A New Iterative Sequence of $(\lambda, \rho)$-Firmly Nonexpansive Multi-Valued Mappings in Modular Function Spaces with Applications

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https://doi.org/10.18280/mmep.100124

Received: 8 September 2022
Accepted: 20 November 2022

Keywords:
firmly nonexpansive mappings, applications in differential equations, iterative sequences, stability, fixed point

ABSTRACT

The fixed point theory is of great importance as it is used as an application for solving differential equations for different types of equations and various applications in physical, engineering, and statistical sciences. This investigation aims to define $(\lambda, \rho)$-firmly nonexpansive multivalued mappings in modular function spaces and to introduce a new iterative algorithm. Accordingly, some results of approximating fixed points for these mappings are proved with an example. Further, the concept of stability is discussed and supported by an example.

1. INTRODUCTION

A lot of academics have worked on the fixed point since it has numerous applications in a wide range of industries. Over the years, many researchers have introduced iterative processes to solve fixed point problems, but the search for more effective and quick methods continues [1]. Modular function spaces were first introduced by Nakano in (1950), and Musielak and Orlicz greatly expanded on them [2]. One of the most important characteristics of the Modular function spaces is that considered a space separate from other spaces and is dealt with by functions.

Khamisi and Kozlowski [3] were the first to discuss the fixed point in modular function spaces in 1990. Researchers in this subject, which is thought to be growing, have tried to approximate the fixed point in modular function spaces. In Mann and Ishikawa iterative processes, Dehaish and Kozlowski [4] proved certain conclusions of approximation fixed point by modular function spaces. For monotone asymptotically nonexpansive mapping in modular function spaces, Alfuraidan and Khamis [5] developed the Fibonacci-Mann iteration with studding. Hussain Khan [6] developed the notion of a strongly nonexpansive mapping from Banach spaces to modular function spaces. Additionally, Panwar [7] recently presented some findings in this area as shown in Figure 1. Only the fixed point theory for single-valued mappings acting in modular function spaces was addressed by Kozlowski [8]. Berinde [9] introduced and studied a tighter idea of nearly stability for fixed point iteration algorithms. Besides, Abed and Abduljabbar [10] demonstrated a general two theorem for the two step iterative sequence of multivalued mappings in a complete convex real modular space. In order to achieve the best approximation in modular spaces, Abed and Sada [11] used fixed point theorems of compact - nonexpansive multivalued mapping. For iteration schemes in multivalued mappings in modular function spaces, Abdul Jabbar and Abed [12] studied convergence. For shared fixed points and convergence, Morwal and Panwar [13] presented a three-step iterative technique in three multivalued - nonexpansive mappings and studded approximation. Okeke and Khan [14] subsequently extended the proof to the class of multivalued -quasi-contractive mappings with studding of stability. Abed and Jabbar [15] developed the idea of normalized duality mapping in actual convex modular spaces. Then, a few of its characteristics have emerged, enabling the handling of outcomes connected to the idea of uniformly smooth convex real modular spaces. In this regards, Abed and Abduljabbar [16] demonstrated convergence for iteration algorithms in multivalued mappings in modular function spaces. Using the Picard-Krasnoselskii hybrid iterative process in these spaces, Okeke et al. [17] proved theorems for -quasi-nonexpansive mappings.

There are many iterative schemes presented by researchers, the aim of the paper to find a new iterative scheme to approximate the fixed point that is faster than the previous iterative scheme.

With the introduction of securely multivalued mappings and a new iterative technique and some comparison, the goal of this work is to extend the findings of prior studies, The following Figure in the study [7]:

![Figure 1](image_url)

Figure 1. Generated of $\rho$-converges to the fixed point by taking different initial values

Here, we construct an iterative sequence of a four-step for $(\lambda, \rho)$-firmly nonexpansive multi-valued mappings and study
it is convergence and stable to fixed point in the framework of modular function spaces confirming the results an example and tables are provided. Further, we mention to utilize our proposed algorithm in solve differential equation as an application.

2. PRELIMINARIES

This section includes some fundamentals, significant definitions, and certain lemmas. Assume that $\Omega$ is a nonempty set and that $\Sigma$ is a nontrivial $\sigma$-algebra of $L^p$ subsets. Considering that $\rho$ is closed in terms of constructing a finite union and has countable crossings and differences, let $\rho$ be nontrivial ring subsets of $\Omega$. Additionally, let us assume that $E \cap A \in \rho$ for any $E \in \rho$ and $A \in \Sigma$, there exists an increasing series of sets $\mathcal{K}_n \in \rho$ such that $\Omega = \bigcup \mathcal{K}_n$. With help from, we denote by $E$ the linear space of all simple functions. The space of all extended measurable functions, or all functions, is denoted by $M_{\infty}$.

\[
 f: \Omega \to [-\infty, \infty] \quad \text{Then,} \quad \{g_n\} \subset E, \quad |g_n| \leq |f| \quad \text{and} \quad g_n(w) \to f \quad \text{for all} \; w \in \Omega, \quad \text{by} \; \lambda \quad \text{the characteristic function of the set} \; \{A\}. \]

**Definition 1** [8]:

Let $\rho: M_{\infty} \to [0, \infty]$ be a nontrivial, convex, and even function.

Then, $\rho$ is a regular convex function pseudo modular if:

(a) $\rho(0) = 0$.

(b) $\rho$ is considered as a monotone, and, $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in M_{\infty}$.

(c) $\rho$ is considered as an orthogonally sub additive, and, $\rho(f_{\lambda} \beta) \leq \rho(f_{\lambda}) + \rho(f_{\gamma})$ for any $\lambda, \beta \leq \Sigma$ as $\lambda \cap \beta$ nonempty, where $f \in M_{\infty}$.

(d) $\rho$ has the Fatou property: $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in M_{\infty}$.

(e) $\rho$ is considered as an order continuous in $E$, and, $n \in E$ and $|g_n(w)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Define $M = \{f \in M_{\infty}, |f(w)| < \infty, \rho - a.e.\}$

where, each $f \in M$ is actually an equivalence class of functions equal $\rho - a.e.$ rather than an individual function.

**Definition 2** [18]:

Let $\rho: M \to [0, \infty]$ possesses the following properties:

1. $\rho(0) = 0, f = 0, \rho - a.e.$

2. $\rho(af) = f(\rho)$, for every scalar $a$

3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ For every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

where: convex modular.

**Definition 3** [3]:

According to definition $2$ $\rho$ is considered as a convex modular in $X$, as well as is named modular function spaces:

$$L_p = \{f \in M: \rho(\lambda f) \to 0 \quad \text{as} \quad \lambda \to 0\}$$

**Definition 4** [8]:

Let $\rho \in \mathbb{R}$ then $\rho$ has $\Delta_2$-condition if $\sup \rho(2f_{\lambda}, D) \to 0$ as $k \to \infty$ and $D \to \emptyset$, and $\sup \rho(f_{\lambda}, D) \to 0$

Here, $\rho$ is regular convex function modular if $\rho(f) = 0$ then $f = 0, \rho - a.e$ the class of all nonzero regular convex function in modular $\Omega$ is denoted by $\mathbb{R}$.

**Definition 5** [8]:

Here, we considered the above definition of $\rho$ on $\Omega$ let $r > 0, c > 0$ define $D(r, e) = \{(f, g): f, g \in L_p, \rho f \leq r, \rho f - g \geq er\}$.

Let $\xi_1(r, e) = \inf \{1 - \frac{1}{2}^{\rho(f_{\lambda})^2} : (f, g) \in D(r, e)\}$

If $D(r, e) \neq \emptyset$ and $\xi_1(r, e) = 1$, If $D(r, e) = \emptyset$

Also, $\rho$ satisfy (UC1) if for every $r > 0, e > 0 \xi_1(r, e) > 0$ then $D(r, e) \neq \emptyset$.

**Definition 6** [8]:

Let $\rho$ be a nonzero regular convex function modular defined on $\Omega$ satisfy (UCU1) $\delta > 0, e > 0$ there exists $\eta_1(r, e) > 0$ depending only on $\delta$ and $e$ such that $\xi_1(r, e) \geq \eta_1(r, e)$ for any $r > \delta$.

**Definition 7** [10, 15]:

Let $\rho \in \mathbb{R}$

1. We say that $\{f_n\}$ is $\rho$-convergent to $f$ if $\rho(f_n - f) \to 0$.

2. A sequence $\{f_n\}$ is $\rho$-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

3. $B \subset L_p$ is named $\rho$ - closed if for any $f_n \in L_p$ the convergence $\rho(f_n - f) \to 0$ and $f$ belongs to $B$.

4. $B \subset L_p$ is named $\rho$-bounded if $\rho$- diameter is finite. $\rho$- diameter define as $\Delta_{\rho}(B) = \sup\{\rho(f - g), f, g \in B\} < \infty$.

5. A set $B \subset L_p$ is named strongly $\rho$-bounded if $\beta > 1$ and $M_p(B) = \sup\{\rho(\rho f - g), f, g \in B\} < \infty$.

6. A set $B \subset L_p$ is called $\rho$-compact if every $f_n \in B$, there exists a subsequence $\{f_{n_k}\}$ and $f \in \rho(f_{n_k} - f) \to 0$.

7. A set $B \subset L_p$ is called $\rho$-a.e. closed if every $f_n \in B$, which $\rho$ - a.e. converges to some $f$, then $f$ be in $B$.

8. A set $B \subset L_p$ is called $\rho$-a.e. -compact if every $f_n \in B$, there exists a subsequence $\{f_{n_k}\}$ $\rho - a.e$ -converges to some $f$ in $B$.

9. Let $f \in L_p$ and $B \subset L_p$, the $\rho$-distance between $f$ and $B$ is defined as:

$$dist_{\rho}(f, B) = \inf\{\rho(f - g), g \in B\}.$$
Lemma 1 [6]:
Let \( \rho \in \mathbb{R} \) satisfy (UUC1) and let \( \{t_n\} \) in \((0, 1)\) be bounded away from 0 and 1, if \( m > 0 \) Then, \( \lim \sup_{n \to \infty} \rho(t_n) \leq m \), \( \lim_{n \to \infty} \rho(t_n) \leq m \), and \( \lim_{n \to \infty} \rho(t_n f_n + (1 - t_n) g_n) = m \), then \( \lim_{n \to \infty} \rho(f_n - g_n) = 0 \).

Lemma 2 [9]:
Let \( \{a_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty} \) be sequence of nonnegative number and \( 0 \leq r < 1 \), such that \( c_n + 1 \leq r c_n + d_n \) for all \( n \geq 0 \)
1- if \( \lim_{n \to \infty} a_n = 0 \), then \( \lim_{n \to \infty} c_n = 0 \)
2- if \( \sum_{n=1}^{\infty} d_n < \infty \), then \( \sum_{n=1}^{\infty} c_n < \infty \)
Here, \( P_\rho(E) \) denotes the family of nonempty \( \rho \)-proximinal, \( \rho \)-bounded subset of \( E \), \( C_\rho(E) \) denotes the family of nonempty \( \rho \)-closed, \( \rho \)-bounded subset of \( E \), and \( H_\rho(\cdot, \cdot) \) - Hausdorff distance on \( C_\rho(E) \)

\[
H_\rho(A, B) = \max \{ \sup_{f \in A} \text{dist}_\rho(f, B), \sup_{g \in B} \text{dist}_\rho(g, A) \} \quad A, B \in C_\rho(L_p)
\]

where, \( \text{dist}_\rho(f, B) = \inf(\rho(f - g), g \in B) \).

Lemma 3 [13]:
Let \( \rho \in \mathbb{R} \) and satisfy \( A, B \in P_\rho(L_p) \) for each \( f \in A \) there exists \( g \) in \( B \) such that \( \rho(f - g) \leq H_\rho(A, B) \).

Definition 11 [13, 14]:
Let \( T : E \to 2^E \) be multivalued mapping said to be \( \rho \)-nonexpansive mapping if \( H_\rho(Tf, Tg) \leq \rho(f - g) \) said to be \( \rho \)- quasi nonexpansive mapping if for \( s \in F_\rho(T) \) of \( T \) in modular spaces:

\[
H_\rho(Tf, s) \leq \rho(f - s)
\]

Finally, we can consider to be \( \rho \)-contraction mapping if there exists constant:

\[
0 \leq k < 1
\]

\[
H_\rho(Tf - Tg) \leq k \rho(f - g)
\]

For all \( f, g \) in \( E \).

Lemma 4 [13]:
Let \( T : E \to 2^E \) be a multivalued mapping and \( P_\rho^T(f) = \{ g \in T : \rho(f - g) = \text{dist}_\rho(f, Tg) \} \) then:
1- \( f \in F_\rho(T) \), and \( f \in T(f) \)
2- \( P_\rho^T = \{ f \} \) and \( f = g \) for each \( g \in P_\rho^T(f) \)
3- \( f \in F_\rho(P_\rho^T(f)) \), and \( f \in P_\rho^T(f) \), further \( F_\rho(T) = F(P_\rho^T(f)) \) where \( F(P_\rho^T(f)) \) denotes the set of fixed points of \( P_\rho^T(f) \).

3. RESULTS AND DISCUSSION

3.1 Convergence results for \((\lambda, \rho)\)-firmly nonexpansive multivalued mappings

Starting with the definition below:

Definition 12: Let \( T : E \to 2^E \) said to be \((\lambda, \rho)\)-firmly nonexpansive multivalued mapping if for \( \lambda \) in \((0, 1)\), \( H_\rho(Tf, Tg) \leq \rho((1 - \lambda)(f - g) + \lambda(u - v)) \), \( u \in Tf, v \in Tg \) said to be \((\lambda, \rho)\)- quasi firmly nonexpansive multivalued mapping if for \( \lambda \) in \((0, 1)\) and \( s \in F_\rho(T) \) is the set of fixed point of \( T \) in modular spaces:

\[
H_\rho(Tf, s) \leq \rho((1 - \lambda)(f - s) + \lambda(u - s)) \quad u \in Tf
\]

Clearly, \((\lambda, \rho)\)- quasi firmly nonexpansive mapping is quasi nonexpansive mapping.

Lemma 5: Every \((\lambda, \rho)\)-firmly nonexpansive mapping is \(\rho\)-nonexpansive mapping:

Proof:
By Definition 12, convexity of \( \rho \) and Lemma 3, obtaining:

\[
H_\rho(Tf, Tg) \leq \rho((1 - \lambda)(f - g) + \lambda(u - v)), u \in P_\rho^T(f), v \in P_\rho^T(g)
\]

\[
\leq (1 - \lambda)\rho(f - g) + \lambda\rho(u - v)
\]

\[
\leq (1 - \lambda)\rho(f - g) + \lambda\rho(Tf, Tg)
\]

Hence \( H_\rho(Tf, Tg) \leq \rho(f - g) \).

Lemma 6:
Let \( \rho \in \mathbb{R} \) and \( E \) be nonempty \( \rho \)-bounded, \( \rho \)-closed and \( E \in L_p \) let \( T : E \to 2^E \) be \((\lambda, \rho)\)-firmly nonexpansive multivalued mapping, then the \( F_\rho(T) \) is convex and closed.

Proof:
Let \( \{ f_n \} \) is a sequence in fixed point set \( F_\rho(T) \) is \( \rho \)-converges to some \( f \) in \( E \), to prove \( f \) fixed point.

\[
\rho\left(f - \frac{Tf - Th}{2}\right) \leq \frac{1}{2}\rho(f - f_n) + \frac{1}{2}\rho(Tf - Th) \leq \frac{1}{2}\rho(f - f_n) + \frac{1}{2}\rho(Tf - Th) \leq \rho(f - f_n) \to 0 \text{ as } n \to \infty.
\]

\[
\rho\left(f - \frac{Tf - Th}{2}\right) = 0, \text{ then } Tf = f, \text{ and } f \in F_\rho(T), \text{ by define } \rho \text{-closed } F_\rho(T)
\]

To prove \( F_\rho(T) \) convex, let \( f, g \in F_\rho(T) \) and \( h = \frac{f + g}{2} \)

\[
\rho(f - Th) = \rho(Th - f) \leq H_\rho(Th, Tf) \leq \rho(h - f)
\]

\[
= \rho\left(f - \frac{g}{2}\right)
\]

(1)

\[
\rho(g - Th) = \rho(Th - g) \leq H_\rho(Th, Tg) \leq \rho(h - g)
\]

(2)

\[
\rho(f - h) = \rho\left(f - \frac{g}{2}\right), \rho(g - h) = \rho\left(\frac{g}{2}\right)
\]

(3)

\[
\rho\left(f - \frac{h + Th}{2}\right) = \rho\left(\frac{1}{2}f - h + \frac{1}{2}(f - Th)\right)
\]

By \( h = \frac{f + g}{2} \) and convexly \( \leq \frac{1}{2}\rho\left(f - \frac{g}{2}\right) + \frac{1}{2}\rho\left(\frac{g}{2}\right) = \rho\left(f - \frac{g}{2}\right)\)

\[
\rho\left(\frac{f - h}{2}\right) \rho\left(g - \frac{h + Th}{2}\right) = \rho\left(f - \frac{g}{2}\right) + \frac{1}{2}\rho(g - Th)
\]

By \( h = \frac{f + g}{2} \) and convexly \( \leq \frac{1}{2}\rho\left(f - \frac{g}{2}\right) + \frac{1}{2}\rho\left(\frac{g}{2}\right) = \rho\left(f - \frac{g}{2}\right)\)

\[
\rho\left(f - \frac{g}{2}\right) \leq \frac{1}{2}\rho\left(f - \frac{h + Th}{2}\right) + \frac{1}{2}\rho\left(h + Th - g\right)
\]

And by above:
\[
p(\mathbf{f} - \frac{h+\mathbf{t}}{2}) \leq \rho \left( \frac{\mathbf{f} - \mathbf{t}}{2} \right), \quad \text{then} \quad p(\mathbf{f} - \frac{h+\mathbf{t}}{2}) = \\
\rho \left( \frac{\mathbf{f} - \mathbf{t}}{2} \right) \tag{4}
\]

By (1), (2), (3), (4), and Lemma 1, \( \rho(\mathbf{h}-\mathbf{t})=0, \mathbf{h} \in P_p(T) \),
then \( P_p(T) \) is convex.

Below, we introduce a new iterative algorithm and then prove convergence results.

Let \( T:E \to 2^E \), and let \( E \) nonempty convex subset of \( L_p \) sequence \( \{ f_n \} \) by the following iterative process:

\[
f_n \in E \\
h_n = (1 - \beta_n) f_n + \beta_n u_n \\
g_n = v_n \\
J_n = (1 - \alpha_n) g_n + \alpha_n w_n \\
\{ \alpha_n \} \text{ and } \{ \beta_n \} \in (0,1), u_n \in P_p^T(f_n), v_n \in P_p^T(h_n), \text{ } w_n \in P_p^T(g_n), m_n \in P_p^T(J_n). \tag{5}
\]

where, \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) in \((0,1), u_n \in P_p^T(f_n), v_n \in P_p^T(h_n), \text{ } w_n \in P_p^T(g_n), m_n \in P_p^T(J_n). \)

**Theorem 1:**
Let \( \rho \in \mathcal{R} \) satisfy (UUC1) and \( \Delta_2 \)-condition, let \( E \) be nonempty \( \rho \)-bounded, \( \rho \)-closed and convex \( E \subset L_p \), and \( T:E \to 2^E \), be \((\alpha, \rho)\)-firmly nonexpansive multivalued mapping, let \( \{ f_n \} \) in \( E \) define by (5) then \( \lim_{n \to \infty} \rho(f_n - s) \) exists for all \( s \) fixed point.

**Proof:**
Let \( s \in P_p(T) \). To prove \( \lim_{n \to \infty} \rho(f_n - s) \) exists.
By Definitions (11, 12), convexity of \( \rho \), and Lemmas (3, 5), we get:

\[
\rho(f_{n+1} - s) = \rho(m_n - s) \leq H_p(P_p^T(J_n), P_p^T(s)) \tag{6}
\]

\[
\rho(J_n - s) \leq \rho((1 - \alpha_n) g_n + \alpha_n w_n - s) \leq (1 - \alpha_n) \rho(g_n - s) + \alpha_n H_p(P_p^T(g_n), P_p^T(s)) \leq \rho(g_n - s) \tag{7}
\]

Also,

\[
\rho(g_n - s) = \rho(v_n - s) \leq H_p(P_p^T(h_n), P_p^T(s)) \tag{8}
\]

Similarly,

\[
\rho(h_n - s) = \rho(\beta_n u_n + (1 - \beta_n) f_n - s) \leq \beta_n H_p(P_p^T(f_n), P_p^T(s)) + (1 - \beta_n) \rho(f_n - s) \tag{9}
\]

By (6), (7), (8) and (9), \( \rho(f_{n+1} - s) \leq \rho(f_n - s) \), so,
\( \lim_{n \to \infty} \rho(f_n - s) \) exists for all \( s \in P_p(T) \).

**Theorem 2:**
Let \( \rho \in \mathcal{R} \) satisfy (UUC1) and \( \Delta_2 \)-condition, let \( E \) be nonempty \( \rho \)-bounded, \( \rho \)-closed and convex \( E \subset L_p \), and \( T:E \to 2^E \), be \((\alpha, \rho)\)-firmly nonexpansive multivalued mapping, let \( \{ f_n \} \) in \( E \) define by (5) then \( \lim_{n \to \infty} \text{dist}_p(f_n, P_p^T(f_n)) = 0 \)

**Proof:**
By Theorem 1 \( \lim_{n \to \infty} \rho(f_n - s) \) exists

Let

\[
\lim_{n \to \infty} \rho(f_n - s) = k, \text{ where } k \geq 0 \tag{10}
\]

By (7), (8), and (9) the following hold:

\[
\rho(h_n - s) \leq \rho(f_n - s) = \lim_{n \to \infty} \rho(h_n - s) \leq k \tag{11}
\]

\[
\lim_{n \to \infty} \rho(g_n - s) \leq k \tag{12}
\]

\[
\lim_{n \to \infty} \rho(J_n - s) \leq k \tag{13}
\]

\[
\rho(v_n - s) \leq H_p(P_p^T(h_n), P_p^T(s)) \leq \rho(h_n - s) \leq \rho(f_n - s) \tag{14}
\]

\[
\lim_{n \to \infty} \rho(v_n - s) \leq \lim_{n \to \infty} \rho(f_n - s) \tag{15}
\]

\[
\rho(w_n - s) \leq H_p(P_p^T(g_n), P_p^T(s)) \leq \rho(g_n - s) \leq \rho(f_n - s) \tag{16}
\]

Then \( \lim_{n \to \infty} \rho(w_n - s) \leq k \)

\[
\rho(m_n - s) \leq H_p(P_p^T(J_n), P_p^T(s)) \leq \rho(f_n - s) \leq \rho(f_n - s) \leq \rho(f_n - s) \tag{17}
\]

Then \( \lim_{n \to \infty} \rho(m_n - s) \leq k \)

Let \( \lim_{n \to \infty} \inf \rho(f_n + 1 - s) = \rho(m_n - s) \leq H_p(P_p^T(J_n), P_p^T(s)) \leq \rho(f_n - s) \leq \rho(g_n - s) \)

So,

\[
\lim_{n \to \infty} \inf \rho(f_n + 1 - s) = \lim_{n \to \infty} \inf \alpha_n \rho(g_n - s) + (1 - \alpha_n) \rho(g_n - s) \]

Then, \( k \leq \lim_{n \to \infty} \inf \alpha_n \rho(g_n - s) + (1 - \alpha) k \) \Rightarrow \( k \leq \alpha \lim_{n \to \infty} \rho(g_n - s) \)

Hence,

\[
k \leq \lim_{n \to \infty} \inf \rho(w_n - s) \tag{18}
\]

By (16) and (18),

\[
\lim_{n \to \infty} \rho(w_n - s) = k \tag{19}
\]

\[
\rho(w_n - s) \leq H_p(P_p^T(g_n), P_p^T(s)) \leq \rho(g_n - s) \tag{20}
\]

Then,

\[
k \leq \rho(g_n - s) \tag{21}
\]

By (12) and (20),

\[
\lim_{n \to \infty} \rho(g_n - s) = k \tag{22}
\]

Since,

\[
\rho(g_n - s) = \rho(v_n - s), \text{ so, } \lim_{n \to \infty} \rho(v_n - s) = k \tag{23}
\]

\[
\rho(v_n - s) \leq H_p(P_p^T(h_n), P_p^T(s)) \leq \rho(h_n - s) \Rightarrow \lim_{n \to \infty} \rho(v_n - s) \leq \lim_{n \to \infty} \rho(h_n - s) \tag{24}
\]

\[
\text{so,}
\]

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\[ k \leq \lim_{n \to \infty} \rho(h_n - s) \]  

By (11) and (28),
\[ \lim_{n \to \infty} \rho(h_n - s) = k \]  

By (24),
\[ \lim_{n \to \infty} \rho(h_n - s) = k \Rightarrow \lim_{n \to \infty} \rho(\beta_n u_n + (1 - \beta_n)f_n - s) = k \]
\[ \lim_{n \to \infty} \rho(\beta_n(u_n - s) + (1 - \beta_n)(f_n - s) = k \]  

By (10), (15), (25) and Lemma 1, \( \lim \rho(f_n - u_n) = 0 \) then \( u_n \in P^f_p(f_n) \). Since \( \text{dist}_p(f_n, P_p^f(f_n)) \leq \lim \rho(f_n - u_n) \), \( \lim_{n \to \infty} \text{dist}_p(f_n, P_p^f(f_n)) = 0 \). This completes the proof.

**Theorem 3:**

Let \( \rho \in \mathbb{R} \) satisfy (UUC1) and \( \Delta \)-condition, let \( E \) be nonempty \( \rho \)-bounded, \( \rho \)-closed and convex \( E \subset L_p \) and \( T : E \to 2^E \) be \( (\iota, \rho) \)-firmly nonexpansive multivalued mapping, let \( \{f_n\} \) in \( E \) define by (5), \( f_0 \) unique fixed point in \( T \), then \( f_n \) converge to fixed point in \( T \).

**Proof:**

By convexity of \( \rho \), Lemma 3, Definitions (11,12) and Lemma 5, implies that:
\[ \rho(h_n - f_0) = \rho((1 - \beta_n)f_n + \beta_n u_n - f_0) \leq (1 - \beta_n)\rho(f_n - f_0) + \beta_n \rho(u_n - f_0) \leq (1 - \beta_n)\rho(f_n - f_0) + \beta_n \rho(f_n - f_0) \leq \rho(f_n - f_0) \]  

Again,
\[ \rho(g_n - f_0) \leq \rho(v_n - f_0) \leq H_p(p_p^f(h_n), P_p^f(f_0)) \leq \rho(h_n - f_0) \leq \rho(f_n - f_0) \]  

Similarity,
\[ \rho(g_n - f_0) = \rho(\alpha_1 g_n + \alpha_0 w_n - f_0) \leq (1 - \alpha_1)\rho(g_n - f_0) + \alpha_1 \rho(w_n - f_0) \leq (1 - \alpha_1)\rho(g_n - f_0) + \alpha_1 \rho(w_n - f_0) \leq \rho(f_n - f_0) \]  

Similarly,
\[ \rho(f_n - f_0) = \rho(\alpha_1 g_n + \alpha_0 w_n - f_0) \leq H_p(p_p^f(h_n), P_p^f(f_0)) \leq \rho(f_n - f_0) \leq \rho(f_n - f_0) \]  

Now, \( \rho(f_n - f_0) = \rho(\alpha_1 g_n + \alpha_0 w_n - f_0) \leq (1 - \alpha_1)\rho(g_n - f_0) + \alpha_1 \rho(w_n - f_0) \leq (1 - \alpha_1)\rho(g_n - f_0) + \alpha_1 \rho(w_n - f_0) \leq \rho(f_n - f_0) \)  

So, \( \rho(f_n - f_0) \leq \rho(f_n - f_0) \), \( \rho(f_n - f_0) \leq \rho(f_n - f_0) \), \( \rho(f_n - f_0) \leq \rho(0) = 0 \), then \( f_n \to f_0 \).

**Theorem 5:**

Let \( \rho \in \mathbb{R} \) satisfy (UUC1) and \( \Delta \)-condition, let \( E \) be nonempty \( \rho \)-compact, \( \rho \)-closed and convex \( E \subset L_p \) and \( T : E \to 2^E \) be \( (\iota, \rho) \)-firmly nonexpansive multivalued mapping, let \( \{f_n\} \) in \( E \) define by (5), \( f_0 \) unique fixed point in \( T \).

**Proof:**

By Theorem 1 \( \lim_{n \to \infty} \rho(f_n - s) \) exists, and \( \lim_{n \to \infty} \rho(f_n - f_0) \) exists and \( s \in F_p(T) \).

Suppose \( f_n \) sequence of \( f_n \), and \( u_k \) sequence in \( F_p(T) \).
\[ \rho(f_n - u_k) \leq 2^{\frac{1}{2k}} \]  

Since, \( \lim_{n \to \infty} \rho(f_n - u_k) = 0 \),
\[ \rho(f_{n+1} - u_k) \leq 2^{\frac{1}{2k}} \]  

Therefore, \( \rho(u_k - u_k) \to 0 \), as \( k \to \infty \).
\[ u_k \text{ is } \rho \text{-cauchy in } F_p(T) \text{, since } \Delta \text{ condition, so } \rho \text{-cauchy } \Rightarrow \rho \text{-convergence.} \]
\[ u_k \to u_k \text{ in } F_p(T) \text{, so } \rho(u_k - s) \to 0 \]

Now, \( \rho(f_n - s) \leq \rho(f_n - u_k) + \rho(u_k - s) \), \( \rho(f_n - u_k) \to 0 \), and \( \rho(u_k - s) \to 0 \). \( f_n \) converges to fixed point \( s \) in \( F_p(T) \).
Theorem 6:
Let $\rho \in \mathbb{R}$ satisfy (UUC1) and $\Delta_2$ -condition, let $E$ be nonempty $\rho$-bounded, $\rho$-closed and convex $E \subset L_p$, and $T: E \to 2^E$ be $(\rho, \rho)$- firmly nonexpansive multivalued mapping, and $T$ satisfied condition (1), let $\{f_n\}$ in $E$ define by (5) then $f_n$ converge to fixed point $s$ of $T$.

Proof:
By Theorem 1 $\lim_{n \to \infty} \rho(f_n - s) = 0$ , nothing to prove, if $\lim_{n \to \infty} \rho(f_n - s) = k, k \geq 0$.
By Theorem 1 $\rho(f_{n+1} - s) \leq \rho(f_n - s)$
Then $\lim_{n \to \infty} \rho(f_n, F(T)) \leq \lim_{n \to \infty} \rho(f_n, F(T))$, hence $\lim_{n \to \infty} \rho(f_n, F(T)) = 0$.
By applying condition (1) and Theorem 2 $\lim_{n \to \infty} \rho(f_n, F(T)) = 0$.
Since $\emptyset(0) = 0$, so, $\lim_{n \to \infty} \rho(f_n, F(T)) = 0$, by Theorem 5, $f_n$ is $\rho$ - converge to fixed point $s$.

3.2 Stability
In this section, firstly we reform the definition of stability as in the study [9], then give results and an example.

Definition 13 [9]:
Let $E$ be an empty convex subset of modular function space $L_p$, and $T: E \to E$, let $x_n \in E$, and $x_{n+1} = f(T, x_n)$ define the iterative schemes which given sequence $x_n \in E$, suppose that $\{x_n\}_{n=1}^{\infty}$ converge to $x \in F(T) \neq \emptyset$, then $\{y_n\}_{n=1}^{\infty}$ be any bounded sequence in $E$ and put $\epsilon_n = \rho(y_n - f(T, x_n))$.
1- The iteration scheme $\{x_n\}_{n=1}^{\infty}$ define by $x_{n+1} = f(T, x_n)$ is displayed to be T-stable on $E$ if $\lim_{n \to \infty} \epsilon_n = 0$ implies that $\lim_{n \to \infty} y_n = x$.
2- The iteration scheme $\{x_n\}_{n=1}^{\infty}$ define by $x_{n+1} = f(T, x_n)$ is displayed to be almost T-stable on $E$ if $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n \to \infty} y_n = x$.
3- The iteration scheme $\{x_n\}_{n=1}^{\infty}$ define by $x_{n+1} = f(T, x_n)$ is considered to be summably almost T-stable on $E$ if and only if $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\sum_{n=1}^{\infty} \rho(y_n - x) < \infty$.

Theorem 7:
Let $\rho \in \mathbb{R}$ satisfy (UUC1) and $\Delta_2$ -condition, let $E$ be nonempty $\rho$-bounded, $\rho$-closed and convex $E \subset L_p$, and $T: E \to 2^E$ be $(\rho, \rho)$- firmly nonexpansive multivalued mapping, and $T$ satisfied condition (1), let $\{f_n\}$ in $E$ define by (5) then $f_n$ is summably almost T-stable.

Proof:
Let $s$ is fixed point of $T$, and $\epsilon_n = \rho(f_{n+1} - s) \leq \rho(f_{n+1} - m_n) + \rho(m_n - s) - \rho(f_{n+1} - m_n) + \rho(m_n - s) \leq \epsilon_n + H_\rho(\rho_n, f_n, T(s)) \leq \epsilon_n + \rho(f_n - s)$.

Substituting $f_n$ and similarity above:
$\leq \epsilon_n + p((1 - \alpha_n)g_n + \alpha_n w_n - s)$
$\leq \epsilon_n + (1 - \alpha_n)\rho(g_n - s) + \alpha_n H_\rho(p_n, g_n, T(s)) \leq \epsilon_n + \rho(g_n - s)$

Substituting $g_n$ and similarity above:
$= \epsilon_n + \rho(v_n - s) \leq \epsilon_n + H_\rho(p_n, h_n, T(s)) \leq \epsilon_n + \rho(h_n - s)$

Substituting $h_n$ and similarity above:
$\leq \epsilon_n + \rho(\beta_n u_n + (1 - \beta_n) f_n - s) \leq \epsilon_n + \rho(f_n - s)$

So, by Lemma 2 and Definition 13, implies that $f_n$ is summably almost T-stable.

Example 1:
The set of real number $\mathbb{R}$ by the space $\rho(f) = |f|$, $\rho$ is satisfy (UUC1) and $\Delta_2$ -condition, $E = \{f \in L_p: 0 \leq f < \infty \}$. $T: E \to E$ $(\rho, \rho)$- firmly nonexpansive mapping and $Tf = \frac{f}{4}$ with $F_0(T) = \{0\}$, and let $\alpha_n, \beta_n = 0.5$ for all $n$.

$f_n = (1 - \beta_n) f_n + \beta_n T f_n$
$g_n = \lambda n T g_n$
$f_{n+1} = T f_n$

Since $\sum_{n=1}^{\infty} \epsilon_n = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{n^3}$, so $\sum_{n=1}^{\infty} \epsilon_n < \infty$ implies that $\sum_{n=1}^{\infty} \rho(\rho_n - s) < \infty$.

The iterative scheme in (5) is summably almost T-stable.

Example 2:
The set of real number $\mathbb{R}$ by the space $\rho(f) = |f|$, $\rho$ is satisfy (UUC1) and $\Delta_2$-condition, $E = [0, \infty]$ define $T: E \to E$ a mapping, $\Phi: [0, \infty) \to [0, \infty),$ $\Phi(r) = \frac{r}{6}$.

And $Tf = \frac{f + 4}{5}$, $F_0(T) = \{1\}$, to prove $\rho(f - T f) \geq \Phi(\rho(f - T f))$ for all $f$ in $E$.
\[\rho(f - T f) = \rho \left( f \frac{f + 4}{5} \right) = \frac{4f + 4}{5} \] , while
\[\rho(\Phi(f - T f)) = \rho(\Phi(f - T f)) = \Phi(\rho(f - T f)) = \Phi(f - T f) = \frac{f + 4}{5}.
\]
Now, prove $T$ is $(\rho, \rho)$- firmly nonexpansive mapping $\rho(T f - g) = \rho \left( f + \frac{4f + 4}{5} \right) = \left\| f - g \right\| \leq \frac{1}{25} \left\| f - g \right\|$, $T$ is $(\rho, \rho)$-firmly nonexpansive mapping when $\lambda = \frac{1}{5}$.

Tables 1, 2, and 3 represent the corresponding results as shown below.
Table 1. Results of $f_n, h_n, g_n$, and $J_n$ where $\alpha_n=\beta_n=0.5$, with $f_1 = 2$

<table>
<thead>
<tr>
<th>Step</th>
<th>$f_n$</th>
<th>$h_n$</th>
<th>$g_n$</th>
<th>$J_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1.6</td>
<td>1.12</td>
<td>1.072</td>
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<tr>
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<td>1.000000204</td>
</tr>
<tr>
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<td>1.000000005</td>
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</tr>
<tr>
<td>6</td>
<td>1.0000000001</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1.84</td>
<td>1.168</td>
<td>1.14112</td>
</tr>
</tbody>
</table>

Table 2. Results of $f_n, h_n, g_n$, and $J_n$ where $\alpha_n=\beta_n=0.2$, with $f_1 = 2$

<table>
<thead>
<tr>
<th>Step</th>
<th>$f_n$</th>
<th>$h_n$</th>
<th>$g_n$</th>
<th>$J_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1.84</td>
<td>1.168</td>
<td>1.14112</td>
</tr>
<tr>
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</tr>
<tr>
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<td>1.000000014</td>
<td>1.000000003</td>
<td>1.000000002</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3. Results of $f_n, h_n, g_n$, and $J_n$ where $\alpha_n=\beta_n=0.8$, with $f_1 = 2$

<table>
<thead>
<tr>
<th>Step</th>
<th>$f_n$</th>
<th>$h_n$</th>
<th>$g_n$</th>
<th>$J_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1.84</td>
<td>1.168</td>
<td>1.14112</td>
</tr>
<tr>
<td>2</td>
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<td>1.00186624</td>
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</tr>
<tr>
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<tr>
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<td>1.00000050</td>
<td>1.00000100</td>
<td>1.00000003</td>
</tr>
<tr>
<td>5</td>
<td>1.00000001</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Through our study of the above tables, it becomes clear that the closer the value of $\alpha_n$ and $\beta_n$ to the fixed point, the faster the approximation to the fixed point.

4. APPLICATION

Since fixed point theory provides useful tools to solve many problems that have applications in different fields of sciences, the studying of iterative algorithms to approximate the solution of differential equations be one of most active studies area. Therefore, this section is devoted to applying the above results to differential equations in way similar to what is presented in the study [3]. In the following, we deal with especial case of algorithm (5):

Let $\rho \in \mathbb{R}$, consider the initial value problem $v: [0, B] \rightarrow E$ and $E$ in $L_p$:

\[ v(0) = g, \quad v'(t) + (I - T)v(t) = 0 \tag{30} \]

where, $g \in E$, $B > 0$, and $T: E \rightarrow E$ such that $P_{\rho, T}$ is $(\rho, \mu)$-firmly nonexpansive mappings, and by define:

\[ Q(t) = 1 - e^{-t} = \int_0^t e^{s-t} \, ds \tag{31} \]

For any $u: [0, B] \rightarrow L_p$ and $B > 0$ then:

\[ S(u)(t) = \int_0^t e^{s-t} u(s) \, ds \tag{32} \]

Also,

\[ S_t(u)(t) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i} v(t_i) \tag{33} \]

The following Lemma and Theorem in the study [3]:

Lemma 6:

Let $\rho \in \mathbb{R}$ be separable let $m: [0, B] \rightarrow L_p$ by two Bochner-integrable $\| \cdot \|_p$-bounded functions where $B > 0$ for $t \in [0, B]$ then:

\[ \rho \left( e^{-t} l(t) + \int_0^t e^{-s} m(s) \, ds \right) \leq e^{-t} \rho \left( l(t) \right) + Q(t) \sup \rho (m(s)) \quad s \in [0, t] \]

Theorem 8:

Let $\rho \in \mathbb{R}$ be separable, let $D$ in $L_p$ be nonempty, convex, $\rho$-bounded, and $\rho$-closed set with vitali property. Let $T: D \rightarrow P_{\rho}(D)$ be multivalued mapping such that $P_{\rho, T}$ is nonexpansive mappings. Let fixed $g \in E$ define sequence of functions $v_n: [0, B] \Rightarrow E$ by the following formula $v_n(0) = g$, $v_{n+1}(t) = e^{-t} g + \int_0^t e^{s-t} (v_n(s)) \, ds$.

Then $t \in [0, B]$ there exists $v(t) \in E$ such that $\rho(v_n(t) - v(t)) \rightarrow 0$ and by the function $v: [0, B] \rightarrow E$ the $\rho(v_n(t) - v(t)) \rightarrow 0$ is solution to (30), moreover $\rho(g - v_n(t)) \leq Q^{n+1}(B) \delta_\rho(E)$.
5. CONCLUSIONS

The modular type conditions are more natural as assumptions of modular type can be checked more easily than their metric or modular counterparts especially in applications for differential operators, approximations and fixed point results. In the current investigation, the concept of $(\lambda, \rho)$-firmly nonexpansive mapping and its relationship with $(\lambda, \rho)$-quasi firmly nonexpansive mapping and nonexpansive mapping have been discussed. In addition, some convergence and stability results by using an iterative scheme in four steps of multivalued mapping in modular function space have been proved. The study suggests to the authors using the iterative scheme in other styles in modular function spaces or another spaces. This study is important as the iterative scheme that was presented is faster in reaching the fixed point than other iterative (see the research [19]). Lastly, we apply this algorithm to solving of a differential equation.

6. FUTURE WORK

We look forward to employing our results (which relate to the convergence of algorithm 5 and equation 30) in practical application in one of the anther branches of sciences such as physical engineering or control. This aim may require cooperation with some colleagues.

REFERENCES


