilon_1, \epsilon_2^k, \epsilon_2^h, \xi$ are constants for which $\epsilon_1 > 0, \epsilon_2^k > 0, \epsilon_2^h > 0, 0 \le \xi < 1, (\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi < 1$. Assume that g, Y, and Ψ are continuous functions, such that:

$$\begin{cases} |g(v,h_1)-g(v,h_2)| \le \epsilon_1|h_1-h_2|, v\in[0,1], h_1,h_2\in\mathbb{R} \\ |\Upsilon(v,\varsigma,h_1)-\Upsilon(v,\varsigma,h_2)| \le \epsilon_2^k|h_1-h_2|, v,\varsigma\in[0,1], h_1,h_2\in\mathbb{R} \\ |\Psi(v,\varsigma,h_1)-\Psi(v,\varsigma,h_2)| \le \epsilon_2^k|h_1-h_2|, v,\varsigma\in[0,1], h_1,h_2\in\mathbb{R} \end{cases}$$

Let $\zeta: [0,1] \to (0,\infty)$ be a continuous non-decreasing function, and satisfies:

$$\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})\zeta(\varsigma)d\varsigma \le \xi\zeta(v) \quad (52)$$

If $\Lambda \in C^1[0,1]$ satisfies (4), with $\theta>0$, then there is a solution $\Theta(v)$ of the system (5) such that:

$$|\Lambda(v)-\Theta(v)| \le \frac{N\zeta(1)}{[1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi]\zeta(0)}\theta, v\in[0,1] \quad (53)$$

Proof. Since ζ is a continuous non-decreasing function,

$$\zeta(v) \le \zeta(1), v\in[0,1] \quad (54)$$

By theorem 3.3, Eqns. (43) and (51), then it can obtain:

$$|\Lambda(v)-\Theta(v)| \leq \frac{\theta}{[1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi]\zeta(0)}\zeta(v)\left|\int_0^v E_{1-\alpha,1}(-(v-\varsigma)^{1-\alpha})d\varsigma\right| \leq \frac{N\zeta(1)}{[1-(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi]\zeta(0)}\theta \tag{55}$$

Theorem 3.5 shows that the system (5) has the U-H stability.

3.3 Illustrative example

Example 1. Let's assume a fractional Volterra-Fredholm system as follows

$$\Theta'(v)+{}^cD_{0+}^{\frac{1}{2}}\Theta(v) = \frac{1}{100}[v\cos\Theta(v) + \Theta(v)\sin v] + \frac{1}{50}\int_0^v \sin\Theta(\varsigma)d\varsigma + \frac{1}{50}\int_0^1 \cos\Theta(\varsigma)d\varsigma, \tag{56}$$

$$\Theta(0) = 0 \tag{57}$$

By comparison with the system (5), it can get:

$$\alpha = \frac{1}{2}, g(v,\Theta(v)) = \frac{1}{100}[v\cos\Theta(v) + \Theta(v)\sin v], \Upsilon(v,\varsigma,\Theta(\varsigma)) = \frac{1}{50}\sin\Theta(\varsigma), \Psi(v,\varsigma,\Theta(\varsigma)) = \frac{1}{50}\sin\Theta(\varsigma), \Psi(v,\varsigma,\Theta(\varsigma)) = \frac{1}{50}\cos\Theta(\varsigma). \tag{58}$$

Then:

$$\begin{cases} |g(v,h_1)-g(v,h_2)| \leq \frac{1}{50}|h_1-h_2|, h_1,h_2 \in \mathbb{R}, v \in [0,1] \\ |\Upsilon(v,\varsigma,h_1)-\Upsilon(v,\varsigma,h_2)| \leq \frac{1}{50}|h_1-h_2|, h_1,h_2 \in \mathbb{R}, v,\varsigma \in [0,1] \\ |\Psi(v,\varsigma,h_1)-\Psi(v,\varsigma,h_2)| \leq \frac{1}{50}|h_1-h_2|, h_1,h_2 \in \mathbb{R}, v,\varsigma \in [0,1] \end{cases} \tag{59}$$

Let $\zeta(v) = e^v$, it can obtain:

$$\int_0^v E_{\frac{1}{2},1}\left(-(v-\varsigma)^{\frac{1}{2}}\right)e^\varsigma\, d\varsigma < e^v - 1 < \frac{3}{4}e^v, v \in [0,1] \tag{60}$$

Here, it has $\epsilon_1 = \epsilon_2^k = \epsilon_2^h = \frac{1}{50}, \xi = \frac{3}{4}$, and $(\epsilon_1+\epsilon_2^k+\epsilon_2^h)\xi = 0.045 < 1$.

It can see that all the conditions in Theorems 3.2 and 3.5 are satisfied. Then, the system (56) is U-H stability, U-H-R stability and semi-U-H-R stability.

4. CONCLUSIONS

The objective of this study was to provide and demonstrate a novel stability theorem for the nonlinear Volterra-Fredholm integro-differential equation with Caputo fractional derivative utilising the weighted space method and fixed-point technique. The study specifically examines the H-U-R stability and semi-U-H-R stability results.

Besides, a class of nonlinear fractional Volterra-Fredholm integro-differential equations with initial conditions is discussed. By means of the Banach fixed-point techniques and weighted space, stability of the fractional nonlinear Volterra–Fredholm system has been tested. An illustrative example that demonstrates the applicability of the results has been included.

Discussing U-H-Mittag-Leffler stability [22] and finite-time stability [23] for the -Hilfer fractional Volterra-Fredholm integro-differential equations with time-varying delay terms would be a delightful extension of the current results. This will be the focus of future research.

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