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A NEW ADJOINT PROBLEM FOR TWO-DIMENSIONAL HELMHOLTZ EQUATION TO CALCULATE TOPOLOGICAL DERIVATIVES OF THE OBJECTIVE FUNCTIONAL HAVING TANGENTIAL DERIVATIVE QUANTITIES

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ABSTRACT

A special topology optimization problem is considered whose objective functional consists of tangential derivative of the potential on the boundary for two-dimensional Helmholtz equation. In order to derive the adjoint problem, the functional of the conventional topology optimizations required a boundary integral of the potential and its flux. For the present objective functional having the tangential derivative, integration by parts is applied to the part having the tangential derivative of the variation of the potential to generate a tractable adjoint problem. The derived adjoint problem is used in the variation of the objective function, and the topological derivative is derived in the conventional expression.
Keywords: adjoint problem, boundary element method, tangential derivative of potential, topological derivative, topology optimization.

1 INTRODUCTION

The boundary element method (BEM) is useful for topology optimization problems for linear problems, especially wave propagation problems and those for open domains [1,2]. Level-set method [3–7], which uses level-set function for expressing and controlling material distribution, is one of the popular methods among topology optimization methods, and the BEM can be effectively used with it by extracting the boundary of the material from the distribution of the level-set function [8].

Topology optimization problem in which BEM is advantageous to use has an objective functional consisting only of a boundary integral of the primary variable and its derivative quantity, which is the normal flux in potential problems and the traction in elastic problems. There is a case in which we have to consider a boundary functional consisting of the stress components. For example, when we detect the defects in the material from the measurement data of stresses on the boundary [9], this inverse problem can be treated as a topology optimization problem of searching the distribution of the material by minimizing a boundary objective functional of the stresses. There are also shape optimizations whose objective functional is defined with stress distribution (e.g. [10]). The boundary functional of stress has not been appropriate as the stress components involve not only the traction, but also the displacement gradients. However, because the displacement gradients are not the boundary quantities either given as the boundary condition or obtained as the direct boundary unknowns, the adjoint boundary value problem for calculating the sensitivity of the objective functional cannot be derived in the conventional manner.

The target of the present paper is derivation of the new adjoint problem required for calculating the topological derivative. As a test case, we consider a two-dimensional problem governed by the Helmholtz equation. The objective functional of the topology optimization problem is assumed to consist only of the tangential derivative of the potential on some part of the boundary. In the derivation process of the topological derivative, which is used for the

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level-set method [6], an adjoint problem is newly derived by integrating by parts some term of the objective functional.

2 BOUNDARY INTEGRAL EQUATIONS

We consider the following boundary value problem for two-dimensional Helmholtz equation:

$$\nabla^2 u(x) + k^2 u(x) = 0, \quad x \in \Omega, \tag{1}$$

$$u(x) = \bar{u}(x), \quad x \in \Gamma_u, \tag{2}$$

$$q(x) = \frac{\partial u}{\partial n}(x) = \bar{q}(x), \quad x \in \Gamma_q, \tag{3}$$

where u is the potential, k is the wave number, q is the normal flux, n is the outward normal direction to the boundary $\partial\Omega(=\Gamma_u \cup \Gamma_q)$, and $\bar{u}(x)$ and $\bar{q}(x)$ are known functions, respectively. When Ω is an open domain, a radiation condition for $u - u^{\mathrm{in}}$, where u^{in} is the incident wave, is also needed in addition to eqns (2) and (3).

The boundary integral equation for two-dimensional Helmholtz equation is given as

$$cu(x) + \int_\Gamma q^*(x,y)u(y)d\Gamma_y + \int_\Gamma u^*(x,y)q(y)d\Gamma_y \tag{4}$$

where c is a constant, and when x is on the smooth part of the boundary, $c = 1/2$; u^* and q^* are the fundamental solution of the Helmholtz equation and its corresponding normal flux, respectively, given as follows:

$$u^*(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(kr), \tag{5}$$

$$q^*(x,y) = \frac{\partial u^*(x,y)}{\partial n(y)} = -\frac{k}{4} H_1^{(1)}(kr)\frac{\partial r}{\partial n}, \tag{6}$$

where $r = |x - y|$, and $H_m^{(1)}$ is the Hankel function of the first kind of the m-th order.

3 LEVEL-SET METHOD AND TOPOLOGICAL DERIVATIVE

We consider using the level-set method [6,8], which uses the level-set function $\phi(x)$ for representing material distribution. The level-set function is defined here as

$$\left.\begin{array}{ll} \phi(x) > 0, & x \in \Omega \\ \phi(x) = 0, & x \in \Gamma \\ \phi(x) < 0, & x \in \bar{\Omega} \end{array}\right\} \tag{7}$$

The boundary of the material can be extracted from the isoline corresponding to $\phi(x) = 0$. Hence, we find the shape of the material domain in accordance with the change of the distribution of $\phi(x)$. The level-set method being used here uses the following differential equation defined in a fixed domain for the evolution of the level-set function:

$$\frac{\partial \phi(x)}{\partial t} = K\left(T(x) + \tau\nabla^2\phi(x)\right) \tag{8}$$

where K and τ are both positive constants and $T(x)$ is the topological derivative which is the rate of the value of the objective functional when an infinitesimal region centered at x is removed from the original domain.

4 TOPOLOGICAL DERIVATIVE FOR STANDARD OBJECTIVE FUNCTIONAL

The objective functional of the standard form for topology optimizations can be written as follows:

$$J = \int_\Gamma f(u,q)\,d\Gamma + \int_\Omega g(u,\nabla u)\,d\Omega, \tag{9}$$

Because the above expression contains a domain integral, when the functional g is expressed by means of Dirac's delta functions, the BEM is a strong tool to use for calculating the field quantities. We consider here a form which is more useful for BEM as

$$J = \int_\Gamma f(u,q)\,d\Gamma. \tag{10}$$

In this case, the topological derivative can be derived by using the same method given in [8] and can be related to the potential u and its gradients ∇u and the adjoint potential $\tilde{u}$ and its gradients $\nabla\tilde{u}$ by

$$T(x) = 2\nabla\tilde{u}(x)\cdot\nabla u(x) - k^2\tilde{u}(x)u(x), \tag{11}$$

where the adjoint potential $\tilde{u}$ is the solution of the following boundary value problem.

$$\nabla^2\tilde{u}(x) + k^2\tilde{u}(x) = 0, \quad x \in \Omega, \tag{12}$$

$$\tilde{u}(x) = -\frac{\partial f(u,q)}{\partial q}(x), \quad x \in \Gamma_u, \tag{13}$$

$$\tilde{q}(x) = \frac{\partial f(u,q)}{\partial u}(x), \quad x \in \Gamma_q. \tag{14}$$

5 TOPOLOGICAL DERIVATIVE FOR A BOUNDARY FUNCTIONAL OF THE TANGENTIAL DERIVATIVE OF THE POTENTIAL

In this section, we consider a non-standard boundary functional of the tangential derivative of the potential as follows:

$$J = \int_\Gamma f(u_{,s}(x))\,d\Gamma_x, \tag{15}$$

where $u_{,s} = \frac{\partial u}{\partial s}$. In eqn (15), although $f(u_{,s})$ is defined over the entire boundary $\Gamma = \partial\Omega$, it is assumed to have a finite support Γ_s in Γ as shown in Fig. 1.

Let us consider the case where an infinitesimal circular region Ω_ϵ of radius ϵ centered at x^{o} is removed from Ω as shown in Fig. 2. Because the potential is the solution of the Helmholtz equation, J is augmented as

$$\begin{aligned}\bar{J} &= J + \int_\Omega \tilde{u}(x)\left(\nabla^2 u(x) + k^2 u(x)\right)d\Omega_x \\ &= \int_\Gamma f(u_{,s}(x))\,d\Gamma_x + \int_\Omega \tilde{u}(x)\left(\nabla^2 u(x) + k^2 u(x)\right)d\Omega_x \end{aligned} \tag{16}$$

where $\tilde{u}(x)$ is a Lagrange multiplier.

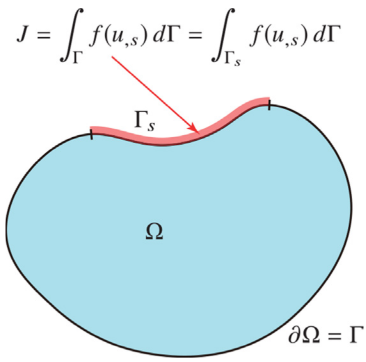

Figure 1: Boundary functional of $u_{,s}$ defined on some part of the boundary.

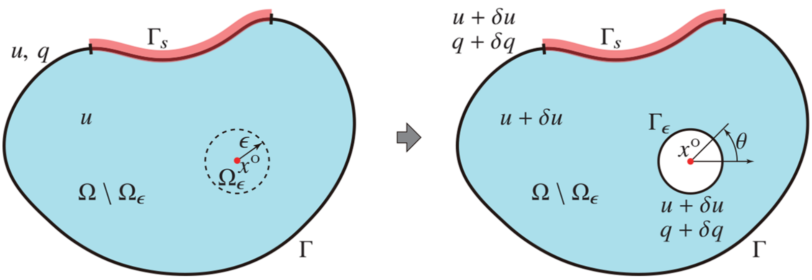

Figure 2: Removal of an infinitesimal circular region.

In what follows, the symbol indicating the point x is omitted unless necessary. Integrating by parts the second term of the right-hand side of eqn (16) gives

$$\bar{J} = \int_{\Gamma} f(u_{,s})\, d\Gamma + \int_{\Gamma} \tilde{u}\, q\, d\Gamma - \int_{\Omega} \nabla\tilde{u} \cdot \nabla u\, d\Omega + k^2 \int_{\Omega} \tilde{u}\, u\, d\Omega. \tag{17}$$

We consider the case in which the boundary condition on Γ_ϵ, which is the newly generated boundary after Ω_ϵ is removed, is the perfect reflection.

By removing an infinitesimal region Ω_ϵ from Ω, u in Ω and on Γ will change to $u + \delta u$ and q on Γ will change to $q + \delta q$. On the newly generated boundary Γ_ϵ, we observe that u at the corresponding point in Ω will change to $u + \delta u$ on Γ_ϵ, and q calculated at the point on the circle of the broken line oriented to the center of the circle x^{o} will change to $q + \delta q$ on Γ_ϵ. The boundary condition on Γ_ϵ is given as

$$q(x) + \delta q(x) = 0, \quad x \in \Gamma_\epsilon. \tag{18}$$

Also, the objective functional is considered only on the original boundary Γ even after a new boundary Γ_ϵ is generated. The variations of u and q cause the change of $\bar{J}$ to $\bar{J} + \delta\bar{J}$, i.e.

$$\begin{aligned}\bar{J} + \delta\bar{J} &= \int_{\Gamma} \left(f(u_{,s}) + \frac{\partial f(u_{,s})}{\partial u_{,s}} \delta u_{,s} \right) d\Gamma \\ &+ \int_{\Gamma} \tilde{u}(q + \delta q)\, d\Gamma + \int_{\Gamma_\epsilon} \tilde{u}(q + \delta q)\, d\Gamma \\ &- \int_{\Omega \setminus \Omega_\epsilon} \nabla\tilde{u} \cdot \nabla(u + \delta u)\, d\Omega + k^2 \int_{\Omega \setminus \Omega_\epsilon} \tilde{u}(u + \delta u)\, d\Omega. \end{aligned} \tag{19}$$

From eqns (19) and (17), we obtain the variation of $\bar{J}$ as follows:

$$\delta\bar{J} = \int_{\Gamma}\left(\frac{\partial f(u_{,s})}{\partial u_{,s}}\delta u_{,s}\right)d\Gamma + \int_{\Gamma}\tilde{u}\,\delta q\,d\Gamma + \int_{\Gamma_{\epsilon}}\tilde{u}\,(q+\delta q)\,d\Gamma$$
$$-\int_{\Omega\setminus\Omega_{\epsilon}}\nabla\tilde{u}\cdot\nabla(\delta u)\,d\Omega + k^2\int_{\Omega\setminus\Omega_{\epsilon}}\tilde{u}\,\delta u\,d\Omega$$
$$+\int_{\Omega_{\epsilon}}\nabla\tilde{u}\cdot\nabla u\,d\Omega - k^2\int_{\Omega_{\epsilon}}\tilde{u}\,u\,d\Omega. \tag{20}$$

By integrating by parts the fourth term on the right-hand side of eqn (20), we have

$$\delta\bar{J} = \int_{\Gamma}\left(\frac{\partial f(u_{,s})}{\partial u_{,s}}\delta u_{,s}\right)d\Gamma + \int_{\Gamma}\tilde{u}\,\delta q\,d\Gamma + \int_{\Gamma_{\epsilon}}\tilde{u}\,(q+\delta q)\,d\Gamma$$
$$-\int_{\Gamma}\tilde{q}\,\delta u\,d\Gamma - \int_{\Gamma_{\epsilon}}\tilde{q}\,\delta u\,d\Gamma + \int_{\Omega\setminus\Omega_{\epsilon}}\left(\nabla^2\tilde{u} + k^2\tilde{u}\right)\delta u\,d\Omega$$
$$+\int_{\Omega_{\epsilon}}\nabla\tilde{u}\cdot\nabla u\,d\Omega - k^2\int_{\Omega_{\epsilon}}\tilde{u}\,u\,d\Omega.$$

$$= \int_{\Gamma_u\cup\Gamma_q}\left(\frac{\partial f(u_{,s})}{\partial u_{,s}}\delta u_{,s}\right)d\Gamma + \int_{\Gamma_u}\tilde{u}\,\delta q\,d\Gamma + \int_{\Gamma_q}\tilde{u}\,\delta q\,d\Gamma + \int_{\Gamma_{\epsilon}}\tilde{u}\,(q+\delta q)\,d\Gamma$$
$$-\int_{\Gamma_u}\tilde{q}\,\delta u\,d\Gamma - \int_{\Gamma_q}\tilde{q}\,\delta u\,d\Gamma - \int_{\Gamma_{\epsilon}}\tilde{q}\,\delta u\,d\Gamma + \int_{\Omega\setminus\Omega_{\epsilon}}\left(\nabla^2\tilde{u} + k^2\tilde{u}\right)\delta u\,d\Omega$$
$$+\int_{\Omega_{\epsilon}}\nabla\tilde{u}\cdot\nabla u\,d\Omega - k^2\int_{\Omega_{\epsilon}}\tilde{u}\,u\,d\Omega. \tag{21}$$

Because on Γ_u and Γ_q, u and q are prescribed as the boundary conditions, respectively, we find $\delta u = 0$ on Γ_u and $\delta q = 0$ on Γ_q. Applying them and eqn (18) to eqn (21), we obtain

$$\delta\bar{J} = \int_{\Gamma_q}\left(\frac{\partial f(u_{,s})}{\partial u_{,s}}\delta u_{,s}\right)d\Gamma$$
$$+\int_{\Gamma_u}\tilde{u}\,\delta q\,d\Gamma - \int_{\Gamma_q}\tilde{q}\,\delta u\,d\Gamma + \int_{\Omega\setminus\Omega_{\epsilon}}\left(\nabla^2\tilde{u} + k^2\tilde{u}\right)\delta u\,d\Omega$$
$$-\int_{\Gamma_{\epsilon}}\tilde{q}\,\delta u\,d\Gamma + \int_{\Omega_{\epsilon}}\left(\nabla\tilde{u}\cdot\nabla u - k^2\,\tilde{u}\,u\right)d\Omega. \tag{22}$$

In eqn (22), $\delta u_{,s}$ on $\Gamma_s \in \Gamma$, δq on Γ_u, δu on Γ_q, and δu in $\Omega\setminus\Omega_{\epsilon}$, and on Γ_{ϵ} are all unknown quantities.

In order to convert $\delta u_{,s}$ on $\Gamma_s \in \Gamma$ to δu for combining it with the third term of the right-hand side of eqn (22), we integrate by parts the term containing $u_{,s}$ to obtain

$$\delta \bar{J} = \frac{\partial f(u_{,s})}{\partial u_{,s}} \delta u \Bigg|_{P_s}^{Q_s}$$
$$-\int_{\Gamma_q} \left[\tilde{q} + \frac{\partial}{\partial s}\left(\frac{\partial f(u_{,s})}{\partial u_{,s}} \right) \right] \delta u \, d\Gamma + \int_{\Gamma_u} \tilde{u} \, \delta q \, d\Gamma + \int_{\Omega \setminus \Omega_\epsilon} \left(\nabla^2 \tilde{u} + k^2 \tilde{u} \right) \delta u \, d\Omega$$
$$-\int_{\Gamma_\epsilon} \tilde{q} \, \delta u \, d\Gamma + \int_{\Omega_\epsilon} \left(\nabla \tilde{u} \cdot \nabla u - k^2 \tilde{u} u \right) d\Omega. \tag{23}$$

where P_s and Q_s are the start and end points of the interval Γ_s.

In eqn (23), there still remains unknown δu at P_s and Q_s, however, by discretizing the boundary into constant elements and extending Γ_s by one element before P_s and after Q_s, we can neglect the first term.

For the second term to fourth term of eqn (23), we consider the following adjoint boundary value problem for $\tilde{u}$:

$$\nabla^2 \tilde{u}(x) + k^2 \tilde{u}(x) = 0, \quad x \in \Omega, \tag{24}$$

$$\tilde{u}(x) = 0, \quad x \in \Gamma_u, \tag{25}$$

$$\tilde{q}(x) = -\frac{\partial}{\partial s}\left(\frac{\partial f(u_{,s})}{\partial u_{,s}}(x) \right), \quad x \in \Gamma_u, \tag{26}$$

By using the adjoint function $\tilde{u}$ in eqn (23), $\delta \bar{J}$ is simplified to result in

$$\delta \bar{J} = -\int_{\Gamma_\epsilon} \tilde{q} \, \delta u \, d\Gamma + \int_{\Omega_\epsilon} \left(\nabla \tilde{u} \cdot \nabla u - k^2 \tilde{u} u \right) d\Omega. \tag{27}$$

In order to evaluate the above integrals, we consider the behaviors of $\tilde{u}$ and δu in the neighborhood of the center point x^{o} of Ω_ϵ.

From the Lebesgue differentiation theorem, when ϵ tends to 0, we observe

$$\int_{\Omega_\epsilon} \left(\nabla \tilde{u} \cdot \nabla u - k^2 \tilde{u} u \right) d\Omega = \pi \epsilon^2 \left(\nabla \tilde{u}^{\mathrm{o}} \cdot \nabla u^{\mathrm{o}} - k^2 \tilde{u}^{\mathrm{o}} u^{\mathrm{o}} \right). \tag{28}$$

where $u^{\mathrm{o}} := u(x^{\mathrm{o}})$ and $\tilde{u}^{\mathrm{o}} := \tilde{u}(x^{\mathrm{o}})$. Because u and $\tilde{u}$ are defined in Ω, we have their Taylor series expansions about x^0 as follows:

$$u(x) = u^{\mathrm{o}} + r(\nabla_x u^{\mathrm{o}} \cos\theta + \nabla_y u^{\mathrm{o}} \sin\theta) + O(r^2) \tag{29}$$

$$\tilde{u}(x) = \tilde{u}^{\mathrm{o}} + r(\nabla_x \tilde{u}^{\mathrm{o}} \cos\theta + \nabla_y \tilde{u}^{\mathrm{o}} \sin\theta) + O(r^2) \tag{30}$$

$$q(x) = -\frac{\partial u(x)}{\partial r} = -(\nabla_x u^{\mathrm{o}} \cos\theta + \nabla_y u^{\mathrm{o}} \sin\theta) + O(r) \tag{31}$$

$$\tilde{q}(x) = -(\nabla_x \tilde{u}^{\text{o}} \cos\theta + \nabla_y \tilde{u}^{\text{o}} \sin\theta) + O(r), \tag{32}$$

where $\nabla_x u^{\text{o}} := \frac{\partial u}{\partial x}(x^{\text{o}})$ and $\nabla_y u^{\text{o}} := \frac{\partial u}{\partial y}(x^{\text{o}})$.

Because δu is defined in $\Omega \setminus \Omega_\epsilon$, we have its asymptotic expansion for x^{o} as follows:

$$\delta u(x) = \frac{a}{r}\cos\theta + \frac{b}{r}\cos\theta + c \times O(\frac{1}{r^2}), \tag{33}$$

which yields

$$\delta q(x) = -\frac{\delta u}{\delta r} = \frac{a}{r^2}\cos\theta + \frac{b}{r^2}\cos\theta - c \times O(\frac{1}{r^3}), \tag{34}$$

where a, b, and c are constants.

On Γ_ϵ, we have the boundary condition $q + \delta q = 0$; therefore, from $q = -\delta q$ for $r = \epsilon$, we have

$$-(\nabla_x u^{\text{o}} \cos\theta + \nabla_y u^{\text{o}} \sin\theta) + O(\epsilon) = -\frac{a}{\epsilon^2}\cos\theta - \frac{b}{\epsilon^2}\cos\theta - c \times O(\frac{1}{\epsilon^3}) \tag{35}$$

By comparing the coefficients of $\cos\theta$, $\sin\theta$, and the remaining order terms, we have

$$a = \epsilon^2 \nabla_x u^{\text{o}} \tag{36}$$

$$b = \epsilon^2 \nabla_y u^{\text{o}} \tag{37}$$

$$c = O(\epsilon^4) \tag{38}$$

Finally, we obtain δu on Γ_ϵ as

$$\delta u = \epsilon(\nabla_x u^{\text{o}} \cos\theta + \nabla_y u^{\text{o}} \sin\theta) + O(\epsilon^2) \tag{39}$$

Thus, the first integral on the right-hand side of eqn (27) is evaluated as follows:

$$\begin{aligned} -\int_{\Gamma_\epsilon} \tilde{q}\, \delta u \, d\Gamma &= \epsilon^2 \int_0^{2\pi} (\nabla_x \tilde{u}^{\text{o}} \cos\theta + \nabla_y \tilde{u}^{\text{o}} \sin\theta)(\nabla_x u^{\text{o}} \cos\theta + \nabla_y u^{\text{o}} \sin\theta) d\theta + O(\epsilon^4) \\ &= (\pi\epsilon^2)(\nabla_x \tilde{u}^{\text{o}} \nabla_x u^{\text{o}} + \nabla_y \tilde{u}^{\text{o}} \nabla_y u^{\text{o}}) + O(\epsilon^4) \\ &= (\pi\epsilon^2)(\nabla \tilde{u}^{\text{o}} \cdot \nabla u^{\text{o}}) + O(\epsilon^4) \end{aligned} \tag{40}$$

Using eqns (40) and (28) yields

$$\delta \bar{J} = \pi\epsilon^2 \left(2\nabla \tilde{u}^{\text{o}} \cdot \nabla u^{\text{o}} - k^2 \tilde{u}^{\text{o}} u^{\text{o}} \right) + O(\epsilon^4). \tag{41}$$

Thus, the topological derivative is obtained as follows:

$$T = \lim_{\epsilon \to 0} \frac{\delta \bar{J}}{\pi \epsilon^2} = 2\nabla \tilde{u}^{\text{o}} \cdot \nabla u^{\text{o}} - k^2 \tilde{u}^{\text{o}} u^{\text{o}}. \tag{42}$$

Thus, the topological derivative is obtained like that of the standard cases of eqn (11) by using the solutions of the new adjoint problem defined by eqns (24) ~ (26).

6 CONCLUDING REMARKS

In this paper, we have investigated a topology optimization problem for the field governed by two-dimensional Helmholtz equation. The objective functional of this problem has been assumed to consist of the tangential derivative of the potential, the solution of the Helmholtz equation, on some part of the boundary. In order to eliminate the tangential derivative of the variation of the boundary potential, integration by parts has been applied, resulting in a new adjoint problem. Further discussions are needed concerning the effective discretization and interpolation of the boundary objective of the tangential derivative for the solution of the derived adjoint problem.

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