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MACH'S PRINCIPLE IS EQUIVALENT TO NEWTON'S FIRST AXIOM (A SURVEY)

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ABSTRACT
The following remarks outline the structure of the Newtonian Mach's principle and its implications for fluid motion and turbulence. This principle can only be understood as part of classical thermomechanics on a global scale and is directly related to both the first axiom of Newtonian mechanics and to global total energy covariance.
Keywords: A Mach's principle, covariant Newtonian-Clausius thermomechanics, implications for fluid turbulence.

1 INTRODUCTORY REMARKS

Fluid mechanics, in its classical form, will be studied under the general heading of continuum mechanics and, as such, is firmly based in Newtonian-Clausius thermomechanics. Relativistic effects are not considered in what follows. This classical thermomechanics is best understood in the covariant formulation which places the consequences of the first axiom of Newton at the very foundation of the theory. Fluid mechanics, being just a subset of general thermomechanics, has the same formal structure as that thermomechanics, but with the distinct constitutive theory appropriate to a viscous fluid. Hence the presentation herein has been kept general without the explicit introduction of any constitutive content.

Classical (Newtonian-Clausius) continuum thermomechanics is based upon a foundation of empirical information obtained from an experimental study of natural events. The mechanical experiments of Galileo Galilei (1564–1642) suggested that a body moving at constant velocity would continue to do so until some external action caused a change in that motion. These experiments were conducted on a flat surface so that they were only influenced by the gravitational field of the earth in a minimal way. Thermal effects were not treated by Galilei (indeed, the classical foundations of thermomechanics were not completed before the work of Clausius and Helmholtz in the 19th century). While both Descartes (1596–1650) and Huygens (1629–1695) had a similar understanding of force free motion; the latter did not publish this work (it was, however, published posthumously in Huygens [1]). The review in Penrose [2] made a link between the interests of Newton in the properties of light propagation and his work in dynamics. In particular should light propagation be treated as wave motion or corpuscular motion? Huygens also had a dual interest in optics and mechanics but often held different opinions to those of Newton.

In the *Principia,* Newton (1642–1727) made the well-documented empirical observation of uniform motion the *first axiom* of his theory of mechanics. To quote as *Axiom N1:*

> *"Every body preserves in its state of being at rest or of moving uniformly straight forward except insofar as it is compelled to change its state by forces impressed"*
>
> Axiom N1

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from the translation by Cohen and Whitman [3]. In an earlier manuscript, usually referred to as *Xa* — see Herivel [4], Newton added the *proviso* that the frictional resistance denies this completely uniform motion. Newton in reference [3] did, however, note that: "… *for it may be that there is no body really at rest, to which the places and the motion of others can be referred* …". The concept expressed in Axiom N1 is in strong contrast to the requirement of Aristotle that bodies only move if forces act upon them. It is recognized, however, that Axiom N1 is an *idealized* statement since totally force-free motion cannot be found in the known universe. The Mach principle derived below is ideal in the same sense.

The absolute spacetime demanded by Newton, as the frame in which motion takes place, is herein, replaced by a fixed coordinate frame, A_c, against which all motion is to be assessed. Axiom N1 fixes the geodesics of spacetime and demands that the geometry be Euclidean. In addition, the axiom does not make any reference to body distortion induced by thermal energy transfer. According to Chang [5] knowledge of thermometry was well established by 1600CE (but without standard temperature scales). This implies that, at the time of Newton, thermal expansion was a well recognized physical effect that should be included in the specification of body motion. However, it is this latter aspect of reality that forces Mach's principle to be *thermomechanical* in nature. The work of Mach (1838–1916) adds further background to the Newtonian Axiom N1 and provides a different interpretation of that theory.

A physical body of interest, $B \in \mathbb{B}$, either fluid or solid, cannot move "uniformly straight forward," under the conditions of Axiom N1, in the real universe due to the presence of the ubiquitous gravitational force engendered by the other bodies in that universe, ϖ Here $\mathbb{B}$ denotes the set of all bodies in ϖ. As noted by Friedman [6], only in a one-body universe (where the external gravitational field vanishes) would it be possible for this uniform motion to exist. It is all these other bodies in the universe that give rise to Mach's principle in the context of Newtonian thermomechanics. The original discussion of Mach was mechanical and can be found in Mach [7]. Mach's principle appears trivial in a one body universe: indeed, it is difficult to understand how motion could even be defined in that one-body universe.

The "principle" of Mach can be stated in the form:

(φ): "… *magnitude of the inertia of any body is determined by the masses of the universe and their distribution.*"

Mach's "principle"

as expressed by Bondi [8]. Mach [7] did not declare this statement as a *"principle"* with which to define Newtonian mechanics. It was Einstein who introduced the terminology *"Mach's principle"*. Penrose [2] added the observation that Mach refers body motion to other bodies in the universe and not to spacetime (absolute or otherwise) as did Newton. The statement (φ) was written specifically for *mechanics* while the present discussion concerns the extension of that concept to the more general case of continuum *thermomechanics:* fluid mechanics, and turbulent flow in particular. The present interest resides in the exploration of the statement (φ) of Mach and placing it in the general context of classical Newton-Clausius-Helmholtz thermomechanics. The latter component of the theory was well established at the time of Mach but its fruits were, clearly, not fully digested: this thinking of Mach was mechanical rather than thermomechanical: which is somewhat surprising in light of Mach [9].

Berkeley [10] raised several concerns about the theory of mechanics as presented by Newton. The most prominent of these was the supposed existence of *absolute space* and *absolute time* against which Newtonian mechanics was to be played out. The discussion in

Sklar [11] draws attention to the ideas of Henry More (1614–1687) and Isaac Barrow (1630–1677) concerning the existence of spacetime and the possible influence of those ideas upon the work of Newton. Mach [7] raised different issues with respect to the first axiom of Newtonian mechanics. The main concern of Mach, at least in the interests of the present discussion, was that no mechanism had been put forward to explain the motion described by that axiom. In other words: *why do bodies possess this inertial motion property* and move at constant velocity when no external forces are applied? Mach suggested that all the other bodies located throughout the entire universe were the cause for such inertial forces in the given body *B*. Thus Mach wrote:

> *"When, accordingly, we say, that a body preserves unchanged its direction and velocity* in space, *our assertion is nothing more or less than an abbreviated reference to* the entire universe."

see: Mach [7] for the full context of this observation. No extension of this assertion was given by Mach (even though the required background was available through the work of Clausius (1822–1887) and Helmholtz (1821–1894)).

The philosophy of Mach was discussed by Bradley [12]. Mach did not explicitly adopt the appellation "*Principle*" for the statement (φ), but did consider all events in ϖ to be connected in some sense. See Brown [13] for general comments upon both Newtonian mechanics and its relationship to relativity theory. Interest herein resides with the global implications of Newtonian mechanics in the context of Mach's comments: specifically, the meaning and implications of the statement (φ).

The theory of Newtonian mechanics adopts a spacetime model, $\mathbb{W}_{st}$, of the form:

$$\mathbb{W}_{st} = \mathbb{T} \circ \mathbb{S}_t; \quad \mathbb{S}_t = \mathbb{R}^3 \equiv \{\mathbf{x}\}; \quad \mathbb{T} = \mathbb{R} \tag{1}$$

$\mathbb{R}$ denotes the set of real numbers, while eqn (1) defines the spacetime required for the Newtonian theory. The length and time scales are both measured relative to a standard coordinate frame Λ_c. Axiom N1 is accepted as the foundation of the Newtonian *thermomechanics,* (with the implied Galilean transformation on space-time). This understanding of an ideal (and artificial) Newtonian universe, ϖ, allows the following deliberations to be written down. The Galilean group, $\mathbb{G}_a$, is accepted as a representation of the first axiom of Newton, and in the present context, is expressed in the form:

$$\mathbf{x}^* = \mathrm{Q}[\mathbf{x} + \mathbf{V}_T t + \mathbf{x}_0] \in \mathbb{R}^3; \qquad t^* = t + t_0 \in \mathbb{R} \tag{2}$$

Hence the velocity, $\mathbf{v}(\mathbf{x}, t)$, and acceleration, $\mathbf{a}(\mathbf{x}, t)$, vectors must enjoy the following pair of transformations:

$$\mathbf{v}^* = d\mathbf{x}^* / dt = \mathrm{Q}[\mathbf{v} + \mathbf{V}_T]; \quad \mathbf{a}^* = \mathrm{Q}\mathbf{a} \tag{2a,b}$$

With $\mathbf{Q} \in \mathbb{SO}_3$ a *constant* orthogonal matrix that defines coordinate orientation change. Both the boost velocity, $\mathbf{V}_T$, and the translation $\mathbf{x}_0$ reside in $\mathbb{R}^3$ and are constant vectors. The notation that was introduced in eqn (1), along with eqn (2), represents the fibre bundle structure of Newtonian spacetime, $\mathbb{W}_{st}$, (as was shown on Figure 1) and discussed in Penrose [14].The time $t \in \mathbb{R}$ is constant on each spatial fibre but $\mathbf{x}$ varies over the whole of $\mathbb{R}^3$ (termed the universe ϖ). Also $t_0 \in \mathbb{R}$ is an arbitrary constant time translation. Not recognized at the time that Newton was preparing the *Principia,* but essential for the theory of *thermomechanics*, was the observation that thermal energy transfer could also cause local body motion. But, of course, such motion is very different from that required by the first axiom of Newton since it

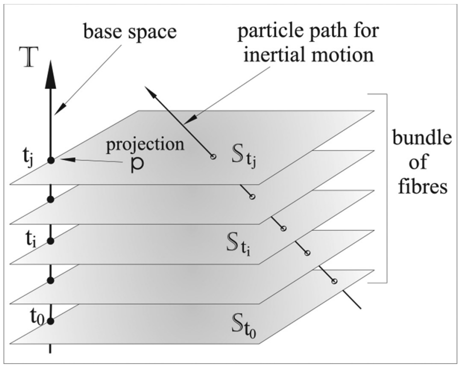

Figure 1: Fibre bundle structure of $\mathbb{W}_{st}$.

mainly concerns the distortion of body geometry. Chang [5] traces the history of this heat induced motion as it evolved into the ubiquitous thermometer applications. The action of thermal energy on body motion is noted here since the principle of Mach in Newtonian mechanics can only be understood in the context of *thermomechanics* where the inertial force on body B is found to be expressed as an integral over the entire external universe.

2 MACH'S PRINCIPLE: THEORETICAL STRUCTURE

From the statement of Mach's principle given in (φ) above, it is appropriate to apply Newtonian mechanics to the entire physical Newtonian universe $\mathcal{W}_N \prec \mathcal{W}$ located within the spatial universe $\mathcal{W}$. This application would not be questioned at the time of Mach (1838–1916) since relativistic theories were not written down before the work of *Einstein (1879–1955).* Indeed, Mach *implied* just that application in Mach [7]. It can be noted that, while Einstein did acknowledge the influence of Mach in his relativity theories, his interests were distinct from those herein (since the Newtonian structure is retained over the entire Newtonian universe, $\mathcal{W}_N$, in the present study) as in Moulden [15]. The notion of a large expanding universe did not arise before the relativistic solutions developed from Einstein's theory were examined by *Lamaitre* and *Friedmann* in the 1920s. The work of Hubble (see Hubble [16] for the initial observations) confirmed this expansion.

The basic assumption of the present theoretical development is that the Newtonian model, $\mathcal{W}_N$, of the physical universe must be isolated in that: *no external forces or energy sources act upon* $\mathcal{W}_N$ from its exterior. By implication, no mass is exchanged between $\mathcal{W}_N$ and its exterior: $\mathcal{W}_N$ (with $\mathcal{W}_N = \gamma_i B_i$) γ $\varnothing$) is closed as shown in Figure 2. Note that the density $\rho(\mathbf{x}, t)$, as a field over $\mathcal{W}_N$, is also a function of temperature, $\theta(\mathbf{x}, t)$, and hence the Mach principle of fluid motion must be a *thermomechanical principle* and cannot be treated solely from within the confines of Newtonian mechanics. The concept associated with the Mach principle is that thermo-dynamic statements, such as the conservation of total energy in the entire universe, must hold in *all inertial frames.* Then Newtonian mechanics and classical thermodynamics are mutually consistent theories.

A foundational requirement in Newtonian mechanics is expressed by the constraint: *the mass,* $M(\mathcal{W}) \equiv \sum_i M(B_i)$, *on the entire universe,* $\mathcal{W}$, *is time invariant with* $M(B_k) = \int_{D_k} \rho_k dV$. This statement is just the requirement: $d\,M(\mathcal{W})/dt = 0$ and must also hold for each of the subbodies,

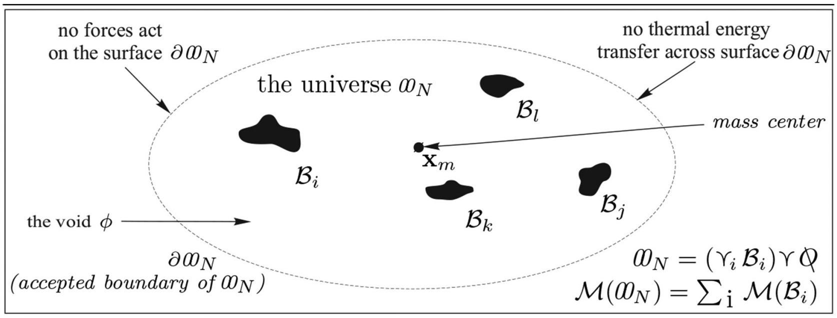

Figure 2: The finite Newtonian model Universe.
Here: $\varpi_N = (\Upsilon_i B_i)\Upsilon\varnothing \subset \varpi, \varpi \equiv \chi_t(\varpi) \subset \mathbb{S}_t \equiv \mathbb{R}^3$.

$B_i \prec \varpi$, in the universe. The void is devoid of mass and does not participate, in any way, in the theory of thermomechanics.

3 THE WORK OF CLAUSIUS AND ITS CONSEQUENCES

It was suggested by Clausius [17] in 1865 that the total energy of the universe, $E(\varpi)$, must be time invariant. In the following, this concept is accepted as applying to the Newtonian universe that was shown in Figure 2. Let $e(\mathbf{x}, t)$ be the specific internal energy distribution over ϖ_N, and $\rho(\mathbf{x}, t)$ the corresponding fluid density. Then the requirement of *Clausius* can be expressed in the form:

$$\begin{aligned} A(\varpi_N) \equiv \frac{dE(\varpi_N)}{dt} &\equiv 0 = \int_{\varpi_N} \rho\left[\tfrac{de}{dt} + \langle \mathbf{v}, \mathbf{a}\rangle\right] dV \\ &\equiv P(\varpi_N) + Q(\varpi_N) \end{aligned} \tag{3}$$

when applied to the entire Newtonian universe. Equation (3) describes the thermodynamics of the universe in the classical context. Here $P(\varpi_N)$ represents the mechanical working of the entire universe and $Q(\varpi_N)$ the corresponding thermal working (see eqs (4a) and (4b) below). Equation (3) holds on the Newtonian sub-universe, ϖ_N since there are, by assumption, no energy sources external to ϖ_N. Let $\mathbf{f}_B^m$ denote the body force per unit mass. Surface couples and body moments are not considered herein (but were included in Moulden [15]). No thermal energy is transferred between the Newtonian universe, ϖ_N, and its exterior. Hence, $\mathbf{q} \perp \mathbf{n}$ on the boundary, $\partial\varpi_N$, of ϖ_N (or else $\mathbf{q}$ vanishes on that boundary). That is, there is no global thermal working of the Newtonian universe: $Q(\varpi_N) = -\int_{\partial\varpi_N} \langle \mathbf{q}, \mathbf{n}\rangle dA \equiv 0$. In addition, forces do no work on the boundary $\partial\varpi_N$ so that there must be $P(\varpi_N) = 0$. Hence, the equality stated in eqn (3). The void, $\varnothing$, the empty space between bodies, is also part of the Newtonian universe, ϖ_N, but does not participate in the energy balance of ϖ_N; it has zero mass and no physical properties. The temperature field, $\theta(\mathbf{x}, t)$, of bodies in ϖ is not part of the discussion herein since entropy considerations are not required, nor included, below.

4 MACH'S PRINCIPLE: ACTUALITY

From application of eqn (2), the Galilean covariance of the energy constraint in eqn (3) requires the transformation (assuming that the internal energy of the universe, $e(\varpi_N, t)$, is also invariant under the Galilean group, $\mathbb{G}_a$):

$$A(\varpi_N) \mapsto A(\varpi_N) + \int_{\varpi_N} \rho \langle \mathbf{a}, \mathbf{V}_T \rangle dV \tag{3a}$$

Herein, D_i represents the space occupied by the body B_i and $\mathbf{a}_i = d\mathbf{v}_i/dt$ the local acceleration vector for motion in the domain D_i occupied by the body B_i. Hence, for the action $A(\varpi_N)$, to be $\mathbb{G}_a$ covariant over the Newtonian universe there must be the identity:

$$\int_{\varpi_N} \rho \langle \mathbf{a}, \mathbf{V}_T \rangle dV \equiv 0 \tag{3b}$$

but, since $\mathbf{V}_T \in \mathbb{R}^3$ is defined to be a universal constant in the Galilean transformation, eqn (3b) reduces to:

$$\int_{\varpi_N} \rho \mathbf{a}\, dV \equiv 0 \Rightarrow \int_{D_j} \rho \mathbf{a}\, dV = -\int_{\varpi_{N-D_j}} \rho \mathbf{a}\, dV \tag{MP}$$

and must be imposed upon Newtonian mechanics for each body B_j as a consequence of $A(\varpi_N)$ being *Galilean covariant.* The implications are discussed below. Equation (MP) only invokes the first axiom of Newtonian mechanics and represents *the statement of Mach's principle* for that mechanics. This is a thermomechanical principle which emerges from the Galilean covariance of eqn (3): it does *not follow* from the second axiom of Newtonian mechanics. Mach did not obtain equation (MP) since the second axiom of Newton was his starting point. The inertial force on a given body, B_j, (in the domain $D_j \prec \varpi_N$) is determined by the presence of all other bodies in the universe. Since $\rho \to \rho$ and $\mathbf{a} \to \mathbf{Qa}$ under the Galilean group $\mathbb{G}$a, equation (MP) holds in all inertial frames.

The above finding does not *explain* the inertial motion of bodies in the context of Newtonian mechanics. At best it provides a global expression by which that inertial motion can, at least in principle, be determined.

5 DEVELOPMENTS

Start the theory of viscous fluid motion from the statement:

Axiom: *Total energy invariance (Clausius)*

In the inertial frame $\Lambda_c = \{(\mathbf{x}, t)\}$ the total energy, $E(\varpi_N)$, of the entire universe ϖ is invariant under time translation. □

which implies that $A(\varpi_N) = 0$ for all time. The definitions required for equation (MP) were given in eqn (3). For $A(\varpi_N)$ to satisfy eqn (3) above, there must be a energy transfer across the boundary, ∂D_i, of each domain D_i given by $P(D_i, t)$ and $Q(D_i, t)$. $\mathbb{I}_\infty$ denotes the index set for the non-void bodies, B_i, in ϖ_N. For each $i \in \mathbb{I}_\infty$:

$$P(D_i, t) = \int_{D_i} \rho_i \langle \mathbf{f}^m_{B_i}, \mathbf{v}_i \rangle dV_i + \int_{\partial D_i} \langle \mathbf{t}_i, \mathbf{v}_i \rangle dA_i \tag{4a, b}$$

$$Q(D_i, t) = -\int_{\partial D_i} \langle \mathbf{q}_i, \mathbf{n}_i \rangle dA_i$$

provided that $\mathbf{t}(\mathbf{x}, t)$ is the stress vector (related to the Cauchy stress tensor by $\mathbf{T}_i\mathbf{n}_i = \mathbf{t}_i$ for outward unit normal $\mathbf{n}_i$ for each body B_i) and $\mathbf{q}_i(\mathbf{x}, t)$ denotes the heat flux vector with $\mathbf{q}_i \to \mathbf{Q}_i \mathbf{q}i$ under $\mathbb{G}_a$. Then, utilizing the divergence theorem there is, in place of eqn (4a) and (4b), the system:

$$P(D_i, t) = \int_{D_i} [\rho_i \langle \mathbf{f}^m_{B_i}, \mathbf{v}_i \rangle + div(\mathbf{T}_i^T \mathbf{v}_i)] dV_i$$

$$Q(D_i, t) = -\int_{D_i} div(\mathbf{q}_i) dV_i \tag{4c,d}$$

Here, in eqn (4c), $\mathbf{f}^{m}_{B_i}$. represents the body force per unit mass acting on body B_i. In other words (as was a basic assumption of the present formulation) there are no *external forces,* or *thermal energy sources,* acting upon the Newtonian universe ϖ_N that effect changes in that universe. Assume in what follows that the Cauchy stress tensor, **T**, is symmetric so that:

$$div(\mathbf{T}^T\mathbf{v}) = \langle div(\mathbf{T}), \mathbf{v}\rangle + trace(\mathbf{TL})$$

with velocity gradient $\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}$. This symmetry assumption on **T** is consistent with the request of covariance under the Galilean group $\mathbb{G}$a as shown in Green and Rivlin [18] as well as Moulden [19]. The explicit form of the Cauchy stress tensor is not required (but the standard transformation $\mathbf{T}^* = \mathbf{QT}\,\mathbf{Q}^T$ is assumed to hold under coordinate change between inertial frames). The covariance of the component $P(D_i,t)$ under the transformation group, $\mathbb{G}_a$, follows from the above:

$$P(D_i,t) \mapsto P(D_i,t) + \int_{D_i} \langle [\mathbf{f}^{\upsilon}_{B_i}(B_i, B_{e_i}) + div(\mathbf{T}_i)], \mathbf{V}_T\rangle dV_i = 0$$

Where the stress field is represented by a Cauchy stress tensor $\mathbf{T}_i$(x,t). Here $\mathbf{f}^{v}_{B_i}$ represents the body force per unit volume. Also: $Q(D_i, t) \to Q(D_i, t)$ under $\mathbb{G}_a$. Using the covariance given in eqn (3a) for $A(\varpi_N$, t) provides the following global constraint:

$$\int_{D_i} [\langle(\rho_i\mathbf{a}_i - \mathbf{f}^{\upsilon}_{B_i}(B_i, B_{e_i}) - div(\mathbf{T}_i)), \mathbf{V}_T\rangle]dV = 0$$

Since the boost velocity, V_T, is constant, there is (for each body B_i):

$$\int_{D_i} [\rho_i\mathbf{a}_i - \mathbf{f}^{\upsilon}_{B_i}(B_i, B_{e_i}) - div(\mathbf{T}_i)]dV_i = 0 \quad \forall i \in \mathbb{I}_\infty$$

the exact form of the symmetric stress tensors, $\mathbf{T}_i$, is not important for the present interests — the Stokes form is most usual in applications to fluid motion. This statement, which holds for all $i \in \mathbb{I}_\infty$, represents the global momentum conservation for the bodies in the Newtonian universe, ϖ_N. From now on, the subscript $|_i$ will be dropped and attention given to a single body and the thermomechanics of that body, so that the localization theorem provides the force balance for the body *B:*

$$\rho\mathbf{a} - \mathbf{f}^{\upsilon}_B(B, B_e) - div(\mathbf{T}) = 0 \tag{5a}$$

as the classical local *linear momentum equation,* valid at all points over body *B*. The Navier Stokes equations follow for the linear viscous fluid model while linear elasticity theory can also be constructed using a different representation for this Cauchy stress tensor **T**.

The local mass conservation equation for the specific body, *B*, can be stated in the standard form:

$$\partial\rho / \partial t + \partial(\rho\upsilon_i) / \partial x_i = 0 \tag{5b}$$

The consequences of eqn (4a) are independent of the nature of the body *B* and hence apply to both fluid and solid bodies. It is only the constitutive content of the theory that distinguishes between these two classes of material.

6 GLOBAL IMPLICATIONS

It is well understood that the ocean tides on Earth are mainly generated by motion of the sun and the Earth's moon. The theory of Mach demands that *all* bodies external to the earth also have some (however small) effect upon tidal motion and upon the motion of every other earth-bound body. Consider just one body, B, in the universe of Figure 2. The result of equation (MP) shows that the integral over the entire domain, D^e, exterior to B:

$$\int_{D^e} \rho \mathbf{a} dV \equiv -\mathbf{f}_I(B, B^e) = \mathbf{f}_B(B, B^e) + \mathbf{f}_S(B, B^e) \tag{6}$$

identifies the inertial force, $\mathbf{f}_I (B, B^e)$, acting on body B; but as noted above, gives no reason for the existence of that inertial motion. The inertial force on body B, $\mathbf{f}_I (B, B^e)$, is defined by the total motion external to that body as given in equation (MP). This is just the principle advocated (but not given explicitly) by Mach. However, the discussion above is based upon thermomechanics: it is not simple Newtonian mechanics as requested by Mach. The inertial force on a body is given by an integral over that part of the universe exterior to itself. For a body of fluid near the center of an isotropic expanding universe, this integral would be very small and the inertial force essentially zero: the acceleration vector, $\mathbf{a}(\mathbf{x},t) \sim \mathbf{0}$ to very good approximation. Such a body of fluid would either be at rest or, at most, in uniform linear motion with velocity, $\mathbf{v}(\mathbf{x},t)$, *constant.*

This last result describes the inertial force on a body at any location in the universe. There is experimental evidence to suggest that the universe is isotropic (at least relative to the solar system — see Raine [20]). As shown in Moulden [21] this isotropy has significant implications. Equation (MP) shows that the body B suffers a fluctuating inertial force due to the presence of velocity fluctuations, $\mathbf{v}(\mathbf{x},t)$, in the external universe. If body B is close to the center of the universe (which is assumed to be essentially isotropic in composition) then these resultant fluctuations will be essentially zero. This need not be the situation for the case where the body of interest is not close to the mass center of ω. The mechanics would not be Newtonian at such a location. Time delay does not enter this Newtonian formulation.

7 FLUID TURBULENCE

The extension of the above theory of turbulent flow was given in Moulden [20], and can be summarized herein. Here, let $\overline{(\cdot)}$ denotes a mean value of some physical quantity and $(\cdot)'$ the corresponding fluctuation. Turbulence, *per se,* is only of interest for the study of fluid body motion. Apply the Reynolds decomposition (see Reynolds [22]) which has the generic form: $\mathbf{f}(B, B^e) \mapsto \overline{\mathbf{f}}(B, B^e) + \mathbf{f}'(BB^e)$. The Reynolds decomposition, given in the usual way, as:

$$\mathbf{v} \mapsto \mathbf{V} + \mathbf{v}'; \qquad P \mapsto \overline{P} + p'; \qquad \rho \mapsto \overline{\rho} + \rho'$$

The body force, may or may not, have a fluctuating component. So that, for the inertial force $\mathbf{f}_I (B, B^e)$, equation (MP) expands to give:

$$\begin{aligned} \mathbf{f}_I(B, B^e) \equiv -\int_{D^e} \rho \mathbf{a}\, dV &= \overline{\mathbf{f}}_I(B, B^e) + \mathbf{f}'_I(BB^e) \\ &\equiv -\int_{D^e} [\overline{\rho}\overline{\mathbf{a}} + \overline{\rho}\mathbf{a}' + \rho'\overline{\mathbf{a}} + \rho'\mathbf{a}'] dV \end{aligned} \tag{7}$$

Break this equality into mean and fluctuating components to find the mean and fluctuating components of the inertial force on body B :

$$\overline{\mathbf{f}}_I(B, B^e) = -\int_{D^e} [\overline{\rho}\overline{\mathbf{a}} + \varepsilon(\rho'\mathbf{a}')] dV \tag{7a}$$

and:

$$\mathbf{f}'_I(B, B^e) = -\int_{D^e} [\overline{\rho}\mathbf{a}' + \rho'\overline{\mathbf{a}} + \rho'\mathbf{a}' - \varepsilon(\rho'\mathbf{a}')]\, dV \tag{7b}$$

where $\varepsilon(\cdot)$ represents the appropriate mean value operator (the exact form of which is not of significance in what follows). The integrals in eqn (7a) and (7b) are taken over the entire external universe relative to body B. This external universe must include interstellar gases as well as all solid bodies. All components of the external universe possess the time dependent acceleration vectors $\bar{\mathbf{a}}(\mathbf{x}, t)$ contained in eqn (7a) and (7b). Hence the fluctuating inertial force acting on body B depends upon the fluctuating acceleration, $\mathbf{a}'$, and density, ρ', fields of the entire external universe. If body B is near the center of ω_N then there will be significant cancellation of the effects from the distant bodies in that universe and $\mathbf{f}'_I\,(B,B^e)$ in eqn (7b) need not be large. Only in such situations would the conditions be Newtonian (or nearly so) with the global norm $|\mathbf{f}_I(B,\, B_e)|_g \approx 0$. This would not be true for bodies located near the edge of ω_N.

The inertial force is balanced with the body and surface forces as in eqn (5a) above. In particular for the body force:

$$\mathbf{f}_B(B,B^e) = \bar{\mathbf{f}}_B(B,B^e) + \mathbf{f}'_B(B,B^e) \equiv \int_{D_e} \bar{\mathbf{f}}^{v}_B\, dV + \int_{D_e} \mathbf{f}'^{v}_B\, dV$$

while the surface force decomposes as:

$$\mathbf{f}_S(B,B^e) = \bar{\mathbf{f}}_S(B,B^e) + \mathbf{f}'_S\,(B,B^e) \equiv \int_{D_e} \frac{1}{\bar{\rho}}\, div(\bar{\mathbf{T}})\, dV + \int_{D_e} \frac{1}{\rho'}\, div(\mathbf{T}')\, dV$$

and the theory is completed when the Cauchy stress tensor $\mathbf{T}(\mathbf{x}, t)$ has been specified for the specific fluid bodies of interest. The mean value field equation then follows and involves the stress tensor $\varepsilon(\rho'\mathbf{a}')$.

8 DISCUSSION

The important finding in the above development, the equivalence between the first axiom of Newton and the principle of Mach emerges, not from Newtonian mechanics, but from the energy considerations of Clausius. Mach's principle is not a simple consequence of Newtonian mechanics — even though Mach thought in those terms. It is explicitly a consequence of the concepts of both Newtonian mechanics and the Clausius notion of global energy conservation. Barbour [23] discusses these issues in a broader framework.

It is evident that the above theory is inadequate in the global real universe since it is founded upon Newtonian mechanics. However, that first axiom of Newton would be a very good approximation for the universe as known at the time that the axiom was written. Newton was aware that the speed of light was finite but not that this fact was required in mechanics on a global scale. In the context of the above development, it is the adoption of *global* Galilean transformations in the theory (as part of Newtonian mechanics as given in eqn 2), that must be called in question. A theory based upon the modern understanding of a non-Newtonian universe would lead to different conclusions to those noted above.

If the real universe were, in fact, unbounded then the above theory would only be meaningful if the integrals over D^e remain bounded over the entire time span of the motion.

9 FINAL COMMENTS

It has been suggested in the development above that the principle of Mach can only be defined in classical thermomechanics if both the first axiom of Newton, and the energy conservation statement of Clausius, are accepted. That is, the principle of Mach is a *thermomechanical principle*. It is further shown that turbulent fluctuations throughout the universe are controlled by the global structure of this Newtonian-Clausius thermomechanics. As such, the paper is a summary, unification and extension of the discussion given in Moulden [15, 21, 24].

The above discussion is restricted to Newtonian mechanics. It has been well understood for a century now that this global application of Newtonian mechanics is not warranted. Hence the above discussion is of interest, only, and not meant to be a realistic representation of the real universe. It does, however, amplify the thoughts of Mach and hence is of interest in its own right. In defense of Mach, it can be recalled that the understanding of compressible fluid mechanics was not well developed at the time that Mach wrote Mach [7]. Of course, Mach had photographed shock waves in supersonic flow but the theory of that particular motion was not fully developed until the work of Taylor [25] and Rayleigh [26] in 1910.

The present interests are very different from those of Zel'dovich [27].

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