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A New Iterative Sequence of (λ, ρ)-Firmly Nonexpansive Multi-Valued Mappings in Modular Function Spaces with Applications

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ABSTRACT

The fixed point theory is of great importance as it is used as an application for solving differential equations for different types of equations and various applications in physical, engineering, and statistical sciences. This investigation aims to define (λ, ρ) - firmly nonexpansive multivalued mappings in modular function spaces and to introduce a new iterative algorithm. Accordingly, some results of approximating fixed points for these mappings are proved with an example. Further, the concept of stability is discussed and supported by an example.

1. INTRODUCTION

A lot of academics have worked on the fixed point since it has numerous applications in a wide range of industries. Over the years, many researchers have introduced iterative processes to solve fixed point problems, but the search for more effective and quick methods continues [1]. Modular function spaces were first introduced by Nakano in (1950), and Musielak and Orlicz greatly expanded on them [2]. One of the most important characteristics of the Modular function spaces is that considered a space separate from other spaces and is dealt with by functions.

Khamisi and Kozlowski [3] were the first to discuss the fixed point in modular function spaces in 1990. Researchers in this subject, which is thought to be growing, have tried to approximate the fixed point in modular function spaces. In Mann and Ishikawa iterative processes, Dehaish and Kozlowski [4] proved certain conclusions of approximation fixed point by modular function spaces. For monotone asymptotically nonexpansive mapping in modular function spaces, Alfuraidan and Khamsi [5] developed the Fibonacci-Mann iteration with studding. Hussain Khan [6] developed the notion of a strongly nonexpansive mapping from Banach spaces to modular function spaces. Additionally, Panwar [7] recently presented some findings in this area as shown in Figure 1. Only the fixed point theory for single-valued mappings acting in modular function spaces was addressed by Kozlowski [8]. Berinde [9] introduced and studied a tighter idea of nearly stability for fixed point iteration algorithms. Besides, Abed and Abduljabbar [10] demonstrated a general two theorem for the two step iterative sequence of multivalued mappings in a complete convex real modular space. In order to achieve the best approximation in modular spaces, Abed and Sada [11] used fixed point theorems of compact - nonexpansive multivalued mapping. For iteration schemes in multivalued mappings in modular function spaces, Abdul Jabbar and Abed [12] studied convergence. For shared fixed points and convergence, Morwal and Panwar [13] presented a three-step iterative technique in three multivalued - nonexpansive mappings and studded approximation. Okeke and Khan [14] subsequently extended the proof to the class of multivalued -quasi-contractive mappings with studding of stability. Abed and Jabbar [15] developed the idea of normalized duality mapping in actual convex modular spaces. Then, a few of its characteristics have emerged, enabling the handling of outcomes connected to the idea of uniformly smooth convex real modular spaces. In this regards, Abed and Abduljabbar [16] demonstrated convergence for iteration algorithms in multivalued mappings in modular function spaces. Using the Picard-Krasnoselskii hybrid iterative process in these spaces, Okeke et al. [17] proved theorems for -quasi-nonexpansive mappings.

There are many iterative schemes presented by researchers, the aim of the paper to find a new iterative scheme to approximate the fixed point that is faster than the previous iterative scheme.

With the introduction of securely multivalued mappings and a new iterative technique and some comparison, the goal of this work is to extend the findings of prior studies, The following Figure in the study [7]:

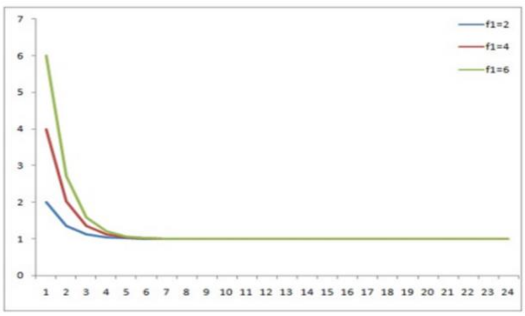

Figure 1. Generated of ρ-converges to the fixed point by taking different initial values

Here, we construct an iterative sequence of a four-step for *(λ, ρ)*- firmly nonexpansive multi-valued mappings and study it is convergence and stabile to fixed point in the framework of modular function spaces confirming the results an example and tables are provided. Further, we mention to utilize our proposed algorithm in solve differential equation as an application.

2. PRELIMINARIES

This section includes some fundamentals, significant definitions, and certain lemmas. Assume that Ω is a nonempty set and that Σ is a nontrivial σ -algebra of L p subsets. Considering that ρ is closed in terms of constructing a finite union and has countable crossings and differences, let ρ be nontrivial ring subsets of Ω. Additionally, let us assume that $E \cap A \in \rho$ for any $E \in \rho$ and $A \in \Sigma$, there exists an increasing series of sets$K_n \in \rho$ such that $\Omega = \cup K_n$. With help from, we denote by E the linear space of all simple functions. The space of all extended measurable functions, or all functions, is denoted by M_∞.

$f: \Omega \longrightarrow [-\infty, \infty]$ Then, $\{g_n\} \subset E$, $|g_n| \leq |f|$ and $g_n(w) \longrightarrow f$ for all $w \in \Omega$, by 1_A the characteristic function of the set A [8].

Definition 1 [8]:

Let $\rho: M_\infty \longrightarrow [0, \infty]$ be a nontrivial, convex, and even function.

Then, ρ is a regular convex function pseudo modular if:

(a) $\rho(0)=0$.

(b) ρ is considered as a monotone, and, $|f(w)|\leq|g(w)|$ for all $w \in \Omega$ implies $\rho(f)\leq\rho(g)$, where $f, g \in M_\infty$.

(c) ρ is considered as an orthogonally sub additive, and, $\rho(f_{1_{A\cup B}}) \leq \rho(f_{1_A}) + \rho(f_{1_B})$ for any $A, B \in \Sigma$ as $A\cap B$ nonempty, where $f \in M_\infty$.

(d) ρ has the Fatou property: $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$,where $f \in M_\infty$.

(e) ρ is considered as an order continuous in E, and, $g_n \in E$ and $|g_n(w)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Define $M = \{f \in M_\infty: |f(w)| < \infty, \rho - a.e\}$ where, each $f \in M$ is actually an equivalence class of functions equal *ρ-a.e.* rather than an individual function.

Definition 2 [18]:

Let $\rho: M \longrightarrow [0,\infty]$ possesses the following properties:

1- $\rho(0)=0, f=0$, *ρ-a.e*

2- $\rho(\alpha f)=\rho(f)$, for every scalar α

3- $\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y)$ For every $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ where: convex modular.

Definition 3 [3]:

According to definition 2 ρ is considered as a convex modular in *X*, as well as is named modular function spaces:

$$L_p = \{f \in M: \rho(\lambda f) \longrightarrow 0 \ as \ \lambda \longrightarrow 0\}$$

Definition 4 [8]:

Let $\rho \in \Re$ then ρ has Δ_2-condition if sup $\rho(2f_n, D) \longrightarrow 0$ as $k\longrightarrow\infty$ and $D \longrightarrow \emptyset$, and sup $\rho(f_n, D) \longrightarrow 0$

Here, ρ is considered regular convex function modular if $\rho(f)=0$ then $f=0$, *a-e* the class of all nonzero regular convex function in modular Ω is denoted by $\Re$.

Definition 5 [8]:

Here, we considered the above definition of ρ on Ω let $r>0$, $\epsilon>0$ define $D(r, \epsilon) = \{(f, g): f, g \in L_P, \rho f \leq r, \rho f - g \geq \epsilon r\}$.

Let $\xi_1(r, \epsilon) = \inf\left\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f, g) \in D(r, \epsilon)\right\}$

If $D(r, \epsilon) \neq \emptyset$ and $\xi_1(r, \epsilon) = 1$, If $D(r, \epsilon) = \emptyset$

Also, ρ satisfy (UC1) if for every r$> 0, \epsilon > 0$ $\xi_1(r, \epsilon) > 0$ then $D(r, \epsilon) \neq \emptyset$.

Definition 6 [8]:

Let ρ be a nonzero regular convex function modular defined on Ω ρ satisfy (UUC1) $\delta \geq 0, \epsilon > 0$ there exists $\eta_1(r, \epsilon) > 0$ depending only on δ and ϵ such that $\xi_1(r, \epsilon) > \eta_1(r, \epsilon) > 0$ for any $r>\delta$.

Definition 7 [10, 15]:

Let $\rho \in \Re$

1- We say that $\{f_n\}$ is ρ-convergent to f if $\rho(f_n - f) \longrightarrow 0$.

2- A sequence $\{f_n\}$ is ρ-Cauchy sequence if $\rho(f_n - f_m) \longrightarrow 0$ as $n, m\longrightarrow\infty$.

3- $B \sqsubset L_p$ is *named* ρ-closed if for any $f_n \in L_p$ the convergence $\rho(f_n - f) \longrightarrow 0$and f belongs to B.

4- $B \sqsubset L_p$ is *named*ρ-bounded if ρ- diameter is finite. ρ-diameter define as $\mathfrak{H}_p(B) = \sup\{\rho(f - g), f \in B, g \in B\} < \infty$.

5- A set $B \sqsubset L_p$ is named strongly ρ-bounded if $\beta > 1$ and $M_p(B) = \sup\{\rho(\beta(f - g)), f \in B, g \in B\} < \infty$.

6- A set $B \sqsubset L_p$ is called ρ-compact if every $f_n \in B$, there exists a subsequence $\{f_{n_k}\}$ and f in $\rho(f_{n_k} - f) \rightarrow 0$.

7- A set $B \sqsubset L_p$ is called *ρ-a.e*, closed if every $f_n \in B$, which$\rho - a.e$, converges to some *f*, then *f* in *B*.

8- A set $B \sqsubset L_p$ is called *ρ-a.e*, -compact if every $f_n \in B$, there exists a subsequence $\{f_{n_k}\}$ $\rho - a.e$ -converges to some *f* in *B*.

9- Let *f* in L_p and $B \sqsubset L_p$, the ρ-distance between *f* and *B* is defined as:

$$dist_p(f, B) = \inf\{\rho(f - g), g \in B\}.$$

Note that, ρ dose note satisfy triangle inequality so ρ-convergence dose not ρ-Cauchy, it is possible that this relationship can be realized if and only if ρ satisfies Δ_2-condition [6].

Definition 8 [6]:

The sequence $\{t_n\}$ is considered to be bounded away from 0 if $\alpha>0$ and t $t_n > \alpha$ for all $n \in N$. Also, the sequence $\{t_n\}$ is considered to be bounded away from 1 if $b < 1$ and $t_n \leq b$for all $n \in N$.

Definition 9 [6]:

$E \subset L_p$, let $T: E \longrightarrow 2^E$ said to be satisfy condition (I) if no decreasing function $\emptyset: [0, \infty) \longrightarrow [0, \infty)$ with $\emptyset(0) = 0, \emptyset(r) > 0$ for all $r \in [0,\infty]$ such that $\rho(f - Tf) \geq \emptyset(dist_\rho\left(f, F_p(t)\right))$ for all $f \in E$.

Definition 10 [13]:

A set $E \subset L_p$ is *named* ρ- proximinal if for each $f \in L_p$ there exists an element g in E. And, $\rho(f - g) = dist_p(f, E) = \inf\{\rho(f - h): h \ in \ E$.

Lemma 1 [6]:

Let $\rho \in \Re$ satisfy (UUC1) and let $\{t_n\}$ in (0, 1) be bounded away from 0 and 1, if $m>0$ Then, $\lim sup_{n\to\infty}\rho(f_n) \leq m$, $\lim sup_{n\to\infty}\rho(g_n) \leq m$, and $lim_{n\to\infty}\rho(t_n f_n + (1-t_n)g_n) = m$, then $\lim_{n\to\infty} \rho(f_n - g_n) = 0$.

Lemma 2 [9]:

Let $\{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty}$ be sequence of nonnegative number and $0 \leq r < 1$, such that $c_{n+1} \leq rc_n + d_n$ for all $n \geq 0$

1- if $lim_{n\to\infty} d_n = 0$, then $lim_{n\to\infty} c_n = 0$

2- if $\sum_{n=0}^{\infty} d_n < \infty$, then $\sum_{n=0}^{\infty} c_n < \infty$

Here, $P_p(E)$ denotes the family of nonempty ρ-proximinal, ρ-bounded subset of E, $C_p(E)$ denotes the family of nonempty ρ-closed, ρ-bounded subset of E, and $H_p(.,.)$ ρ- Hausdorff distance on $C_p(E)$

$$H_p(A,B) = \max\{ sup_{f\in A}\, dist_p\,(f,B), \\ sup_{g\in B}\, dist_p\,(g,A)\}\ A,B\ \in C_p(L_p)$$

where, $dist_p(f,B) = \inf\{\rho(f-g), g \in B\}$.

Lemma 3 [13]:

Let $\rho \in \mathcal{R}$ and satisfy $A, B \in P_p(L_p)$ for each f *in* A there exists g *in* B such that $\rho(f-g) \leq H_p(A,B)$.

Definition 11 [13, 14]:

Let $T: E \longrightarrow 2^E$ is multivalued mapping said to be ρ-nonexpansive mapping if $H_p(Tf,Tg) \leq \rho(f-g)$ said to be ρ- quasi nonexpansive mapping if for $s \in F_p(T)$ of T in modular spaces:

$$H_p(Tf,s) \leq \rho(f-s)$$

Finally, we can consider to be ρ-contraction mapping if there exists constant:

$$0 \leq k < 1 \\ H_p(Tf - Tg) \leq k\rho(f-g)$$

For all *f, g* in *E*.

Lemma 4 [13]:

Let $T: E \longrightarrow 2^E$ be a multivalued mapping and $P_p^T(f) = \{g \in T: \rho(f-g) = dist_\rho(f,Tf)\}$ then:

1- $f \in F_p(T)$, and $f \in T(f)$

2- $P_p^T = \{f\}$ and $f = g$ for each $g \in P_p^T(f)$

3- $f \in F_p\left(P_p^T(f)\right)$, and $f \in P_p^T(f)$, further $F_p(T) = F(P_p^T(f))$ where $F(P_p^T(f))$ denotes the set of fixed points of $P_p^T(f)$.

3. RESULTS AND DISCUSSION

3.1 Convergence results for (λ, ρ)- firmly nonexpansive multivalued mappings

Starting with the definition below:

Definition 12:

Let $T: E \longrightarrow 2^E$ said to be *(λ, ρ)*- firmly nonexpansive multivalued mapping if for λ in (0,1), $H_p(Tf,Tg) \leq \rho[(1-\lambda)(f-g) + \lambda(u-v)\,]$, $u \in Tf, v \in Tg$ said to be *(λ, ρ)*-quasi firmly nonexpansive multivalued mapping if for λ in (0,1) and $s \in F_p(T)$ is the set of fixed point of T in modular spaces:

$$H_p(Tf,s) \leq \rho[(1-\lambda)(f-s) + \lambda(u-s)] \\ u \in Tf$$

Clearly, *(λ, ρ)*- quasi firmly nonexpansive mapping is quasi nonexpansive mapping.

Lemma 5:

Every (*λ, ρ*) - firmly nonexpansive mapping is *ρ*-nonexpansive mapping:

Proof:

By Definition 12, convexity of ρ and Lemma 3, obtaining:

$$H_p(Tf,Tg) \leq \rho[(1-\lambda)(f-g) + \lambda(u-v)], u \in P_p^T(f), v \in P_p^T(g) \\ \leq (1-\lambda)\rho(f-g) + \lambda\rho(u-v) \\ \leq (1-\lambda)\rho(f-g) + \lambda H_p(Tf,Tg)$$

Hence $H_p(Tf,Tg) \leq \rho(f-g)$.

Lemma 6:

Let $\rho \in \Re$ and E be nonempty ρ-bounded, ρ-closed and $E \subset L_p$ let $T: E \longrightarrow 2^E$ be *(λ, ρ)*- firmly nonexpansive multivalued mapping, then the $F_p(T)$ is convex and closed.

Proof:

Let $\{f_n\}$ is a sequence in fixed point set $F_p(T)$ is ρ-converges to some f in E, to prove f fixed point.

$\rho\left(\frac{f-Tf}{2}\right) \leq \frac{1}{2}\rho(f-f_n) + \frac{1}{2}\rho(Tf-f_n) \quad \leq \frac{1}{2}\rho(f-f_n) + \frac{1}{2}H_p(Tf,Tf_n) \ \leq \frac{1}{2}\rho(f-f_n) + \frac{1}{2}\rho(f-f_n) \ = \rho(f-f_n) \longrightarrow 0\ as\ n \longrightarrow \infty$.

$\rho\left(\frac{f-Tf}{2}\right) = 0$, then $Tf = f, and\ f \in F_p(T)$, by define ρ – closed $F_p(T)$ is closed.

To prove $F_p(T)$ convex, let *f, g* in $F_p(T)$ and $h = \frac{f+g}{2}$:

$$\rho(f-Th) = \rho(Th-f) \leq H_p(Th,Tf) \leq \rho(h-f) \\ = \rho\left(\frac{f-g}{2}\right) \tag{1}$$

$$\rho(g-Th) = \rho(Th-g) \leq H_p(Th,Tg) \leq \rho(h-g) = \rho\left(\frac{f-g}{2}\right) \tag{2}$$

$$\rho(f-h) = \rho\left(\frac{f-g}{2}\right), \rho(g-h) = \rho\left(\frac{f-g}{2}\right) \tag{3}$$

$$\rho\left(f - \frac{h+Th}{2}\right) = \rho(\frac{1}{2}(f-h) + \frac{1}{2}(f-Th))$$

By $h = \frac{f+g}{2}$ and convexly $\leq \frac{1}{2}\ \rho\left(\frac{f-g}{2}\right) + \frac{1}{2}\ \rho\left(\frac{f-g}{2}\right) = \rho\left(\frac{f-g}{2}\right) \rho\left(g - \frac{h+Th}{2}\right) = \rho(\frac{1}{2}(g-h) + \frac{1}{2}(g-Th)$

By $h = \frac{f+g}{2}$ and convexly $\leq \frac{1}{2}\,\rho\left(\frac{f-g}{2}\right) + \frac{1}{2}\,\rho\left(\frac{f-g}{2}\right) = \rho\left(\frac{f-g}{2}\right)$

$\rho\left(\frac{f-g}{2}\right) \leq \frac{1}{2}\rho\left(f - \frac{h+Th}{2}\right) + \frac{1}{2}\rho\left(\frac{h+Th}{2} - g\right)$

$\rho\left(\frac{f-g}{2}\right) \leq \rho\left(f - \frac{h+Th}{2}\right)$

And by above:

$$\rho\left(f-\frac{h+Th}{2}\right)\leq\rho\left(\frac{f-g}{2}\right), \text{ then } \rho\left(f-\frac{h+Th}{2}\right)=\rho\left(\frac{f-g}{2}\right) \quad (4)$$

By (1), (2), (3), (4), and Lemma 1, $\rho(h\text{-}Th)=0$, h in $F_p(T)$, then $F_p(T)$ is convex.

Below, we introduce a new iterative algorithm and then prove convergence results.

Let $T: E \longrightarrow 2^E$, and let E nonempty convex subset of L_p sequence $\{f_n\}$ by the following iterative process:

$$\begin{gathered} f_1 \in E \\ h_n = (1-\beta_n)f_n + \beta_n u_n \\ g_n = v_n \\ J_n = (1-\alpha_n)g_n + \alpha_n w_n \\ f_{n+1} = m_n,\ n\epsilon N \end{gathered} \quad (5)$$

where, $\{\alpha_n\}$and $\{\beta_n\}$ in (0,1), $u_n \in P_\rho^T(f_n), v_n \in P_\rho^T(h_n)$, $w_n \in P_\rho^T(g_n), m_n \in P_\rho^T(J_n)$.

Theorem 1:

Let $\rho \in \Re$ satisfy (UUC1) and Δ_2-condition, let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be (λ, ρ)- firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by (5) then $lim_{n\to\infty}\rho(f_n - s)$ exists for all s fixed point.

Proof:

Let $s \in F_p(T)$**.** To prove $lim_{n\to\infty}\rho(f_n - s)$ exists.

By Definitions (11, 12), convexity of ρ, and Lemmas (3,5), we get:

$$\begin{aligned}\rho(f_{n+1}-s)=\rho(m_n-s)&\leq H_p(P_p^T(J_n),P_p^T(s))\\ &\leq \rho(J_n-s)\end{aligned} \quad (6)$$

$$\begin{aligned}\rho(J_n-s)\leq\rho\big((1-\alpha_n)g_n+\alpha_n w_n\big)-s)&\\ \leq (1-\alpha_n)\rho(g_n-s)&\\ +\alpha_n H_p(P_p^T(g_n),P_p^T(s))&\\ \leq\rho(g_n-s)&\end{aligned} \quad (7)$$

Also,

$$\begin{aligned}\rho(g_n-s)=\rho(v_n-s)&\leq H_p(P_p^T(h_n),P_p^T(s))\\ &\leq\rho(h_n-s)\end{aligned} \quad (8)$$

Similarly,

$$\begin{aligned}\rho(h_n-s)=\rho(\beta_n u_n+(1-\beta_n)f_n-s)&\\ \leq\beta_n H_p(P_p^T(f_n),P_p^T(s))&\\ +(1-\beta_n)\rho(f_n-s)\leq\rho(f_n-s)&\end{aligned} \quad (9)$$

By (6), (7), (8) and (9), $\rho(f_{n+1}-s) \leq \rho(f_n - s)$, so, $lim_{n\to\infty}\rho(f_n - s)$ exists for all $s \in F_p(T)$.

Theorem 2:

Let $\rho \in \Re$ satisfy (UUC1) and Δ_2-condition, let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $: E \longrightarrow 2^E$, be (λ,ρ)- firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by (5) then $lim_{n\to\infty}\, dist_\rho\rho\big(f_n, P_p^T(f_n)\big) = 0$

Proof:

By Theorem 1 $lim_{n\to\infty}\rho(f_n - s)$ exists

Let

$$lim_{n\to\infty}\rho(f_n-s)=k, \text{ where } k\geq 0 \quad (10)$$

By (7), (8), and (9) the following hold:

$$\rho(h_n-s)\leq\rho(f_n-s)\Rightarrow\ lim_{n\to\infty}\rho(h_n-s)\leq k \quad (11)$$

$$lim_{n\to\infty}\rho(g_n-s)\leq k \quad (12)$$

$$lim_{n\to\infty}\rho(J_n-s)\leq k \quad (13)$$

$$\begin{gathered}\rho(v_n-s)\leq H_p(P_p^T(h_n),P_p^T(s))\leq\rho(h_n-s)\\ \leq\rho(f_n-s)\\ \lim_{n\to\infty}\rho(v_n-s)\leq\lim_{n\to\infty}\rho(f_n-s)\leq k\end{gathered} \quad (14)$$

$$\begin{gathered}\rho(u_n-s)\leq H_p\left(P_p^T(f_n),P_p^T(s)\right)\leq\rho(f_n-s),\\ \text{Then } \lim_{n\to\infty}\rho(u_n-s)\leq k\end{gathered} \quad (15)$$

$$\begin{gathered}\rho(w_n-s)\leq H_p(P_p^T(g_n),P_p^T(s))\leq\rho(g_n-s)\\ \leq\rho(f_n-s)\\ \text{Then } \lim_{n\to\infty}\rho(w_n-s)\leq k\end{gathered} \quad (16)$$

$$\begin{gathered}\rho(m_n-s)\leq H_p(P_p^T(J_n),P_p^T(s))\leq\rho(J_n-s)\\ \leq\rho(f_n-s)\\ \text{Then } \lim_{n\to\infty}\rho(m_n-s)\leq k\end{gathered} \quad (17)$$

Let $\lim_{n\to\infty}\alpha_n=\alpha$, $\rho(f_{n+1}-s)=\rho(m_n-s)\leq H_p(P_p^T(J_n),P_p^T(s))\leq\rho(J_n-s)\leq\rho(\alpha_n w_n+(1-\alpha_n)g_n-s)\leq\alpha_n\rho(w_n-s)+(1-\alpha_n)\rho(g_n-s)$.

So, $\lim_{n\to\infty} inf\rho(f_{n+1}-s)\leq\lim_{n\to\infty}\inf[\alpha_n\rho(w_n-s)+(1-\alpha_n)\rho(g_n-s)]$.

Then, $k\leq\lim_{n\to\infty}\inf\ \alpha_n\rho(w_n-s)+(1-\alpha)k \quad \Rightarrow \alpha k\leq\alpha\lim_{n\to\infty}\inf\ \rho(w_n-s)$.

Hence,

$$k\leq\lim_{n\to\infty}\inf\ \rho(w_n-s) \quad (18)$$

By (16) and (18),

$$\begin{gathered}\lim_{n\to\infty}\rho(w_n-s)=k\\ \rho(w_n-s)\leq H_p(P_p^T(g_n),P_p^T(s))\leq\rho(g_n-s)\end{gathered} \quad (19)$$

Then,

$$k\leq\rho(g_n-s) \quad (20)$$

By (12) and (20),

$$\lim_{n\to\infty}\rho(g_n-s)=k \quad (21)$$

Since,

$$\begin{gathered}\rho(g_n-s)=\rho(v_n-s),\text{so, }\lim_{n\to\infty}\rho(v_n-s)=k\\ \rho(v_n-s)\leq H_p(P_p^T(h_n),P_p^T(s))\leq\rho(h_n-s)\\ \Rightarrow\lim_{n\to\infty}\rho(v_n-s)\\ \leq\lim_{n\to\infty}\rho(h_n-s)\end{gathered} \quad (22)$$

so,

$$k \le \lim_{n\to\infty} \rho(h_n - s) \quad (23)$$

By (11) and (23), then:

$$\lim_{n\to\infty} \rho(h_n - s) = k \quad (24)$$

By (24),

$$\lim_{n\to\infty} \rho(h_n - s) = k \Rightarrow \lim_{n\to\infty} \rho(\beta_n u_n + (1-\beta_n) f_n - s) = k$$
$$\lim_{n\to\infty} \rho(\beta_n (u_n - s) + (1-\beta_n)(f_n - s) = k \quad (25)$$

By (10), (15), (25) and Lemma1, $\lim_{n\to\infty} \rho(f_n - u_n) = 0$ then $u_n \in P_p^T(f_n)$. Since $dist_\rho \rho(f_n, P_p^T(f_n)) \le \lim_{n\to\infty} \rho(f_n - u_n)$, $lim_{n\to\infty}\, dist_\rho \rho(f_n, P_p^T(f_n))$ =0. This completes the proof.

Theorem 3:
Let $\rho \in \Re$ satisfy (UUC1) and Δ_2 -condition, let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be *(λ, ρ)*- firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by (5), f_0 unique fixed point in *T*, then f_n converge to fixed point in *T*.

Proof:
By convexity of ρ, Lemma 3, Definitions (11,12) and Lemma 5, implies that:

$$\begin{aligned}\rho(h_n - f_0) &= \rho((1-\beta_n) f_n + \beta_n u_n) - f_0) \\ &\le (1-\beta_n)\rho(f_n - f_0) + \beta_n H_p(P_p^T(f_n), P_p^T(f_0)) \\ &\le (1-\beta_n)\rho(f_n - f_0) + \beta_n \rho(f_n - f_0) \\ &\le \rho(f_n - f_0)\end{aligned} \quad (26)$$

Again,

$$\begin{aligned}\rho(g_n - f_0) &\le \rho(v_n - f_0) \\ &\le H_p(P_p^T(h_n), P_p^T(f_0)) \\ &\le \rho(h_n - f_0) \\ &\le \rho(f_n - f_0)\end{aligned} \quad (27)$$

Similarity,

$$\begin{aligned}\rho(J_n - f_0) &= \rho((1-\alpha_n) g_n + \alpha_n w_n - f_0) \\ &\le (1-\alpha_n)\rho(g_n - f_0) + \alpha_n H_p(P_p^T(g_n), P_p^T(f_0)) \\ &\le (1-\alpha_n)\rho(g_n - f_0) + \alpha_n \rho(g_n - f_0) \\ &\le \rho(f_n - f_0)\end{aligned} \quad (28)$$

Similarity,

$$\begin{aligned}\rho(f_{n+1} - f_0) = \rho(m_n - f_0) &\le H_p(P_p^T(J_n), P_p^T(f_0)) \\ &\le \rho(J_n - f_0) \\ \rho(f_n - f_0) &\le \rho(f_{n-1} - f_0)\end{aligned} \quad (29)$$

Since $\rho(f_1 - f_0) \le \rho(f_0 - f_0)$, so, $\rho(f_n - f_0) \le \rho(f_0 - f_0)$, $\rho(f_n - f_0) \le \rho(0) = 0$, then $f_n \longrightarrow f_0$.

Theorem 4:
Let $\rho \in \Re$ satisfy (UUC1) and Δ_2 -condition, let E be nonempty ρ-compact, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be *(λ, ρ)*- firmly nonexpansive multivalued mapping, let $\{f_n\}$ n E define by (5) then f_n converge to fixed point of *T*.

Proof:
Since E is ρ-compact there exists subsequence f_{n_k} of f_n and $\rho(f_{n_k} - f) = 0$.

$lim_{n\to\infty} \rho(f_{n_k} - f) = 0$, to prove f fixed point.

Let g fixed point $g \in P_p^T(f), g_k \in P_p^T(f_{n_k})$.

By Lemma 3, Definitions (7, 11), and Lemma 5, we get

$$\begin{aligned}\rho\left(\frac{f-g}{3}\right) &= \rho[\frac{f - f_{n_k}}{3} + \frac{f_{n_k} - g_k}{3} + \frac{g_k - g}{3}] \\ &\le \frac{1}{3}\rho(f - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - g_k) + \frac{1}{3}\rho(g_k - g) \\ &\le \frac{1}{3}\rho(f - f_{n_k}) + \frac{1}{3} dist_\rho(f_{n_k}, P_p^T(f_{n_k})) + \frac{1}{3} H_p(P_p^T(f_{n_k}), P_p^T(f)) \\ &\le \frac{1}{3}\rho(f - f_{n_k}) + \frac{1}{3} dist_\rho(f_{n_k}, P_p^T(f_{n_k})) + \frac{1}{3}(f_{n_k} - f) dist_\rho(f_{n_k}, P_p^T(f_{n_k})) = 0\end{aligned}$$

by Theorem 2, $\rho\left(\frac{f-g}{3}\right) = 0$, therefore *f*=*g*, then *T* has unique fixed point *f*, f_n converge to fixed point of *T*.

Theorem 5:
Let $\rho \in \Re$ satisfy (UUC1) and Δ_2 -condition, let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be *(λ, ρ)*- firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by (5) then f_n converge to fixed point *s* of *T* if and only if lim $inf_{n\to\infty}\, dist_p(f_n, F(T)) = 0$, where $dist_p(f_n, F(T)) = \inf\{\rho(f - s), s \in F_p(T)\}$.

Proof:
Let f_n converge to fixed point *s* of *T*, to prove lim $inf_{n\to\infty}\, dist_p(f_n, F(T)) = 0$.

Since $f_n \longrightarrow s$, then lim $_{n\to\infty}$ $dist_p(f_n, s) = 0$.

Since $dist_p(f_n, F_p(T)) \le dist_p(f_n, s)$, then lim $inf_{n\to\infty}\, dist_p(f_n, F_p(T)) = 0$.

If lim $inf_{n\to\infty}\, dist_p(f_n, F_p(T)) = 0$, to prove $\rho(f_n - s) = 0$.

By Theorem 1 $lim_{n\to\infty} \rho(f_n - s)$ exists, then $lim_{n\to\infty} \rho(f_n - F_p(T))$ exists and s$\in F_p(T)$.

Suppose f_{n_k} s any subsequence of f_n, and u_k sequence in $F_p(T)$.

$\rho(f_{n_k} - u_k) \le \frac{1}{2^k}$, Since, lim $inf_{n\to\infty}\, dist_p\left(f_n, F_p(T)\right) = 0$.

$\rho(f_{n+1} - u_k) \le \rho(f_n - u_k) \le \frac{1}{2^k}$.

$\rho(u_{k+1} - u_k) \le \rho(u_{k+1} - f_{n+1}) + \rho(f_{n+1} - u_k) \le \frac{1}{2^{k+1}} + \frac{1}{2^k} \le \frac{1}{2^{k-1}}$.

$\rho(u_{k+1} - u_k) \longrightarrow 0$, as $k \longrightarrow \infty$.

u_k is ρ-cauchy in $F_p(T)$, since Δ_2 condition, so ρ-cauchy$\Leftrightarrow$ ρ-converge.

u_k is ρ-convergence in $F_p(T)$, so $\rho(u_k - s) \longrightarrow 0$.

Now, $\rho(f_{n_k} - s) \le \rho(f_{n_k} - u_k) + \rho(u_k - s)$, $\rho(f_{n_k} - u_k) \longrightarrow 0$, and $\rho(u_k - s) \longrightarrow 0$,$f_n$ Converges to fixed point *s* in $F_p(T)$.

Theorem 6:

Let $\rho \in \Re$ satisfy (UUC1) and Δ_2-condition, let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$ be *(λ,ρ)*- firmly nonexpansive multivalued mapping, and *T* satisfied condition (I), let $\{f_n\}$ in *E* define by (5) then f_n converge to fixed point *s* of *T*.

Proof:

By Theorem 1 $lim_{n\to\infty}\rho(f_n - s)$ exists, s is fixed point.

If $lim_{n\to\infty}\rho(f_n - s) = 0$, nothing to prove, if $lim_{n\to\infty}\rho(f_n - s) = k, k \geq 0$.

By Theorem 1$\rho(f_{n+1} - s) \leq \rho(f_n - s)$

Then $, dist_\rho(f_{n+1}, F_p(T)) \leq dist_\rho(f_n, F_p(T))$, hence $lim_{n\to\infty} dist_\rho(f_n, F_p(T))$ exists.

By applying condition (I) and Theorem 2

$lim_{n\to\infty}\emptyset(dist_\rho\left(f_n, F_p(T)\right) \leq lim_{n\to\infty}(f_n, P_p^T(f_n)) = 0.$

Since $\emptyset(0) = 0$, so, $lim_{n\to\infty} dist_\rho(f_n, F_p(T))$ = 0, by Theorem 5, f_n is ρ -converge to fixed point s.

3.2 Stability

In this section, firstly we reform the definition of stability as in the study [9], then give results and an example.

Definition 13 [9]**:**

Let E be anon empty convex subset of modular function space L_p, and *T:E⟶E*, let $x_1$in E, and $x_{n+1} = f(T, x_n)$ define the iterative schemes which given sequence x_n in E, suppose that $\{x_n\}_{n=1}^\infty$ converge to $x \in F_p(T) \neq \emptyset$, let $\{y_n\}_{n=1}^\infty$ be any bounded sequence in E and put $\varepsilon_n = \rho(y_{n+1} - f(T, x_n))$.

1- The iteration scheme $\{x_n\}_{n=1}^\infty$ define by $x_{n+1} = f(T, x_n)$ is displayed to be T-stable on E if $lim_{n\to\infty}\varepsilon_n = 0$ implies that $lim_{n\to\infty} y_n = x$.

2- The iteration scheme $\{x_n\}_{n=1}^\infty$ define by $x_{n+1} = f(T, x_n)$ is displayed to be almost T-stable on E if $\sum_{n=1}^\infty \varepsilon_n < \infty$ implies that $lim_{n\to\infty} y_n = x$.

3- The iteration scheme $\{x_n\}_{n=1}^\infty$ define by $x_{n+1} = f(T, x_n)$ is considered to be summably almost T-stable on E if and only if $\sum_{n=1}^\infty \varepsilon_n < \infty$ implies that $\sum_{n=1}^\infty \rho(y_n - x) < \infty$.

Theorem 7:

Let $\rho \in \Re$ satisfy (UUC1) and Δ_2-condition, let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be *(λ, ρ)*- firmly nonexpansive multivalued mapping, and *T* satisfied condition (I), let $\{f_n\}$ in *E* define by (5) then f_n is summably almost T-stable.

Proof:

Let s is fixed point of T, and $\varepsilon_n = \rho(f_{n+1} - m_n)$, by (5), convexity of ρ, Lemma 3, Definitions (11, 12), and Lemma 5, implies that:

$$\begin{aligned}\rho(f_{n+1} - s) = \rho((f_{n+1} - \ m_n) + (m_n - s))\\ \leq \rho(f_{n+1} - \ m_n) + \rho(m_n - s)\\ \leq \varepsilon_n + H_p(P_p^T(J_n), P_p^T(s))\\ \leq \varepsilon_n + \rho(J_n - s)\end{aligned}$$

Substituting J_n and similarity above:

$$\begin{aligned}\leq \varepsilon_n + p\big((1 - \alpha_n)g_n + \alpha_n w_n\big) - s))\\ \leq \varepsilon_n + (1 - \alpha_n)\rho(g_n - s) + \alpha_n H_p(P_p^T(g_n), P_p^T(s))\\ \leq \varepsilon_n + \rho(g_n - s)\end{aligned}$$

Substituting g_n and similarity above:

$$\begin{aligned}= \varepsilon_n + \rho(v_n - s)\\ \leq \varepsilon_n + H_p(P_p^T(h_n), P_p^T(s))\\ \leq \varepsilon_n + \rho(h_n - s)\end{aligned}$$

Substituting h_n and similarity above:

$$\begin{aligned}\leq \varepsilon_n + \rho(\beta_n u_n + (1 - \beta_n)f_n - s)\\ \leq \varepsilon_n + \rho(f_n - s)\\ \rho(f_{n+1} - s) \leq \varepsilon_n + \rho(f_n - s)\end{aligned}$$

So, by Lemma 2 and Definition 13, implies that f_n is summably almost T-stable.

Example 1:

The set of real number $\Re$ by the space *ρ(f)=|f|*, ρ is satisfy (UUC1) and Δ_2 -condition, $E = \{f \in L_p : 0 \leq f < \infty\}$, *T:E⟶E (λ, ρ)*- firmly nonexpansive mapping and $Tf = \frac{f}{4}$, with $F_p(T) = \{0\}$, and let α_n, β_n = 0.5 for all *n*.

$$\begin{aligned}h_n = (1 - \beta_n)f_n + \beta_n T f_n\\ g_n = T h_n\\ J_n = (1 - \alpha_n)g_n + \alpha_n T g_n\\ f_{n+1} = T J_n\end{aligned}$$

Let $f_n = \frac{n+1}{n+2}$

$h_n = \frac{1}{2}\frac{n+1}{n+2} + \frac{1}{2}\frac{n+1}{4(n+2)} = \frac{5}{8}\frac{(n+1)}{(n+2)}$, then $Th_n = \frac{5(n+1)}{32(n+2)}$

$f_{n+1} = T(\frac{1}{2}(\frac{5(n+1)}{32(n+2)}) + \frac{1}{2}(\frac{5(n+1)}{128(n+2)})$, so $f_{n+1} = \frac{25(n+1)}{1024(n+2)}$

$\varepsilon_n = \rho(f_{n+1} - f(T, f_n))$

$= \rho(\frac{n+2}{n+3} - \frac{25(n+1)}{1024(n+2)}) = \left|\frac{n+2}{n+3} - \frac{25(n+1)}{1024(n+2)}\right| = \left|\left(1 - \frac{25}{1024}\right) + (\frac{25}{1024(n+2)} - \frac{1}{n+3})\right| \leq \left|1 - \frac{25}{1024}\right| + \left|\frac{25}{1024(n+2)} - \frac{1}{n+3}\right| \leq \left|\frac{1}{n+2} - \frac{1}{n+3}\right| = \frac{1}{n+2} - \frac{1}{n+3}$

Since $\sum_{n=1}^\infty \varepsilon_n \leq \sum_{n=1}^\infty(\frac{1}{n+2} - \frac{1}{n+3})$, so $\sum_{n=1}^\infty \varepsilon_n < \infty$

$\sum_{n=1}^\infty \rho(f_n - s)$ s fixed point

$= \sum_{n=1}^\infty \left|\frac{n+1}{n+2} - 0\right| = \sum_{n=1}^\infty \frac{n+1}{n+2} = \sum_{n=1}^\infty(1 - \frac{n+1}{n+2}) < \infty$

The iterative scheme in (5) is summably almost T-stable.

Example 2:

The set of real number $\Re$ by the space *ρ(f)=|f|*, ρ is satisfy (UUC1) and Δ_2-condition, *E=[0, 3]* define *T:E⟶E* a mapping, $\emptyset: [0, \infty) \longrightarrow [0, \infty)$, $\emptyset(r) = \frac{r}{6}$

And $Tf = \frac{f+4}{5}$, $F_p(T) = \{1\}$, to prove $\rho(f - Tf) \geq \emptyset(dist_p(f, F_p(T))$ for all *f in E*.

$\rho(f - Tf) = \rho\left(f - \frac{f+4}{5}\right) = \frac{4f+4}{5}$, while $\emptyset(dist_p(f, F_p(T)) = \emptyset(dist_p(f, \{1\}) = \phi[\rho(f - 1)] = \frac{f-1}{6}$.

Now, prove *T* is *(λ, ρ)*- firmly nonexpansive mapping

$\rho(Tf - Tg) = \rho\left(\frac{f+4}{5} - \frac{g+4}{5}\right) = \left|\frac{1}{5}(f - g)\right| \leq \left|\frac{21}{25}(f - g)\right| \leq \rho(\frac{4}{5}(f - g) + \frac{1}{5}\left(\frac{1}{5}(f - g)\right))$, T is *(λ, ρ)*-firmly nonexpansive mapping when $\lambda = \frac{1}{5}$.

Tables 1, 2, and 3 represent the corresponding results as shown below.

Table 1. Results of f_n, h_n, g_n, and J_n where α_n= β_n= 0.5, with $f_1 = 2$

Step	f_n	h_n	g_n	J_n
1	2	1.6	1.12	1.072
2	1.0144	1.00864	1.001728	1.0010368
3	1.00020736	1.000124416	1.00024883	1.00001493
4	1.000002986	1.000001701	1.00000034	1.000000204
5	1.000000041	1.000000025	1.000000005	1.000000003
6	1.000000001	1	1	1
7	1	1	1	1

Table 2. Results of f_n, h_n, g_n, and J_n where α_n= β_n=0.2, with $f_1 = 2$

Step	f_n	h_n	g_n	J_n
1	2	1.84	1.168	1.14112
2	1.028224	1.02370816	1.004741632	1.003982971
3	1.000796594	1.000669138	1.000133828	1.000112415
4	1.000022483	1.000018885	1.000003777	1.000003173
5	1.00000635	1.000000533	1.000000107	1.000000090
6	1.000000018	1.000000014	1.000000003	1.000000002
7	1	1	1	1

Table 3. Results of f_n, h_n, g_n, and J_n where α_n= β_n=0.8, with $f_1 = 2$

Step	f_n	h_n	g_n	J_n
1	2	1.36	1.072	1.02592
2	1.005184	1.00186624	1.000373248	1.000134369
3	1.000026874	1.0000009674	1.000001935	1.00000696
4	1.000000139	1.00000050	1.00000010	1.00000003
5	1.00000001	1	1	1
6	1	1	1	1

Through our study of the above tables, it becomes clear that the closer the value of α_n and β_n o the fixed point, the faster the approximation to the fixed point.

4. APPLICATION

Since fixed point theory provides useful tools to solve many problems that have applications in different fields of sciences, the studying of iterative algorithms to approximate the solution of differential equations be one of most active studies area. Therefore, this section is devoted to applying the above results to differential equations in way similar to what is presented in the study [3]. In the following, we deal with especial case of algorithm (5):

Let $\rho \in \Re$, consider the initial value problem *v: [0, B] ⟶E* and E in L_p:

$$\begin{gathered} v(0) = g \\ v^{\backslash}(t) + (I - T)v(t) = 0 \end{gathered} \tag{30}$$

where, $g \in E$, *B>0*, and *T: E⟶E* such that P_ρ^T is *(λ, ρ)*-firmly nonexpansive mappings, and by define:

$$Q(t) = 1 - e^{-t} = \int_0^t e^{s-t}\, ds \tag{31}$$

For any $u: [0, B] \longrightarrow L_p$ and $B > 0$ then:

$$S(u)(t) = \int_0^t e^{s-t}\, u(s) ds \tag{32}$$

Also,

$$S_t(u)(t) = \sum_{i=0}^{n-1} (t_{i+1} - t_i) e^{t_i - t} v(t_i) \tag{33}$$

The following Lemma and Theorem in the study [3]:

Lemma 6:

Let $\rho \in \Re$ be separable let $m, l: [0, B] \longrightarrow L_p$ by two Bochner-integrable $\|.\|_\rho$-bounded functions where *B>0* for $t \in [0, B]$ then:

$$\rho\left(e^{-t} l(t) + \int_0^t e^{s-t}\, m(s) ds\right) \leq e^{-t}\rho\big(l(t)\big) + Q(t) \sup_{s \in [0,t]} \rho\big(m(s)\big)$$

Theorem 8:

Let $\rho \in \Re$ be separable, let D in L_p be nonempty, convex, ρ-bounded, and ρ-closed set with vitali property. Let $T: D \longrightarrow P_\rho(D)$ be multivalued mapping such that P_ρ^T is nonexpansive mappings. Let fixed $g \in E$ define sequence of functions $v_n: [0, B] \Longrightarrow E$ by the following formula $v_0(0) = g$, $v_{n+1}(t) = e^{-t} g + \int_0^t e^{s-t} T(v_n(s)) ds$.

Then $t \in [0, B]$ there exists $v(t) \in E$ such that $\rho(v_n(t) - v(t)) \longrightarrow 0$ and by the function $v: [0, B] \longrightarrow E$ the $\rho(v_n(t) - v(t)) \longrightarrow 0$ is solution to (30), moreover $\rho(g - v_n(t)) \leq Q^{n+1}(B)\delta_\rho(E)$.

5. CONCLUSIONS

The modular type conditions are more natural as assumptions of modular type can be checked more easily than their metric or modular counterparts especially in applications for differential operators, approximations and fixed point results. In the current investigation, the concept of (λ, ρ)-firmly nonexpansive mapping and its relationship with (λ, ρ)- quasi firmly nonexpansive mapping and nonexpansive mapping have been discussed. In addition, some convergence and stability results by using an iterative scheme in four steps of multivalued mapping in modular function space have been proved. The study suggests to the authors using the iterative scheme in other styles in modular function spaces or another spaces. This study is important as the iterative scheme that was presented is faster in reaching the fixed point than other iterative (see the research [19]). Lastly, we apply this algorithm to solving of a differential equation.

6. FUTURE WORK

We look forward to employing our results (which relate to the convergence of algorithm 5 and equation 30) in practical application in one of the anther branches of sciences such as physical engineering or control. This aim may require cooperation with some colleagues.

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