# FREE VIBRATIONS OF STEPPED NANO-BEAMS

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ABSTRACT
Free vibrations of beams and rods made of nano-materials are investigated. It is assumed that the dimensions of cross sections of nano-beams are piecewise constant and that the beams are weakened with cracks. It is expected that the vibrational behaviour of the nano-material can be described within the non-local theory of elasticity and that the crack induces additional local compliance. The latter is coupled with the stress intensity coefficient at the crack tip.
*Keywords: beam, crack, non-local elasticity, nano-material, vibration.*

## 1 INTRODUCTION

In the recent decade there has been considerable progress in the use of nano-plates and nano-beams due to the need of micro- and nano-electromechanical systems. It is known that the behaviour of nano-structures can be modelled with non-local theories of elasticity (see Eringen [1, 2], Reddy [3]). Lim [4, 5] has developed non-local bending theories and applied these for quasistatically loaded nano-beams. Buckling of nano-beams was studied with the help of non-linear non-local models by Reddy [3], Emam [6], Challamel *et al.* [7]. Analytical solutions for the transverse vibration of simply supported and clamped at both ends nano-beams with axial force are obtained by Li *et al*. [8], Lu *et al.* [9]. In Ref. [10] the Ritz method is accommodated for buckling and vibration of non-local beams. The vibrations of nano-beams with cracks are studied in Ref. [11] making use of the non-local theory of elasticity. In the present paper the free vibrations of nano-beams clamped at both ends are investigated. It is assumed that the nano-beams have stepped cross sections and that the beams are weakened with cracks located at the corners of steps.

## 2 PROBLEM FORMULATION

Let us consider natural vibrations of a nano-beam of length $l$. The edges of the beam at $x = 0$ and $x = l$ are fully clamped. The coordinate axis $Ox$ coincides with the axis of corresponding straight beam; the onset of coordinates is located at the centre of the left-hand end of the beam. It is assumed that the nano-beam has rectangular cross sections with the width $b$ and the height

$$h = \begin{cases} h_0, x \in (0,a), \\ h_1, x \in (a,l). \end{cases} \tag{1}$$

In eqn. (1) the quantities $h_0, h_1$ and $a$ are considered as given numbers. The nano-beam is weakened with a crack of length $c$ at $x = a$. The crack length is expected to be constant. Thus the crack area is

$$S_c = cb. \tag{2}$$

The aim of the study is to determine the eigenfrequencies of natural vibrations of the nano-beam and to clarify the sensitivity of eigenfrequencies on the geometrical parameters of the beam and on the physical parameters of the material. The material of the nano-beam is


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assumed to be an elastic material obeying the constitutional equations of a non-local theory of elasticity (see Eringen [2], Lellep & Lenbaum [12]).

## 3 EQUATION OF MOTION

It was shown in the previous study by the authors [12] that the constitutional equations of non-local elasticity lead to the equation

$$M - \eta \frac{\partial^2 M}{\partial x^2} = M_C, \tag{3}$$

where $M$ stands for the bending moment and $M_c$ is the moment calculated by the rules of the classical bending theory of thin-walled beams. Evidently (see Ref. [12]) the bending moment $M_c$ and the deflection $w$ are coupled as

$$M_C = -EI \frac{\partial^2 w}{\partial x^2}, \tag{4}$$

where $E$ is the Young's modulus and $I$ stands for the moment of inertia of the cross section of the beam. In eqn (3) $\eta$ is a material constant (it is connected with the dimensions of the lattice of the nano-material). Combining eqns (3) and (4) one can write

$$M = \eta \frac{\partial^2 M}{\partial x^2} - EI \frac{\partial^2 w}{\partial x^2}. \tag{5}$$

On the other hand, it follows from the equilibrium equations (see Ref. [8]), that

$$\frac{\partial^2 M}{\partial x^2} = \mu \frac{\partial^2 w}{\partial t^2} - N \frac{\partial^2 w}{\partial x^2}. \tag{6}$$

In eqn (6) $\mu = \rho bh$ stands for the mass per unit length of the nano-beam and $\rho$ being the density of the material. Here $t$ denotes time and $N$ is the axial tension applied at the edges of the beam. Substituting eqn (6) into eqn (5) results in

$$M = \eta\left(-Nw'' + \mu\ddot{w}\right) - EIw'', \tag{7}$$

where the notation

$$w' = \frac{\partial w}{\partial x},$$
$$\dot{w} = \frac{\partial w}{\partial t} \tag{8}$$

is used. Combining eqns (6) and (7) one easily obtains

$$\eta\left(-Nw^{IV} + \mu\ddot{w}''\right) - EIw^{IV} = \mu\ddot{w} - Nw''. \tag{9}$$

The latter can be presented in the form

$$\left(\eta N + EI\right)w^{IV} + \mu\left(\ddot{w} - \eta\ddot{w}''\right) - Nw'' = 0. \tag{10}$$

This is the equation of motion for nano-beams. Taking $\eta = 0$ in eqn (10), one obtains

$$EIw^{IV} - Nw'' + \mu\ddot{w} = 0. \tag{11}$$

The latter is the equation of free vibrations of beams in the classical beam theory. The eqn (10) can be solved with the method of separation of variables making use of appropriate boundary conditions. In the case of beams clamped at both ends the boundary conditions are

$$w\left(0,t\right) = 0, w'\left(0,t\right) = 0 \tag{12}$$

and

$$w\left(l,t\right) = 0, w'\left(l,t\right) = 0. \tag{13}$$

## 4 SOLUTION OF THE GOVERNING EQUATIONS

For solution of the linear fourth order differential equation with partial derivatives let us assume that

$$w\left(x,t\right) = W_j\left(x\right)\cdot T\left(t\right), \tag{14}$$

where $j = 0$ for $x \in \left(0,a\right)$ and $j = 1$ if $x \in \left(a,l\right)$. In eqn (14) the function $W_j\left(x\right)$ depends only on $x$ and the function $T\left(t\right)$ is the function of time $t$. Differentiating eqn (14) with respect to $x$ and $t$ one can easily recheck that

$$\frac{\partial w}{\partial x} = W_j'\left(x\right)\cdot T\left(t\right),$$

$$\frac{\partial^2 w}{\partial x^2} = W_j''\left(x\right)\cdot T\left(t\right),$$

$$\frac{\partial^3 w}{\partial x^3} = W_j'''\left(x\right)\cdot T\left(t\right),$$

$$\frac{\partial^4 w}{\partial x^4} = W_j^{IV}\left(x\right)\cdot T\left(t\right),$$

$$\frac{\partial w}{\partial t} = W_j\cdot \dot{T}\left(t\right),$$

$$\frac{\partial^2 w}{\partial t^2} = W_j\cdot \ddot{T}\left(t\right). \tag{15}$$

In eqns (15) on has to take $j = 0$, if $x \in \left(0,a\right)$ and $j = 1$, if $x \in \left(a,l\right)$. Substituting the derivatives in eqns (15) into eqn (10), one obtains

$$\left(\eta N + EI_j\right)W_j^{IV}\cdot T + \mu_j\left(W_j\cdot\ddot{T} - \eta W_j''\cdot\ddot{T}\right) - NW_j''\cdot T = 0. \tag{16}$$

The separation of variables in eqn (16) leads to the equation

$$\frac{\left(\eta N + EI_j\right)\dfrac{W_j^{IV}}{W_j} - N\dfrac{W_j''}{W_j}}{\left(\eta\dfrac{W_j''}{W_j} - 1\right)\mu_j} = \frac{\ddot{T}}{T} = -\omega^2, \tag{17}$$

where $\omega$ stands for the frequency of natural vibrations. It immediately follows from eqn (17) that

$$\ddot{T} + \omega^2 T = 0 \tag{18}$$

and

$$\left(\eta N + EI_j\right)W_j^{IV} - W_j''\left(N - \mu_j\eta\omega^2\right) - \mu_j\omega^2 W_j = 0. \tag{19}$$

It is reasonable to assume that at the initial time instant $w\left(x,0\right) = 0\ ;\dot{w}\left(x_*,0\right) = v_0$. Taking eqn (14) into account, one can assume that

$$T\left(0\right) = 0, \dot{T}\left(0\right) = 1. \tag{20}$$

Evidently, the particular solution of eqn (18) satisfying the initial conditions in eqn (20) can be presented as

$$T\left(t\right) = \frac{1}{\omega}\sin\left(\omega t\right). \tag{21}$$

The eqn (19) is a linear fourth order differential equation with constant coefficients. The characteristic equation corresponding to eqn (19) has the form

$$\left(\eta N + EI_j\right)\lambda^4 - \lambda^2\left(N - \mu_j\eta\omega^2\right) - \mu_j\omega^2 = 0. \tag{22}$$

The roots of the characteristic equation (22) are

$$\lambda_{1,2} = \pm\vartheta_j\ ;\lambda_{3,4} = \pm i\beta_j, \tag{23}$$

where $i$ is the imaginary unit,

$$\vartheta_j = \sqrt{\frac{1}{2\left(\eta N + EI_j\right)}\left[N - \mu_j\eta\omega^2 + A_j\right]} \tag{24}$$

and

$$\beta_j = \sqrt{\frac{-N + \mu_j\eta\omega^2 + A_j}{2\left(\eta N + EI_j\right)}}. \tag{25}$$

In eqns (24) and (25) for the briefness sake the notation

$$A_j = \sqrt{\left(N - \mu_j\eta\omega^2\right)^2 + 4\mu_j\omega^2\left(\eta N + EI_j\right)} \tag{26}$$

is used. Taking eqns (22)–(26) into account, one can present the general solution of eqn (19) as

$$W(x) = C_{1j} \cosh \vartheta_j x + C_{2j} \sinh \vartheta_j x + C_{3j} \cos \beta_j x + C_{4j} \sin \beta_j x \tag{27}$$

for $x \in S_j$, provided $S_0 = [0,a]$, $S_1 = [a,l]$. In order to define particular solutions for regions $S_0$ and $S_1$, one has to satisfy the boundary conditions in eqns (12) and (13), also the boundary conditions for $M(x,t)$ and the appropriate continuity requirements for the deflection and bending moment.

## 5 INTERMEDIATE CONDITIONS

For modelling the influence of the crack on the eigenfrequencies the method of distributed line springs suggested by Dimarogonas and his co-workers [13] will be used. According to this method, one has to introduce the additional compliance $K$, so that

$$w'(a+0,t) - w'(a-0,t) = -KM(a,t), \tag{28}$$

where

$$w'(a \pm 0,t) = \lim_{x \to a \pm 0} w'(x,t). \tag{29}$$

Making use of eqns (7) and (14), one can introduce functions $m_j(x)$, so that

$$M(x,t) = m_j(x) T(t), \tag{30}$$

where

$$m_j(x) = -\eta \mu_j \omega^2 W - (\eta N + EI_j) W'' \tag{31}$$

for $x \in S_j$ ; $j = 0,1$. The relations in eqns (14), (30) and (31) admit to present the requirement in eqn (28) as

$$W'(a+) - W'(a-) = K\left(\eta \mu_0 \omega^2 W(a-) + (\eta N + EI_0) W''(a-)\right). \tag{32}$$

Since the bending moment $M$ is considered at $x = a$, at any time instant, one gets $m_1(a+) = m_0(a-)$, or according to eqn (31)

$$\left[\eta \mu \omega^2 W(a)\right] + \left[(\eta N + EI) W''(a)\right] = 0, \tag{33}$$

here square brackets denote the finite jump of corresponding quantity. This means that

$$\left[y(a)\right] = y(a+0) + y(a-0) \tag{34}$$

for any variable $y = y(x)$. Due to their physical background, it is clear that the deflection $W(x)$ and the shear force $Q = M'$ are continuous as well. Thus one has

$$\left[W(a)\right] = 0 \tag{35}$$

and

$$\left[\eta \mu \omega^2 W'(a)\right] + \left[(\eta N + EI) W'''(a)\right] = 0. \tag{36}$$

Following Dimarogonas [13], also Lellep and Kraav [14], Lellep and Liyvapuu [15], the additional compliance caused by the crack is evaluated as

$$K = \frac{6\pi h\left(1-\nu^2\right)}{EI} f\left(s\right), \tag{37}$$

where $I = \min\left(I_0, I_1\right)$, $h = \min\left(h_0, h_1\right)$ and

$$f\left(s\right) = \int_0^s y F^2\left(y\right) dy. \tag{38}$$

The function $F = F\left(s\right)$ is obtained with the help of experimental data. In the current paper, as in [12]

$$F\left(s\right) = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.8s^4. \tag{39}$$

## 6 NATURAL FREQUENCIES OF THE NANO-BEAM

In order to define the eigenfrequencies of the vibration, one has to specify the constants $C_{ij}\left(i = 1,\ldots,4\ ; j = 0,1\right)$ in eqn (27). Making use of the boundary conditions in eqns (12) and (13) and taking into account eqns (14) and (27), one easily obtains

$$C_{30} = -C_{10},$$

$$C_{40} = -\frac{\vartheta_0}{\beta_0} C_{20} \tag{40}$$

and

$$C_{11} \cosh \vartheta_1 l + C_{21} \sinh \vartheta_1 l + C_{31} \cos \beta_1 l + C_{41} \sin \beta_1 l = 0,$$

$$C_{11} \vartheta_1 \sinh \vartheta_1 l + C_{21} \vartheta_1 \cosh \vartheta_1 l - C_{31} \beta_1 \sin \beta_1 l + C_{41} \beta_1 \cos \beta_1 l = 0. \tag{41}$$

Due to the continuity of the displacement $W$ and the shear force at $x = a$, one has to take into account the requirements

$$\left[W\left(a\right)\right] = 0,$$

$$\left[M'\left(a\right)\right] = 0, \tag{42}$$

where according to eqn (30) $M' = m_j'\left(x\right)T\left(t\right)$. It is worthwhile to mention, that the total set of boundary and intermediate requirements, which consists of eqns (32), (33) and (40) – (42), is a linear system of algebraic equations with respect to unknown constants $C_{10},\ldots,C_{40}\ ; C_{11},\ldots,C_{41}$. Since it is a linear homogenous system, the non-trivial solution exists under the condition, that the determinant $\Delta$ of this vanishes. Equalizing $\Delta = 0$ leads to the equation for determination of natural frequencies.

## 7 DISCUSSION

The obtained set of algebraic equations is solved numerically with the aid of the computer program Mathcad. Numerical results are obtained for the nano-beam with dimensions $l = 500nm, b = 10nm$ and the material constant $\sqrt{\eta} = 2nm$. The results of calculations are presented in Figs 1–5, where the lowest eigenfrequency $\omega$ is depicted. Here the stepped nano-beam with thicknesses $h_0 = 50nm$, $h_1 = \gamma \cdot h_0$ is investigated, where $s = \frac{c}{h_0}$ and $a$ is the point at which the crack occurs. In Figs 1– 4 the density of the material is $\rho = 7850\frac{kg}{m^3}$ and Young's modulus $E = 200GPa$. Figures 1 and 3 correspond to the case where $\gamma = 1.2$. Different curves in Figs 1 and 3 correspond to the crack extension $s = 0.2$ ;$s = 0.3$ and $s = 0.4$. It can be seen from Figs 1 and 3 that the shorter is the crack the higher is the natural frequency of vibration. The dependence of the eigenfrequency on the axial force $N$ and the crack location $a$ can be seen in Figs 1 and 3, respectively. In Figs 2 and 4 different curves correspond to

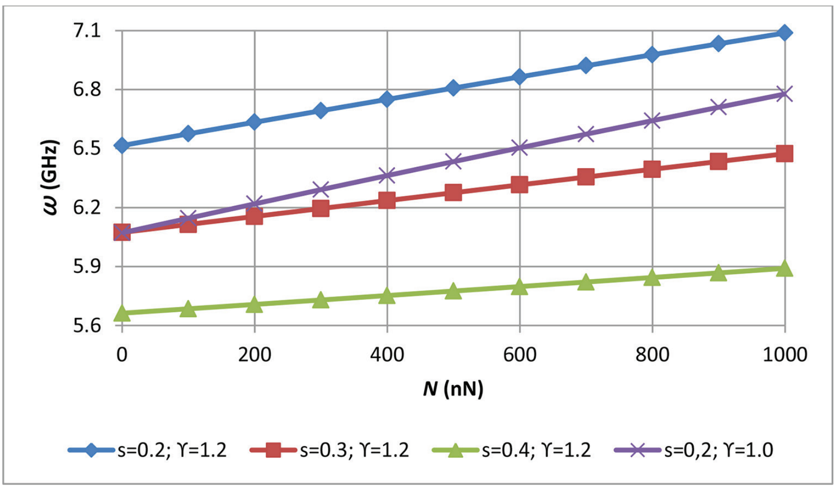


Figure 1: Influence of the axial force on the eigenfrequency for different crack extensions.

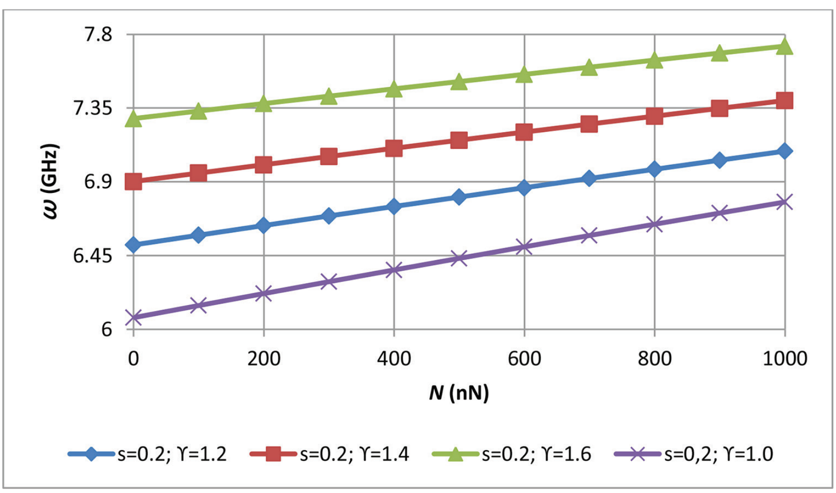


Figure 2: Influence of the axial force on the eigenfrequency for different ratios of thicknesses.

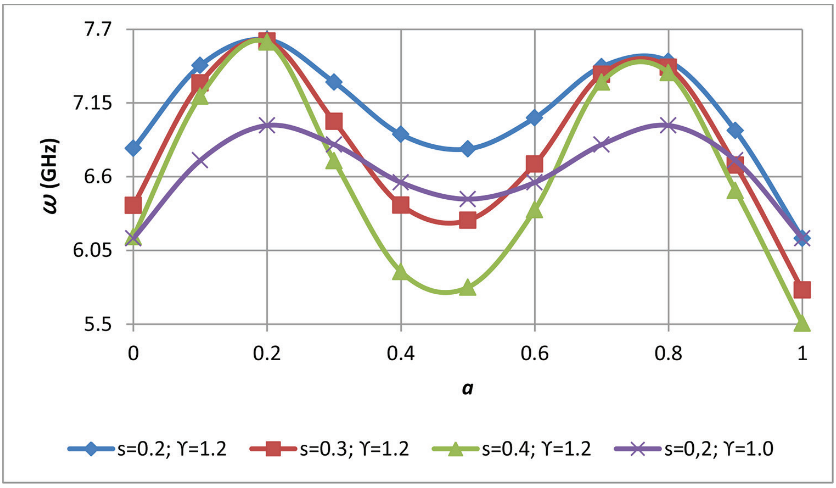


Figure 3: Influence of the crack location on the eigenfrequency for different crack extensions.

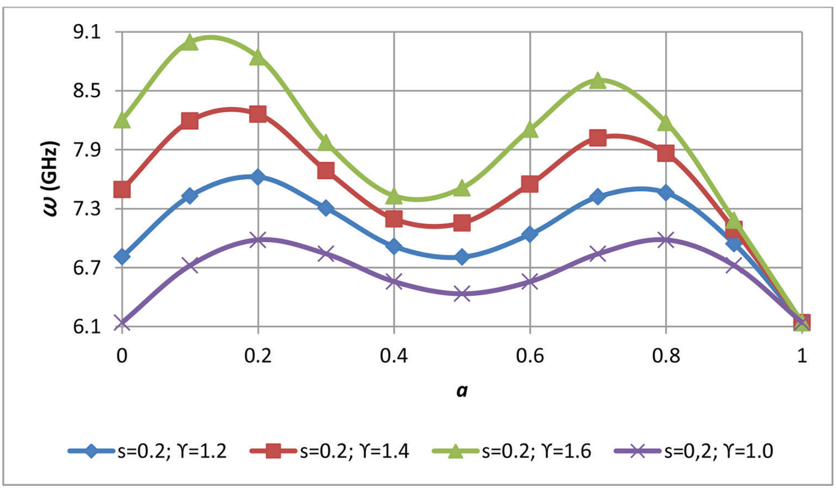


Figure 4: Influence of the crack location on the eigenfrequency for different ratios of thicknesses.

the different ratios of thicknesses and a fixed crack extension $s = 0.2$. It shows from Figs 2 and 4 that a higher ratio of thickness corresponds to a higher frequency. The eigenfrequency versus the axial force $N$ (see Fig. 2) and the step location $a$ (see Fig. 4) can be observed here as well. Also an example for the case of a nano-beam with constant thickness is presented in each of Figs 1–4. In Fig. 5 the dependence of the eigenfrequency on the material of the nano-beam is depicted for a case where $\gamma = 1.2$, $s = 0.2$ and the axial force $N = 500nN$ . Different curves in Fig. 5 correspond to a nano-beam made of iron, copper, zinc or nickel, respectively.

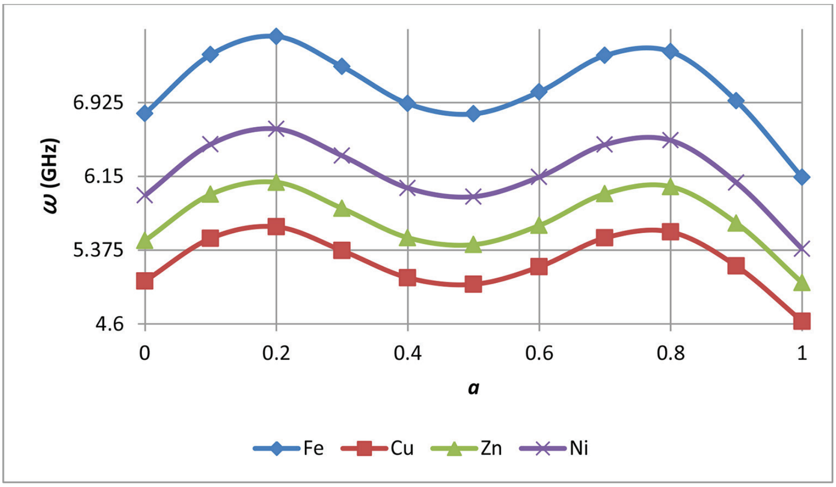


Figure 5: Influence of the crack location on the eigenfrequency for nano-beams made of different materials.

## 8 CONCLUDING REMARKS

A vibration analysis of nano-beams clamped at both ends was undertaken in the case of stepped beams with stable cracks or crack-like defects. It was shown that the defects have essential influence on the eigenfrequencies of nano-beams. Calculations carried out showed, that the lowest eigenfrequencies correspond to nano-beams with more severe defects. It was revealed that the higher is the tension applied to the nano-beam the higher is the frequency of natural vibrations.

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