KANSA RBF METHOD FOR NONLINEAR PROBLEMS

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ABSTRACT

We apply the Kansa–radial basis function (RBF) collocation method to two– dimensional nonlinear boundary value problems. The system of nonlinear equations resulting from the Kansa–RBF discretization is solved by directly applying a standard nonlinear solver. In a natural way, the value of the shape parameter in the RBFs employed in the approximation is included in the unknowns to be determined. The numerical results of some examples are presented and analysed.

Keywords: collocation, Kansa method, radial basis functions

1 INTRODUCTION

Radial basis function (RBF) methods have become popular in recent years in approximation theory as well as in the numerical solution of partial differential equations. The most widely used RBF method for the latter class of problems is the RBF collocation method due to Kansa [1], known as the *Kansa method*. The popularity of this method is due to its meshlessness which means that only a set of points is required in the discretization of the continuous problem. This renders the implementation of the method particularly easy, especially for problems in complex geometries and/or in three dimensions. A disadvantage of the method is the (unknown) optimal choice of the shape parameter which is found in most RBFs. Various techniques have been proposed for the determination of an appropriate value of the shape parameter, see e.g. [2–5].

We apply the Kansa RBF method to two-dimensional nonlinear boundary value problems. The discretization of such problems leads to systems of nonlinear equations which may be solved by using standard software. Moreover, we include the (unknown) value of the shape parameter in the set of unknowns of the nonlinear problem. The solution of the nonlinear problem thus yields not only the coefficients in the RBF approximation but, also, an appropriate value of the shape parameter. The implementation of the proposed technique is very simple and the results of several numerical experiments reveal it is satisfactorily accurate.

2 THE PROBLEM

We consider the boundary value problem in $\mathbb{R}^2$

$$\mathcal{L}u = f \quad \text{in} \quad \Omega, \tag{1a}$$

subject to the boundary condition

$$\mathcal{B}u = g \quad \text{on} \quad \partial\Omega, \tag{1b}$$

where $\mathcal{L}$ is a nonlinear elliptic operator and $\mathcal{B}$ is a linear (or nonlinear) operator describing the boundary condition.

3 THE KANSA METHOD

In the Kansa method [1] we approximate the solution u of boundary value problem (1) by a linear combination of RBFs [6–8]

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$$u_{\mathrm{N}}(x,y)=\sum_{\mathrm{n}=1}^{\mathrm{N}} a_{\mathrm{n}}\phi_{\mathrm{n}}(x,y), \quad (x,y)\in\bar{\Omega}. \tag{2}$$

In this work we shall be using the *normalized multiquadric* basis function

$$\phi_{\mathrm{n}}(x,y)=\Phi(r_{\mathrm{n}})=\sqrt{(cr_{\mathrm{n}})^2+1}, \text{ where } r_{\mathrm{n}}^2=(x-x_{\mathrm{n}})^2+(y-y_{\mathrm{n}})^2, \tag{3}$$

and the points $\{(x_{\mathrm{n}},y_{\mathrm{n}})\}_{\mathrm{n}=1}^{\mathrm{N}}$ are usually referred to as *centers*. Moreover, c is the *shape parameter* and such parameters are often present in RBFs and the determination of their optimal value remains a major challenge.

The coefficients $\{a_{\mathrm{n}}\}_{\mathrm{n}=1}^{\mathrm{N}}$ in equation (2) are determined from the collocation equations

$$\mathcal{L}u_{\mathrm{N}}(x_{\mathrm{m}},y_{\mathrm{m}})=f(x_{\mathrm{m}},y_{\mathrm{m}}), \quad \mathrm{m}=1,\ldots,\mathrm{M}_{\mathrm{int}}, \tag{4a}$$

$$\mathcal{B}u_{\mathrm{N}}(x_{\mathrm{m}},y_{\mathrm{m}})=g(x_{\mathrm{m}},y_{\mathrm{m}}), \quad \mathrm{m}=\mathrm{M}_{\mathrm{int}}+1,\ldots,\mathrm{M}_{\mathrm{int}}+\mathrm{M}_{\mathrm{bry}}, \tag{4b}$$

where $\mathrm{M}_{\mathrm{int}}+\mathrm{M}_{\mathrm{bry}}=\mathrm{M}$ and the points $\{(x_{\mathrm{m}},y_{\mathrm{m}})\}_{\mathrm{m}=1}^{\mathrm{M}}\in\bar{\Omega}$ are the *collocation points*. Note that, in general, the collocation points are not the same as the centers and M≥N.

Since the operator $\mathcal{L}$ is nonlinear, the system of M equations resulting from equations (4a)–(4b) is nonlinear and will be solved in a (nonlinear) least squares sense. Moreover, we shall include the value of the shape parameter c of the RBF in the unknowns. We shall therefore be minimizing of the functional

$$\begin{aligned} S(a,c):=&\sum_{\mathrm{m}=1}^{\mathrm{M}_{\mathrm{int}}}[\mathcal{L}u_{\mathrm{N}}(x_{\mathrm{m}},y_{\mathrm{m}})-f(x_{\mathrm{m}},y_{\mathrm{m}})]^2\\ &+\sum_{\mathrm{m}=\mathrm{M}_{\mathrm{int}}+1}^{\mathrm{M}}[\mathcal{B}u_{\mathrm{N}}(x_{\mathrm{m}},y_{\mathrm{m}})-g(x_{\mathrm{m}},y_{\mathrm{m}})]^2, \end{aligned} \tag{5}$$

where the vector $\boldsymbol{a}=[a_1, a_2, \ldots, a_{\mathrm{N}}]^T$. Thus the goal is to determine not only the vector of coefficients $\boldsymbol{a}$ but also the value of the shape parameter c. The minimization of functional (5) is carried out using the MATLAB© [9] optimization toolbox routine **`lsqnonlin`** which solves nonlinear least squares problems using, by default, a subspace trust region method. The routine **`lsqnonlin`** does not require the user to provide the gradient and, in addition, it offers the option of imposing lower and upper bounds on the elements of the vector of unknowns $x=[\boldsymbol{a}, c]^T$ through the vectors **`lb`** and **`ub`**. We can thus easily impose the constraints on the values of the coefficients and in particular on the values of the shape parameter. In all numerical experiments we chose the constraints $0 < c < 8$ and took the initial vector of coefficients $\boldsymbol{a} = 0$.

4 NUMERICAL RESULTS

In the numerical examples considered in this section, we calculated the approximate solution u_{N} at L test points on a grid in $\bar{\Omega}$. In the case of Examples 1 and 2 the analytical solution is known. Hence, we calculated the maximum relative error E defined by

$$E=\frac{\|u-u_{\mathrm{N}}\|_{\infty,\bar{\Omega}}}{\|u\|_{\infty,\bar{\Omega}}} \tag{6}$$

and the root mean square error (RMSE) *from*

$$\varepsilon = \sqrt{\frac{\sum_{l=1}^{L}[u(x_l, y_l) - u_{\mathrm{N}}(x_l, y_l)]^2}{L}}. \tag{7}$$

4.1 Example 1

We first consider an example from [10, 11] where

$$\mathcal{L}u = -\varepsilon^2 \Delta u - u + u^3 \text{ in } \Omega,$$

subject to the homogeneous Dirichlet boundary condition

$$\mathcal{B}u = u = 0 \text{ on } \partial\Omega,$$

where $\Omega = (0, 1) \times (0, 1)$, the exact solution is given by

$$u(x, y) = (1 + \mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-x/\varepsilon} - \mathrm{e}^{(x-1)/\varepsilon})(1 + \mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-y/\varepsilon} - \mathrm{e}^{(y-1)/\varepsilon})$$

and ε is a known constant. Clearly, the function f in (1a) is obtained by calculating $\mathcal{L}u$ for the u given above. We chose a uniform $M \times M$ grid in $\overline{\Omega}$ for the collocation points and a uniform $N \times N$ grid in $\overline{\Omega}$ for the centres. This arrangement means that we have a total of $\mathrm{M} = M^2$ collocation points and $\mathrm{N} = N^2$ centres. Moreover, we have $\mathrm{M}_{\mathrm{int}} = M^2 - 4M + 4$ interior collocation points and $\mathrm{M}_{\mathrm{bry}} = 4M–4$ boundary points.

The set of test points was taken to be a uniform 51 × 51 grid in $\overline{\Omega}$. Then, for the routine **lsqnonlin** we chose the initial value of the shape parameter $c_0 = 4$ and the maximum number of iterations **niter** equal to 100. We performed calculations for $\varepsilon = 0.1$, 0.15, 0.25, 0.5 and 1. It was observed that the accuracy of the results obtained was poor for $\varepsilon = 0.1$. However, when using continuation from $\varepsilon = 1$ to $\varepsilon = 0.5$ to $\varepsilon = 0.25$ to $\varepsilon = 0.15$ to $\varepsilon = 0.1$, satisfactory results were obtained as is shown in Table 1. By continuation we mean that the final values of all the unknown parameters in **lsqnonlin** for $\varepsilon = 1$ were saved and used as the initial values for the unknown parameters in **lsqnonlin** for $\varepsilon = 0.5$ and so on.

Table 1: Example 1: Results with continuation.

M	N	M	N	ε	c	E	RMSE
15	14	225	196	1.000	1.959	8.616(-3)	2.544(-5)
				0.500	2.078	5.492(-3)	1.911(-4)
				0.250	2.444	1.207(-2)	4.176(-3)
				0.150	2.784	1.548(-2)	6.299(-3)
				0.100	2.906	2.937(-2)	1.408(-2)
16	15	256	225	1.000	4.006	3.502(-3)	5.884(-5)
				0.500	4.049	4.258(-3)	5.331(-4)
				0.250	4.095	8.841(-3)	3.863(-3)
				0.150	4.175	1.331(-2)	7.524(-3)
				0.100	4.210	2.736(-2)	1.341(-2)

(*Continued*)

Table 1: (*Continued*)

M	N	M	N	ε	c	E	RMSE
22	21	484	441	1.000	4.016	7.360(-4)	1.271(-5)
				0.500	4.096	9.611(-4)	1.237(-4)
				0.250	4.280	2.643(-3)	1.153(-3)
				0.150	4.420	1.076(-2)	5.360(-3)
				0.100	4.658	1.434(-2)	6.366(-3)
24	23	576	529	1.000	4.014	6.191(-4)	7.844(-6)
				0.500	4.123	7.583(-4)	8.109(-5)
				0.250	4.311	4.237(-3)	1.041(-3)
				0.150	4.411	1.595(-2)	5.992(-3)
				0.100	4.686	1.821(-2)	5.606(-3)

4.2 Example 2

We next consider an example from [12, 13] where the governing equation is

$$\mathcal{L}u = \Delta u - 4u^3 = 0 \quad \text{in} \quad \Omega, \tag{8}$$

subject to Dirichlet boundary conditions corresponding to the exact solution

$$u(x,y) = \frac{1}{4+x+y}.$$

The computational domain Ω is peanut shaped and its boundary $\partial\Omega$ is defined parametrically by

$$x = r(\vartheta)\cos\vartheta,\ y = r(\vartheta)\sin\vartheta,\ \text{where } r(\vartheta) = 0.3\sqrt{\cos 2\vartheta + \sqrt{1.1 - \sin^2 2\vartheta}},$$

$0 \le \vartheta \le 2\pi$. The collocation points are distributed as follows:

$$(x_{ij}, y_{ij}) = r_j\sqrt{\cos 2\vartheta_i + \sqrt{1.1 - \sin^2 2\vartheta_i}}\,\left(\cos\vartheta_i, \sin\vartheta_i\right), \tag{9}$$

$$\vartheta_i = \frac{2\pi(i-1)}{M_\vartheta},\ i = 1,\dots,M_\vartheta,\ \text{and } r_j = \frac{0.3j}{M_r},\ j = 1,\dots,M_r.$$

A more uniform distribution of points is obtained as follows, see e.g [14]. We first calculate the length of the boundary curve from

$$S = \int_0^{2\pi} \sqrt{r(\vartheta)^2 + r'(\vartheta)^2 d\vartheta},$$

which may be evaluated using the MATLAB© routine **quadl** which evaluates an integral using adaptive Lobatto quadrature within a user–prescribed accuracy. Since there will be M_ϑ

collocation points on the boundary, we take the length of each subsegment to be S/M_ϑ. The angles ϑ_k which give such a distribution are obtained by choosing $\vartheta_1 = 0$ and then solving, serially for $k = 1,\ldots, M_\vartheta - 1$, the nonlinear equations

$$F(t) = \sqrt{(r(t)\cos t - r(\vartheta_k)\cos\vartheta_k)^2 + (r(t)\sin t - r(\vartheta_k)\sin\vartheta_k)^2} - \frac{S}{M_\vartheta} = 0,$$

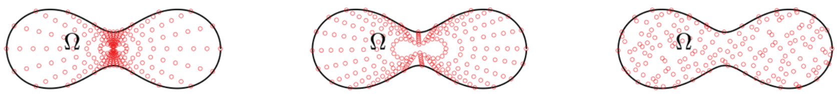
to yield $t = \vartheta_k$, $k = 2,\ldots, M_\vartheta$, respectively. The solution of the nonlinear equations may be carried out using the MATLAB© routine **`fzero`**. The angles ϑ_k ,$k = 1,\ldots, M_\vartheta$, define equally spaced points on the boundary curve.

The collocation points are now distributed as follows:

$$(x_{ij}, y_{ij}) = r_j\sqrt{\cos 2\vartheta_i + \sqrt{1.1 - \sin^2 2\vartheta_i}}\,(\cos\vartheta_i, \sin\vartheta_i,$$

$$i = 1,\ldots, M_\vartheta,\ j = 1,\ldots, M_r,\ \text{where } r_j = 0.3\left(\frac{j}{M_r}\right)^p,\ j = 1,\ldots, M_r.$$

The parameter $p \in (0,1)$ is chosen so that the collocation points are pushed away from the centre of the domain to yield a more uniform distribution of collocation points. An appropriate value of p was found to be 2/3. The centres are chosen in an identical way with M_ϑ and M_r replaced by N_ϑ and N_r, respectively. This arrangement means that we have a total of $\mathrm{M} = M_\vartheta M_r$ collocation points and $\mathrm{N} = N_\vartheta N_r$ centres. Moreover, we have $\mathrm{M}_{\mathrm{int}} = M_\vartheta(M_r - 1)$ interior collocation points and $\mathrm{M}_{\mathrm{bry}} = M_\vartheta$ boundary points. An even more uniform distribution of collocation points and centres maybe obtained by using the *Halton* points [8, Appendix A.1]. These may be generated using the MATLAB© code **`haltonseq.m`**. In this approach we take M_{b} boundary collocation points and M_{i} interior points, and N_{b} boundary centres points and N_{i} interior centres. A typical distribution of collocation points using the first distribution, the more uniform distribution and the Halton points is presented in Fig. 1. The maximum relative error E and RMSE were calculated on a 21×21 grid in $\overline{\Omega}$ generated using a distribution similar to (9) and clearly different than the collocation points. The maximum number of iterations in **`lsqnonlin`** was set to 100 and we took $c_0 = 4$. In Table 2 we present some numerical results for various numbers of degrees of freedom. The corresponding results with the improved distribution of collocation points and centres are presented in Table 3. The results obtained with comparable numbers of degrees of freedom using Halton points are presented in Table 4.



(a) First distribution (b) Improved distribution (c) Halton points

Figure 1: Example 2. Computational domain and typical distributions of collocation points.

Table 2: Example 2: Results for various numbers of degrees of freedom.

M_ϑ	M_r	M	N_ϑ	N_r	N	c	E	RMSE
12	10	120	10	8	80	2.095	3.574(-3)	2.395(-4)
14	12	168	12	10	120	2.134	4.099(-3)	2.550(-4)
16	14	224	14	12	168	2.730	7.854(-3)	6.251(-4)
18	16	288	16	14	224	2.862	6.227(-3)	4.396(-4)
20	18	360	18	16	288	2.854	8.104(-3)	6.215(-4)
22	20	440	20	18	360	3.086	5.796(-3)	2.717(-4)
24	22	528	22	20	440	3.102	5.296(-3)	2.931(-4)
26	24	624	24	22	528	3.172	7.541(-3)	5.154(-4)
28	26	728	26	24	624	2.770	4.691(-3)	3.456(-4)
30	28	840	28	26	728	2.668	3.088(-3)	2.255(-4)

Table 3: Example 2: Results for various numbers of degrees of freedom with improved distribution of points.

M_ϑ	M_r	M	N_ϑ	N_r	N	c	E	RMSE
12	10	120	10	8	80	2.416	9.236(-3)	1.192(-3)
14	12	168	12	10	120	2.830	9.333(-3)	1.162(-3)
16	14	224	14	12	168	2.934	7.185(-3)	6.291(-4)
18	16	288	16	14	224	2.367	2.620(-3)	1.979(-4)
20	18	360	18	16	288	2.380	1.820(-3)	1.406(-4)
22	20	440	20	18	360	2.762	3.773(-3)	3.266(-4)
24	22	528	22	20	440	3.064	2.369(-3)	1.059(-4)
26	24	624	24	22	528	3.168	2.274(-3)	1.327(-4)
28	26	728	26	24	624	3.133	2.054(-3)	1.571(-4)
30	28	840	28	26	728	3.197	2.046(-3)	1.648(-4)

Table 4: Example 2: Results for various numbers of degrees of freedom with Halton points.

M_b	M_i	M	N_b	N_i	N	c	E	RMSE
12	67	79	10	42	52	2.426	4.365(-3)	2.459(-4)
14	107	121	12	67	79	2.358	3.143(-3)	2.234(-4)
16	158	174	14	107	121	2.485	2.244(-3)	1.240(-4)
18	206	224	16	158	174	3.117	6.190(-3)	2.983(-4)
20	247	267	18	206	224	3.162	2.011(-3)	1.597(-4)
22	300	322	20	247	267	3.192	3.187(-3)	2.046(-4)
24	370	394	22	300	322	3.171	3.519(-3)	1.976(-4)
26	452	478	24	370	394	3.173	4.830(-3)	2.480(-4)
28	545	573	26	452	478	3.109	3.735(-3)	1.908(-4)
30	616	646	28	545	573	3.095	1.760(-3)	1.446(-4)
32	692	724	30	616	646	3.164	1.825(-3)	1.557(-4)
34	786	820	32	692	724	3.091	2.929(-3)	1.960(-4)

5 CONCLUSIONS

A Kansa–RBF collocation method has been applied for the solution of nonlinear boundary value problems. The discretization leads to a system of nonlinear equations which is solved using the nonlinear least–squares minimization MATLAB© routine **`lsqnonlin`**. The determination of the shape parameter, which remains a challenging problem in the application of Kansa–RBF collocation methods, is resolved by taking the unknown value of the shape parameter to be part of the unknowns in **`lsqnonlin`**. The main advantage of the proposed technique is the ease with which it can be implemented while the results of several numerical experiments reveal that it leads to accurate numerical results with a relatively small number of iterations.

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