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Ritz Variational Method for Solving the Elastic Buckling Problems of Thin-Walled Beams with Bisymmetric Cross-Sections

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ABSTRACT

The Ritz variational method was used in this study to solve the lateral torsional buckling problem of simply supported thin-walled beam with doubly – symmetric cross-section. Two considered cases of loading were uniform bending moments applied at the two ends, and a point load applied vertically at the midspan. The problem was presented in variational form as the problem of minimizing the total potential energy functional, Π, with respect to the unknown parameters of the generalized displacement modal functions. The total potential energy functional was found to be a function of two unknown displacement buckling functions $v(x)$ and $\phi(x)$ and their derivatives with respect to the longitudinal coordinate axis. Suitable displacement buckling functions that satisfy the Dirichlet boundary conditions at the ends were used as trial functions to obtain the Ritz variational problem as the minimization of Π with respect to the generalized buckling modal displacement amplitudes c_{1n} and c_{2n}. The Ritz variational equations were obtained as the minimum conditions for Π with respect to c_{1n} and c_{2n}. The equations were solved for the two cases considered and the buckling moments found for the nth buckling mode from solving the resulting system of homogeneous algebraic equations. It was found that the expressions obtained for the buckling moments in each considered case were the exact expressions obtained by other researchers in literature who solved using classical mathematical methods. It was further found that for each considered case the critical buckling moment occurred at the first buckling mode, and the critical buckling moment expressions for each case agreed with exact solutions from the literature. The effectiveness of the Ritz variational method was thus illustrated for stability problems of thin-walled beams with Dirichlet boundary conditions.

1. INTRODUCTION

Thin-walled beams, columns and beam columns which are frequently used in bridge structures are prone to lateral torsional buckling (LTB). LTB is the stability failure of a thin-walled beam loaded in the plane of its strong axis, and submitted to uncontrolled excessive simultaneous lateral deflection and twisting about the weaker axis [1-21]. The load at which LTB occurs, can be much smaller than the load that causes the development of its full bending moment capacity. In order thus to avoid premature and sudden failures of thin-walled beams, the analysis of lateral torsional load buckling capacities of thin-walled beams that have greater major axis bending stiffness than minor axis bending stiffness or have large laterally unbraced lengths need to be investigated [22-32].

1.1 Methods of solving LTB problems

Three methods are used to determine the critical elastic LTB load of beams and beam – columns. They are (i) closed form (mathematical) methods; (ii) numerical methods (approximate methods) and (iii) energy or variational methods [22].

1.2 Closed form (mathematical) methods

The mathematical methods involve finding closed form solutions to the differential equations of equilibrium for the stability problem. The differential equation of equilibrium for the lateral torsional buckling problem of a beam or beam – column subjected to end moments about its major axis can be solved mathematically by considering the boundary conditions. However, the mathematical solutions obtained are often too complex or involve series with infinite number of terms, and closed form solutions are obtained in only a few cases. Analytical solutions for the flexural and lateral torisonal buckling stability of beams and beam - columns were presented by Brown [8], Timoshenko and Gere [1], Vlasov [9], Chen and Lui [10] and Bazant and Cedolin [11] for I beams under some representative load cases. Sapkas and Kollar [12] and Mohri et al. [13] also presented analytical solutions for lateral buckling of beams with mono-symmetric cross-sections.

1.3 Numerical (approximate) methods

When mathematical/analytical solutions are not possible to obtain due to the complicated nature of the governing equation of equilibrium introduced by material non homogeneity or non linearity, the use of numerical, approximate approaches/(methods) become necessary. Some of the approximate methods that are used are finite difference method, finite element method (FEM), finite strip method and finite integral method.

1.4 Energy (variational) methods

Energy methods are based on energy principles for solving the governing equation of equilibrium for the stability problem. Energy method is based on the principle that states that the additional strain energy during the LTB is equal to the additional work done by the applied forces. In this method an approximate buckled shape function which satisfies the kinematic boundary conditions and corresponds to the real mode shape is assumed and substituted into the energy equation in order to determine the stability equation. The Rayleigh –Ritz method is a classical method based on energy principles widely used for the static, dynamic and buckling analysis of structures and the solution of boundary value problems [17, 33, 34].

1.5 Review of previous works

Juliusz [18] used the Ritz method to calculate the critical buckling moment of a tapered steel I-beam with simply supported ends. In particular, Juliusz [18] considered the lateral – torsional buckling of beams with tapered flanges and web.

Soltani and Asgarian [23] used the finite difference method (FDM) to solve the lateral – torsional stability problem of simply supported thin-walled beams with mono-symmetric cross-section subjected to bending loads. They derived the differential equations of beams with linear behaviour by applying the stationarity condition to the total potential energy functional with the effects of initial stresses and load eccentricities from the shear centre considered. They used the central finite difference expressions for the corresponding derivatives to express the governing domain ordinary differential equation to a finite difference expression, with the boundary conditions also expressed in finite difference form. They found the FDM to be a most powerful technique to solve the governing differential equations especially for cases with variable coefficients. Numerical examples were used to illustrate the effectiveness of the FDM.

Ma et al. [19] presented a study of elastic lateral distortional buckling of cantilever monosymmetrical I-beams, using the Rayleigh – Ritz method. The Rayleigh – Ritz method has advantages over the conventional finite element method because it is mesh free and requires only $6 \times n$ degrees of freedom; hence the solution process is fast.

Though the problem of LTB of bisymmetric I-beams has a well established solution, the same problem with singly symmetric I-beam has not [21]. A closed form solution of the critical elastic lateral torsional buckling moment for simply supported doubly symmetric I-beam loaded by equal and opposite end moments was developed by Timoshenko and Gere [1]. The formula was adopted by many design codes and specifications. At the end supports the beam was free to warp, but torsional and lateral deflection were restrained. The buckling solution for simply supported singly symmetric section with equal and opposite end moments was first developed by Goodier [35, 36].

Ike et al. [37] have used the Laplace transformation method to solve elastic buckling problems of moderately thick beams under various boundary conditions. Fourier cosine series method has been used by Ike et al. [38] to solve the generalized elastic thin-walled column buckling problem for Dirichlet boundary conditions. A modified single finite Fourier cosine integral transform method has been used by Ike et al. [39] to find the critical buckling loads of first order shear deformable beams with fixed ends.

Onah et al. [40] presented closed-form solutions to the elastic stability problems of moderately thick beams for various boundary conditions. Oguaghamba et al. [41] applied the method of finite Fourier sine integral transformation for solving the elastic stability problems of thin-walled beams with doubly-symmetric cross-sections and Dirichlet boundary conditions.

Oguaghamba and Ike [42] applied the Galerkin-Vlasov method to obtain the exact solution to the eigenvalue problem of elastic stability of Kirchhoff plate with one free edge and three simply supported edges under uniform uniaxial compression. Onyia et al. [43] used the Kantorovich variant of the Galerkin method to study the elastic buckling problems of thin rectangular SCSC plates.

Onyia et al. [44] presented elastic buckling solutions to the eigenvalue problems of SSCF and SSSS rectangular thin plates using the one-dimensional finite Fourier sine integral transform method. Onyia et al. [45] also applied the Galerkin-Vlasov method to solve the elastic buckling problems of SSCF and SSSS thin plates under uniform uniaxial compressive loadings.

Ike et al. [46] used the Generalized Integral Transform Method (GITM) to solve the stability problem of rectangular thin plate with two opposite clamped edges and the other edges simply supported. Ike [47] used the Variational Ritz-Kantorovich-Euler-Lagrange method to develop solutions to the elastic stability problem of rectangular Kirchhoff plate with clamped boundaries. Onah et al. [48] derived elastic buckling solutions for uniaxially compressed CCSS thin plate by using one-dimensional finite Fourier sine integral transformation technique.

In this work, the Ritz variational method is used as a mathematical and numerical analysis tool to solve the variational problem of the lateral – torsional buckling analysis of simply supported thin-walled beams with doubly symmetric cross-sections. Two types of load cases were considered, namely:

- uniform (constant) bending moment applied at the ends, and
- vertical point load applied at the midspan of the beam.

2. THEORETICAL FRAMEWORK

2.1 Variational presentation

The lateral torsional buckling problem of thin-walled beams with bisymmetric cross-sections can be presented in variational form as the problem of minimizing the total potential energy functional Π. The total potential energy functional Π for a thin-walled elastic beam buckling problem is given as the sum of the strain energy expression due to bending Saint Venant torsion and warping, and the potential energy due to applied load and is given by:

$$\Pi = \frac{1}{2}\int_0^l EI_z(v''(x))^2 dx + \frac{1}{2}\int_0^l EI_w(\phi''(x))^2 dx + \frac{1}{2}\int_0^l GJ(\phi'(x))^2 dx + \int_0^l M_y\phi(x)v''(x)dx \quad (1)$$

Alternatively,

$$\Pi = \int_0^l \left(\frac{1}{2}EI_z(v''(x))^2 + \frac{1}{2}EI_w(\phi''(x)^2 + \frac{1}{2}GJ(\phi'(x)^2 + M_y\phi(x)v''(x)\right)dx \quad (2)$$

where, l is the length of the beam, x is the longitudinal coordinate axis, the primes denote derivative with respect to x, I_w is the Saint Venant warping constant or the warping constant, J is the Saint Venant torsion constant, $v(x)$ is the displacement, $\phi(x)$ is the rotational displacement about the longitudinal coordinate axis, I_z is the moment of inertia in the weak axis. E is the Young's modulus of elasticity, G is the shear modulus or the modulus of rigidity, M_y is he applied bending moment.

2.2 Equilibrium presentation

The equilibrium equations are obtained from the total potential energy functional by setting the first variation of Π equal to zero. Thus,

$$\delta\Pi(v''(x)\phi(x), \phi'(x), \phi''(x)) = 0 \quad (3)$$

$$\begin{aligned}\delta\Pi &= \int_0^l (EI_z v''(x) + M\phi)'' dv dx + \int_0^l [(EI_w\phi''(x)'' - (GJ\phi'(x)' + Mv''(x))d\phi dx \\ &+ [(EI_z v'' + M\phi]dv'(x)]_0^l - [(EI_z v'' + M\phi)'dv'(x)]_0^l + [EI_w\phi''(x)d\phi'(x)]_0^l \\ &- \left[((EI_w\phi''(x))' - GJ\phi'(x))d\phi(x)\right]_0^l = 0\end{aligned} \quad (4)$$

The differential equations of equilibrium are given by the first two terms in the expression for the first variation of Π. They are:

$$(EI_z v''(x) + M\phi(x))'' = 0 \quad (5)$$

$$(EI_w\phi''(x))'' - (GJ\phi'(x))' + Mv''(x) = 0 \quad (6)$$

where, the primes denote differentiation with respect to x.

3. METHODOLOGY

3.1 Ritz variational method for thin-walled elastic lateral buckling problems for simply supported ends

A simply supported thin-walled beam with bisymmetric cross-section under uniform bending moment, M, applied at the ends as shown in Figure 1 is considered.

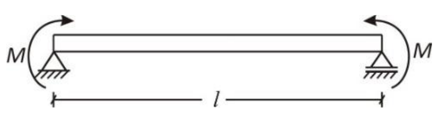

Figure 1. Thin-walled beam with bisymmetric cross-sections subjected to constant moment at the ends

The boundary conditions for the displacement $v(x)$ and the rotation $\phi(x)$ are:

$$v(0) = v''(0) = v(l) = v''(l) = 0 \quad (7)$$

$$\phi(0) = \phi''(0) = \phi(l) = \phi''(l) = 0 \quad (8)$$

Suitable displacement functions $v(x)$ and $\phi(x)$ that automatically satisfy the boundary conditions are:

$$v(x) = \sum_{n=1}^{\infty} c_{1n}\sin\frac{n\pi x}{l} \quad (9)$$

$$\phi(x) = \sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l} \quad (10)$$

where, c_{1n} is the buckling modal amplitude of $v(x)$ for the nth buckling mode, c_{2n} is the buckling modal amplitude of $\phi(x)$ for the nth buckling mode.

The total potential energy functional for thin-walled beams with bisymmetric cross sections under constant moment for the case of simply supported ends is then:

$$\begin{aligned}\Pi = \int_0^l \Bigg\{\frac{1}{2}EI_z &\left\{\left(\frac{d^2}{dx^2}\sum_{n=1}^{\infty} c_{1n}\sin\frac{n\pi x}{l}\right)^2 + \frac{1}{2}EI_w\left(\frac{d^2}{dx^2}\sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l}\right)^2 + \right.\\ \frac{1}{2}GJ&\left(\frac{d}{dx}\sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l}\right)^2 + M\left(\sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l}\right)\left(\frac{d^2}{dx^2}\sum_{n=1}^{\infty} c_{1n}\sin\frac{n\pi x}{l}\right)\Bigg\}dx\end{aligned} \quad (11)$$

The Ritz variational equations are obtained by using the conditions for minimization of Π, thus:

$$\frac{\partial\Pi}{\partial c_{1n}} = 0 \quad (12)$$

$$\frac{\partial\Pi}{\partial c_{2n}} = 0 \quad (13)$$

3.2 Ritz variational method for simply supported beam under load at midspan

The boundary conditions are also given by Eqns. (7) and (8) and the displacement functions are given by Eqns. (9) and (10). However,

$$M(x) = \frac{Px}{2} \qquad 0 \le x \le \frac{l}{2} \quad (14)$$

Then,

$$\begin{aligned}\Pi = \int_0^l \Bigg\{&\frac{1}{2}EI_z\left(\frac{d^2}{dx^2}\sum_{n=1}^{\infty} c_{1n}\sin\frac{n\pi x}{l}\right)^2 + \frac{1}{2}EI_w\left(\frac{d^2}{dx^2}\sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l}\right)^2 + \\ &\frac{1}{2}GJ\left(\frac{d}{dx}\sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l}\right)^2\Bigg\}dx + 2\int_0^{l} \frac{Px}{2}\left(\sum_{n=1}^{\infty} c_{2n}\sin\frac{n\pi x}{l}\right)\left(\frac{d^2}{dx^2}\sum_{n=1}^{\infty} c_{1n}\sin\frac{n\pi x}{l}\right)dx\end{aligned} \quad (15)$$

The extremum conditions Eqns. (12) and (13) are applied to obtain a minimum for Π in Eq. (15).

4. RESULTS

4.1 Lateral torsional buckling of thin-walled beams with bisymmetric cross-sections – case of constant bending moment and simply supported ends

Simplifying Eq. (11),

$$\Pi = \left(\sum_{n=1}^{\infty} c_{1n}\right)^2 \frac{1}{2} EI_z \left(\frac{n\pi}{l}\right)^4 I_1 + \left(\sum_{n=1}^{\infty} c_{2n}\right)^2 \frac{1}{2} EI_w I_1 \left(\frac{n\pi}{l}\right)^4 + \left(\sum_{n=1}^{\infty} c_{2n}\right)^2 \frac{1}{2} GJ \left(\frac{n\pi}{l}\right)^2 I_2 + \sum_{n=1}^{\infty} c_{1n} \sum_{n=1}^{\infty} c_{2n} \left(-\left(\frac{n\pi}{l}\right)^2\right) MI_1 \tag{16}$$

where,

$$I_1 = \int_0^l \sin^2 \frac{n\pi x}{l} dx \tag{17}$$

$$I_2 = \int_0^l \cos^2 \frac{n\pi x}{l} dx = I_1 = \frac{l}{2} \tag{18}$$

$$\frac{\partial \Pi}{\partial c_{1n}} = \left(\sum_{n=1}^{\infty} c_{1n}\right) EI_z \left(\frac{n\pi}{l}\right)^4 I_1 - \sum_{n=1}^{\infty} c_{2n} \left(\frac{n\pi}{l}\right)^2 MI_1 = 0 \tag{19}$$

$$\frac{\partial \Pi}{\partial c_{1n}} = a_{11} c_{1n} + a_{12} c_{2n} = 0 \tag{20}$$

where,

$$a_{11} = EI_z \left(\frac{n\pi}{l}\right)^4 I_1 \tag{21}$$

$$a_{12} = -\left(\frac{n\pi}{l}\right)^2 MI_1 \tag{22}$$

$$\frac{\partial \Pi}{\partial c_{2n}} = \left(\sum_{n=1}^{\infty} c_{2n}\right) EI_w \left(\frac{n\pi}{l}\right)^4 I_1 + \left(\sum_{n=1}^{\infty} c_{2n}\right) GJ \left(\frac{n\pi}{l}\right)^2 I_2 - \sum_{n=1}^{\infty} c_{1n} \left(\frac{n\pi}{l}\right)^2 MI_1 = 0 \tag{23}$$

$$\frac{\partial \Pi}{\partial c_{2n}} = a_{21} c_{1n} + a_{22} c_{2n} = 0 \tag{24}$$

where,

$$a_{21} = -\left(\frac{n\pi}{l}\right)^2 MI_1 \tag{25}$$

$$a_{22} = EI_w \left(\frac{n\pi}{l}\right)^4 I_1 + GJ \left(\frac{n\pi}{l}\right)^2 I_2 = \left(EI_w \left(\frac{n\pi}{l}\right)^4 + GJ \left(\frac{n\pi}{l}\right)^2\right) I_1 \tag{26}$$

For the nth buckling mode, the homogeneous equation obtained is given by:

$$\begin{Bmatrix} EI_z \left(\frac{n\pi}{l}\right)^4 I_1 & -\left(\frac{n\pi}{l}\right)^2 MI_1 \\ -\left(\frac{n\pi}{l}\right)^2 MI_1 & \left(EI_w \left(\frac{n\pi}{l}\right)^4 + GJ \left(\frac{n\pi}{l}\right)^2\right) I_1 \end{Bmatrix} \begin{Bmatrix} c_{1n} \\ c_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{27}$$

For nontrivial solutions at the nth buckling mode the characteristic equation is obtained as:

$$\begin{vmatrix} EI_z \left(\frac{n\pi}{l}\right)^4 & -\left(\frac{n\pi}{l}\right)^2 M \\ -\left(\frac{n\pi}{l}\right)^2 M & \left(EI_w \left(\frac{n\pi}{l}\right)^4 + GJ \left(\frac{n\pi}{l}\right)^2\right) \end{vmatrix} = 0 \tag{28}$$

Expansion of the determinant yields:

$$EI_z \left(\frac{n\pi}{l}\right)^4 \left(EI_w \left(\frac{n\pi}{l}\right)^4 + GJ \left(\frac{n\pi}{l}\right)^2\right) - M^2 \left(\frac{n\pi}{l}\right)^4 = 0 \tag{29}$$

Hence,

$$M^2 \left(\frac{n\pi}{l}\right)^4 = EI_z \left(\frac{n\pi}{l}\right)^4 \left(EI_w \left(\frac{n\pi}{l}\right)^4 + GJ \left(\frac{n\pi}{l}\right)^2\right) \tag{30}$$

Simplifying,

$$M^2 = EI_z \left(EI_w \left(\frac{n\pi}{l}\right)^4 + GJ \left(\frac{n\pi}{l}\right)^2\right)$$

Further simplification yields:

$$M^2 = EI_z \left(\frac{n\pi}{l}\right)^2 \left(EI_w \left(\frac{n\pi}{l}\right)^2 + GJ\right)$$

Hence,

$$\begin{aligned} M &= \left(\frac{n\pi}{l}\right) \sqrt{\left\{EI_z \left(EI_w \left(\frac{n\pi}{l}\right)^2 + GJ\right)\right\}} = \\ M &= \left(\frac{n\pi}{l}\right) \sqrt{EI_z GJ \left(1 + \frac{EI_w}{GJ} \left(\frac{n\pi}{l}\right)^2\right)} \end{aligned} \tag{31}$$

The critical buckling moment M_{cr} is obtained as the least value of M and this occurs when $n = 1$.

Hence,

$$M_{cr} = M(n=1) = \frac{\pi}{l} \sqrt{EI_z GJ + EI_z EI_w \left(\frac{\pi}{l}\right)^2} \tag{32}$$

The critical buckling moment of beams under constant moment at the ends is expressed as:

$$\begin{aligned} M_{cr} &= \frac{\pi}{l} \sqrt{EI_z GJ \left(1 + \frac{EI_w}{GJ} \frac{\pi^2}{l^2}\right)} \\ M_{cr} &= \frac{\pi}{l} \sqrt{EI_z GJ} \sqrt{\left(1 + \frac{EI_w \pi^2}{GJ l^2}\right)} \end{aligned} \tag{33}$$

$$M_{cr} = K_{b_1} \frac{\sqrt{EI_z GJ}}{l} \tag{34}$$

K_{b1} is a parameter defined in terms of G, E, I_w, and J, as:

$$K_{b_1} = \pi\sqrt{\left(1+\frac{EI_w\pi^2}{GJl^2}\right)} \quad (35)$$

The values of K_{b1} are tabulated in terms of $\left(\frac{l^2GJ}{EI_w}\right)$ and presented in Table 1 together with values of K_{b1} obtained previously by Timoshenko and Gere [1].

Table 1. Values of K_{b1} for various values of $\frac{GJl^2}{EI_w}$

$\frac{GJl^2}{EI_w}$	**Present results**	**Timoshenko and Gere [1]**
0	∞	∞
0.1	31.3681	31.3681
1.0	10.3575	10.3575
2	7.6534	7.6534
4	5.8499	5.8499
6	5.1093	5.1093
8	4.6953	4.6953
10	4.4284	4.4284
12	4.2411	4.2411
16	3.9947	3.9947
20	3.8393	3.8393
24	3.7321	3.7321
28	3.6536	3.6536
32	3.5936	3.5936
36	3.5462	3.5462
40	3.5078	3.5078
100	3.2930	3.2930
∞	π	π

4.2 Lateral torsional buckling of simply supported thin-walled beam with bisymmetric cross-section – case of point load *P* at the midspan

The case of a point load P acting at the midspan of a simply supported thin-walled beam with doubly symmetric cross-section as shown in Figure 2 is considered.

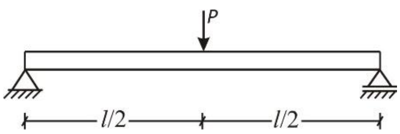

Figure 2. Lateral – torsional buckling of a thin-walled beam with bisymmetric cross-section and simply supported ends with point load P acting at the midspan

The simplification of the total potential energy functional Π expressed by Eq. (15) is given as:

$$\Pi = \left(\sum_{n=1}^{\infty} c_{1n}\right)^2 \frac{1}{2} EI_z\left(\frac{n\pi}{l}\right)^4 I_1 + \left(\sum_{n=1}^{\infty} c_{2n}\right)^2 \frac{1}{2} EI_w\left(\frac{n\pi}{l}\right)^4 I_1 + \left(\sum_{n=1}^{\infty} c_{2n}\right)^2 \frac{1}{2} GJ\left(\frac{n\pi}{l}\right)^2 I_2 - P\sum_{n=1}^{\infty} c_{1n}\sum_{n=1}^{\infty} c_{2n}\left(\frac{n\pi}{l}\right)^2 I_3 \quad (36)$$

where,

$$I_3 = \int_0^{l/2} x\sin^2\frac{n\pi x}{l}dx \quad (37)$$

$$I_3 = \frac{\frac{(n\pi)^2 l^2}{2} - l\sin n\pi - l^2\cos(n\pi) + l^2}{8n^2\pi^2} \quad (38)$$

$$I_3 = \frac{\frac{(n\pi)^2 l^2}{2} + 2l^2}{8n^2\pi^2} = \frac{l^2}{16} + \frac{l^2}{4n^2\pi^2} \quad (39)$$

where, $I_1 = I_2 = \frac{l}{2}$ is obtained from Eq. (18):

$$\frac{\partial\Pi}{\partial c_{1n}} = \left(\sum_{n=1}^{\infty} c_{1n}\right) EI_z\left(\frac{n\pi}{l}\right)^4 \frac{l}{2} - P\sum_{n=1}^{\infty} c_{2n}\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) = 0 \quad (40)$$

$$\frac{\partial\Pi}{\partial c_{2n}} = \left(\sum_{n=1}^{\infty} c_{2n}\right) EI_w\left(\frac{n\pi}{l}\right)^4 \frac{l}{2} + \sum_{n=1}^{\infty} c_{2n} GJ\left(\frac{n\pi}{l}\right)^2 \frac{l}{2} - P\sum_{n=1}^{\infty} c_{1n}\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) = 0 \quad (41)$$

For the nth buckling mode, the equilibrium equations are found as:

$$c_{1n} EI_z\left(\frac{n\pi}{l}\right)^4 \frac{l}{2} - Pc_{2n}\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) = 0 \quad (42)$$

$$-Pc_{1n}\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) + c_{2n}\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l}{2} = 0 \quad (43)$$

The equilibrium equations are expressed in matrix form as:

$$\begin{pmatrix} EI_z\left(\frac{n\pi}{l}\right)^4 \frac{l}{2} & -P\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) \\ -P\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) & \left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l}{2} \end{pmatrix}\begin{pmatrix} c_{1n} \\ c_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (44)$$

For non trivial solutions, the characteristic buckling equation is obtained as:

$$\begin{vmatrix} EI_z\left(\frac{n\pi}{l}\right)^4 \frac{l}{2} & -P\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) \\ -P\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right) & \left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l}{2} \end{vmatrix} = 0 \quad (45)$$

Expansion of the determinant yields:

$$EI_z\left(\frac{n\pi}{l}\right)^4 \frac{l}{2}\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l}{2} - \left(-P\left(\frac{n\pi}{l}\right)^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right)\right)^2 = 0 \quad (46)$$

Thus,

$$P^2\left(\frac{n\pi}{l}\right)^4\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right)^2 = EI_z\left(\frac{n\pi}{l}\right)^4\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l^2}{4} \quad (47)$$

Simplifying,

$$P^2\left(\frac{l^2}{16} + \frac{l^2}{4n^2\pi^2}\right)^2 = EI_z\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l^2}{4} \quad (48)$$

Simplifying further,

$$P^2 l^4\left(\frac{1}{16} + \frac{1}{4n^2\pi^2}\right)^2 = P^2 l^4\left(\frac{n^2\pi^2 + 4}{16n^2\pi^2}\right)^2 = EI_z\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right)\frac{l^2}{4} \quad (49)$$

Hence,

$$P^2 = \left(\frac{16n^2\pi^2}{4+n^2\pi^2}\right)^2 \frac{1}{l^4}\frac{l^2}{4} EI_z\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right) \quad (50)$$

Simplifying further,

$$P^2 = \left(\frac{16n^2\pi^2}{4+n^2\pi^2}\right)^2 \frac{EI_z}{4l^2}\left(EI_w\left(\frac{n\pi}{l}\right)^4 + GJ\left(\frac{n\pi}{l}\right)^2\right) \quad (51)$$

Hence,

$$P^2 = \left(\frac{8n^2\pi^2}{4+n^2\pi^2}\right)^2 \left(\frac{n\pi}{l}\right)^2 \frac{EI_z}{l^2}\left(EI_w\left(\frac{n\pi}{l}\right)^2 + GJ\right) \quad (52)$$

The buckling moments are:

$$M^2 = \frac{P^2l^2}{16} = \left(\frac{8n^2\pi^2}{4+n^2\pi^2}\right)^2 \left(\frac{n\pi}{l}\right)^2 \frac{l^2}{16}\frac{EI_z}{l^2}\left(EI_w\left(\frac{n\pi}{l}\right)^2 + GJ\right) \quad (53)$$

Simplifying,

$$M^2 = \left(\frac{2n^2\pi^2}{4+n^2\pi^2}\right)^2 \left(\frac{n\pi}{l}\right)^2 EI_z\left(EI_w\left(\frac{n\pi}{l}\right)^2 + GJ\right) \quad (54)$$

Hence,

$$M = \left(\frac{2n^2\pi^2}{4+n^2\pi^2}\right) \frac{n\pi}{l} \sqrt{EI_z\left(EI_w\left(\frac{n\pi}{l}\right)^2 + GJ\right)} \quad (55)$$

The lowest buckling moment in this loading case called the critical buckling moment $M_{cr(p)}$ is obtained at $n = 1$ and is given by:

$$M_{cr(p)} = M(n=1) = \left(\frac{2\pi^2}{4+\pi^2}\right)\frac{\pi}{l}\sqrt{EI_z\left(EI_w\left(\frac{\pi}{l}\right)^2 + GJ\right)} \quad (56)$$

$$M_{cr(p)} = 1.423199\frac{\pi}{l}\sqrt{EI_z\left(EI_w\left(\frac{\pi}{l}\right)^2 + GJ\right)} \quad (57)$$

$$M_{cr(p)} = 1.423199 M_{cr(m)} \quad (58)$$

where, $M_{cr(m)}$ is the critical buckling moment obtained for constant bending moment M applied at the beam ends.

$$M_{cr(p)} = \left(\frac{2\pi^2}{4+\pi^2}\right)\frac{\pi}{l}\sqrt{EI_zGJ\left(1+\frac{EI_w}{GJ}\frac{\pi^2}{l^2}\right)} = \frac{P_{cr}l}{4} \quad (59)$$

P_{cr} is the critical buckling load.

$$P_{cr} = \frac{4M_{cr}}{l} = \left(\frac{8\pi^2}{4+\pi^2}\right)\frac{\pi}{l^2}\sqrt{EI_zGJ\left(1+\frac{EI_w\pi^2}{GJl^2}\right)} \quad (60)$$

$$P_{cr} = K_{b_2}\frac{\sqrt{GJEI_z}}{l^2} \quad (61)$$

K_{b2} is a parameter expressed in terms of G, E, J and I_w.

$$K_{b_2} = \left(\frac{8\pi^2}{4+\pi^2}\right)\pi\sqrt{\left(1+\frac{EI_w\pi^2}{GJl^2}\right)} \quad (62)$$

Table 2 shows the values of K_{b2} for various corresponding values of $\frac{GJl^2}{EI_w}$.

Table 2. Values of K_{b2} for various values of $\frac{GJl^2}{EI_w}$

$\frac{GJl^2}{EI_w}$	**Present work**	**Timoshenko and Gere [1]**
0.4	90.62 (7.12%)	84.60
4	33.303 (4.40%)	31.90
8	26.729 (4.41%)	25.60
16	22.741 (4.32%)	21.80
24	21.246 (4.66%)	20.30
32	20.457 (4.37%)	19.60
48	19.637 (4.45%)	18.80
64	19.214 (4.99%)	18.30
80	18.956 (4.92%)	18.10
96	18.781 (4.41%)	17.90
160	18.428 (5.30%)	17.50
240	18.248 (4.87%)	17.40
320	18.158 (5.57%)	17.20
400	18.10 (5.23%)	17.2

Relative difference between the present results for K_{b2} and corresponding results by Timoshenko and Gere [1] are enclosed in brackets in Table 2.

5. DISCUSSION

The Ritz variational method was successfully used in the work to solve the elastic buckling problems of simply supported thin-walled beams with bisymmetric cross-sections. Two cases were considered. The first case considered a simply supported thin-walled beam with bisymmetric cross-section subjected to uniform bending moments applied at the ends. The second case considered a simply supported thin-walled beam with bisymmetric cross-section subjected to a vertical point load P applied at the midspan.

The elastic buckling problem of thin-walled beam with bisymmetric cross-section was presented in variational form using the calculus of variations, as the problem of minimizing the total potential energy functional Π, expressed as the sum of the strain energy expressions due to flexure, Saint Venant torsion and warping and the potential energy of the applied load with respect to the unknown generalised displacement amplitudes of the displacement functions. The total potential energy functional expressed by Eq. (1) is a function of two unknown displacement functions, $v(x)$ and $\phi(x)$ and their derivatives.

For the thin-walled beam with simply supported ends considered in the study, the Dirichlet boundary conditions at the ends are given by Eqns. (7) and (8). Suitable displacement functions that satisfy all the Dirichlet boundary conditions at the simply supported ends are given by Eqns. (9) and (10) and were used as the trial functions in the Ritz variational formulation where the total potential energy functional was expressed in terms of the generalised buckling modal displacement amplitudes as Eq. (11) for the case of uniform bending moment applied at the ends. The Ritz variational equations obtained from the conditions for extremum of the total potential energy were obtained from extremizing Π with respect to c_{1n} and c_{2n} which are expressed as Eqns. (12) and (13).

For the case of thin-walled beam with bisymmetric cross-section subjected to point load applied at midspan, the Ritz formulation of the total potential energy functional presented as Eq. (15) was found to depend on the buckling modal displacement amplitudes c_{1n} and c_{2n}. For the first case considered the total potential energy functional was found to be expressible in the simplified form presented as Eq. (16). The Ritz equations of equilibrium were found from extremization of Π as the system of two equations – Eqns. (19) and (23). The Ritz equations of equilibrium were expressed in matrix form as Eq. (27) which is a homogeneous system of algebraic equations. The condition for nontrivial solution which is that $c_{1n} \neq 0,\quad c_{2n} \neq 0$ were used to obtain the characteristic buckling equation at the nth buckling mode from the varnishing of the coefficient matrix as Eq. (28). Expansion of the determinant yielded the characteristic buckling equation as Eq. (29). Simplification and solution gave the expression for the buckling moment as Eq. (34). The critical buckling moment for the case of simply supported thin-walled beams subjected to uniform bending moment M applied at the ends was found to correspond to the first buckling mode and found as Eq. (35).

For the second case considered, which is the elastic buckling of thin-walled beam with bisymmetric cross-section subjected to a point load applied at midspan, the total potential energy functional Π obtained from simplifying Eq. (15) was found as Eq. (36). The Ritz equations of equilibrium for this second case, were obtained by enforcing exteremum of Π with respect to c_{1n} and c_{2n} as Eqns. (40) and (41) respectively. The Ritz variational equations of equilibrium for the nth buckling mode were found as the system of equations – Eqns. (42) and (43) and presented in matrix form as Eq. (44).

The characteristic buckling equation for the nth buckling mode obtained from the condition for nontrivial solutions was found as Eq. (45). Expansion of the equation resulted in the Eq. (46) as the characteristic buckling equation. Solution of Eq. (46) gave the expression for P^2 as Eq. (52). The expression for the maximum bending moment M which occurs at $x = l/2$ expressed in terms of the applied point load P was used to obtain the expression for the square of the buckling moment M^2 for the nth buckling mode as Eqns. (53) or (54) when simplified. The expression for the buckling moment at the nth buckling mode was thus obtained as Eq. (55). The lowest buckling moment was found to occur at the first buckling mode when $n = 1$, and was obtained as Eq. (57). It was found that for the same length l, the critical buckling moment for a simply supported thin-walled beam with bisymmetric cross-section subjected to a point load applied at midspan of the beam is related to the critical buckling moment for a simply supported thin-walled beam with bisymmetric cross-section subjected to constant bending moment applied at the ends by Eq. (58).

The closed-form expression obtained for the critical buckling moment of the simply supported thin-walled beam with doubly symmetrical cross-section for the case of constant moment at the ends is given by Eq. (32). The equation is identical with the expression previously obtained by Timoshenko and Gere [1]. The expression is further expressed in terms of the parameter K_{b1} defined in terms of G, E, J and I_w as Eq. (34). The values of K_{b1} are calculated for various values of $\frac{l^2GJ}{EI_w}$ and tabulated as shown presented in Table 1, together with previously obtained values from the result from Timoshenko and Gere [1]. Table 1 shows excellent agreement of the present study and the results from Timoshenko and Gere [1].

Similarly, the present results for the critical buckling moment $M_{cr}(p)$ of thin-walled beam with doubly-symmetric cross-section and simply supported ends for the case of point load at midspan is given by Eq. (59). The critical buckling load P_{cr} was calculated from Eq. (59) as the expression given in Eq. (60).

P_{cr} is further expressed in terms of a parameter K_{b2} defined in terms of E, G, J, I_w and l as Eq. (61). The parameter K_{b2} defined by Eq. (62) is calculated for various values of $\frac{l^2GJ}{EI_w}$ and presented in Table 2, along with previous results of K_{b2} obtained by Timoshenko and Gere [1]. Table 2 shows that the relative difference between the P_{cr} obtained for thin-walled doubly symmetric beams under point load at midspan in this work and the previous work by Timoshenko and Gere [1] varies from 4.32% for $\frac{l^2GJ}{EI_w} = 16$ to a maximum relative difference of 7.12% for $\frac{GJl^2}{EI_w} = 0.40$. For $4 < \frac{GJl^2}{EI_w} < 240$, the relative difference of P_{cr} obtained in the present work and the work by Timoshenko and Gere [1] is less than 5%, which is acceptable.

The Ritz variational method has thus been shown to be an effective analytical technique for solving the elastic buckling problems of thin-walled bisymmetric beams with Dirichlet boundary conditions.

6. CONCLUSION

The following conclusions are made from the study:

(i) The Ritz variational method is a good analytical and numerical tool for the determination of the elastic buckling moments of simply supported thin-walled beam with bisymmetric cross-section for the two cases of uniform bending moment applied at the ends, and point load applied at the midspan of the beam.

The lateral torsional buckling problem of thin-walled beam with bisymmetric cross-section is a variational problem presented and formulated as the extremization of the total potential energy functional with respect to the generalized buckling displacement mode amplitudes.

(ii) Trial buckling displacement mode shape functions that are exact shape functions for the displacement functions, and satisfy all the Dirichlet boundary conditions at the ends lead to exact solutions of the lateral torsional buckling problem. The Ritz variational method simplified the variational problem by transforming the problem to an algebraic eigenvalue problem represented by a system of homogeneous algebraic equations.

(iii) The conditions for nontrivial solutions are used to obtain the characteristic buckling equations as a determinantal equation from the vanishing of the determinant of the coefficient matrix.

The eigenvalues of the characteristic buckling equation were used to obtain the *n* buckling moments for the *n* buckling modes.

(iv) The critical buckling moment in each considered case was obtained as the least of the *n* buckling moments and occurred at the first buckling mode.

The expressions obtained for the *n*th buckling moments and the critical buckling moment in each case considered were the same as exact expressions previously obtained using various other methods by Timoshenko and other scholars.

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NOMENCLATURE

Π	total potential energy functional
x	longitudinal coordinate axis of the beam
l	length of the thin-walled beam

I_w	Saint Venant warping constant or the warping constant
J	Saint Venant torsion constant
$v(x)$	displacement
$\phi(x)$	rotational displacement about the longitudinal coordinate axis
I_z	moment of inertia in the weak axis
E	Young's modulus of elasticity
G	Shear modulus or the modulus of rigidity
M	constant or uniform applied moment at the beam ends
M_y	applied bending moment
$\partial\Pi$	first variation of Π
$\phi'(x) = \frac{d\phi(x)}{dx}$	first derivative of $\phi(x)$ with respect to x
$\phi''(x) = \frac{d^2\phi(x)}{dx^2}$	second derivative of $\phi(x)$ with respect to x
c_{1n}	buckling modal amplitude of $v(x)$ for the nth buckling mode
c_{2n}	buckling modal amplitude of $\phi(x)$ for the nth buckling mode
$\sum_{n=1}^{\infty}$	sum of
n	buckling mode number
P	vertical point load applied at the midspan
I_1, I_2, I_3	integrals defined in the paper
$a_{11}, a_{12}, a_{21}, a_{22}$	coefficients of the Ritz equations
$M_{cr(m)}$	critical buckling moment for the case of thin-walled beam with doubly symmetric cross-section and subjected to uniform moments at the ends
$M_{cr(p)}$	critical buckling moment for thin-walled beam with doubly symmetric cross-section subjected to vertical point load P applied at midspan
LTB	lateral torsional buckling
FDM	finite difference method
FEM	finite element method
$\int$	integration notation
$\frac{\partial\Pi}{\partial c_{1n}}$	partial derivative of Π with respect to c_{1n}
$\frac{\partial\Pi}{\partial c_{2n}}$	partial derivative of Π with respect to c_2
$v''(x) = \frac{d^2v(x)}{dx^2}$	second derivative of v(x) with respect to x