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Efficient Solutions for Nonlinear Diffusion Equations Appeared as Models of Physical Problems

Kamel Al-Khaled*, Safa Nayef Taha

Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, Irbid 22110, Jordan

Corresponding Author Email: kamel@just.edu.jo

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ABSTRACT

The differential transform technique (DTM) looks promise for dealing with functional problems. Recent articles have demonstrated the DTM's efficiency in tackling a wide range of issues in many disciplines. In this paper, (DTM) is used to develop approximate, and exact solutions for some nonlinear diffusion equations. Nonlinear diffusion equations are used to describe processes and behaviours in fields of biology, heat transfer, chemical reactions, and mathematical physics. The differential transform method linked with Laplace transform and Pad'e approximation is used to improve some known results. The obtained solutions are compared with the exact known solutions, showing excellent agreement. The differential transformation method was used in conjunction with the use of the Laplace transform and the Pad'e approximation method, for the purpose of improving some calculations in the hope of obtaining a more accurate solution. The results were presented in the form of tables or graphics for the purpose of comparing the calculated solution and comparing it with some of the precise solutions presented previously. The results showed the accuracy of the agreement between the two solutions. This gives us the opportunity to use the method under consideration to find solutions to unknown problems and thus ensure the credibility of the calculated solution.

1. INTRODUCTION

Nonlinear phenomena have substantial applications in applied mathematics, physics, and engineering concerns. Nonlinear phenomena have substantial applications in applied mathematics, physics, and engineering concerns. In the following, we highlight some of the significant applications of nonlinear diffusion equations [1]:

1. Phonon transport [2]:

It is possible to investigate phonon transport in silicon structures that generate internal heat. The linear heat diffusion equation, according to the data, severely underestimates temperature distribution at nanoscales in the presence of an external heat source. It is addressed how the temperature distribution inside a silicon thin film changes when heated by a pulsed laser, an electron beam, or due to near-field thermal radiation effects.

2. Biology [3]:

Mathematical biology is a rapidly expanding and well-known field that corresponds to the most fascinating current application of mathematics. As biology grows more quantitative, the application of mathematics will become more prevalent. The biological sciences complexity necessitates multidisciplinary participation.

3. Archeology [4]:

Both archeological evidence and Homer's Iliad indicate to a nonlinear mathematical model of the Trojan War, which occurred approximately 1180 BC. The number of warriors, the fighting rate factors, and so on, people per hectare, and other pertinent statistics are approximated.

4. Galactic civilizations [5]:

Potential theory investigates galactic civilizations' interstellar diffusion. For nonlinear partial differential and difference equations that describe a number of relevant models drawn from blast wave physics, soil science, and notably population biology, numerical and analytical solutions are developed. In this paper, we study the approximate solutions via the use of differential transform method (DTM) to the nonlinear diffusion equation.

$$u_t = (D(u)u_x)_x \tag{1}$$

where, $D(u)$ is the diffusion term, which plays a prominent role in many engineering or physical applications. In general equation (1) suggested as mathematical models of physical problems in many fields for which the diffusion coefficient $D(u)$ plays a great importance in identifying and studying many phenomena and qualities, such as, applications of fluid flows [6], mathematical model for some physical phenomena [7], ground water modelling [8], industrial processes [9], mathematical biology [10], oil pollutions [11], and diffusion dynamics [12, 13], all of these applications can be represented as a mathematical model by Eq. (1) by changing the type of diffusion function $D(u)$, including exponential, polynomial powers $D(u) = u^n$, or even rational functions $D(u) = 1/(u^2+1)$. It is worth noting that these above-mentioned forms of $D(u)$ represent solutions to mathematical models with interesting engineering or physical applications, which may be new, and it may be difficult to study their characteristics, as we will see in the last section of the numerical applications. Therefore, it is important in this research to find approximate solutions for Eq. (1) in order to study some engineering or physical phenomena by changing the diffusion coefficient *D(u)*. There are many previous studies that discuss different methods with the aim of obtaining solutions to the nonlinear equation in Eq. (1), some of them are approximate and others are exact and obtained in a closed form solutions [14]. These solutions include different formulas for the diffusion coefficient *D(u)*. We are having difficulty locating their analytical solutions. Recently, various intriguing approximate analytical solutions, such as references [15-17], have been offered. An approximate solution of nonlinear fractional diffusion equation using the q-homotopy analysis transform method used [18]. A novel method called variational iteration method is proposed to solve nonlinear partial differential equations [19-21]. Because of the many types that the diffusion coefficient takes, we will focus on three types.

1. A situation that represents a rapid diffusion represented in the case of $D(u) = u^n,\ n < 0$. For $n = \frac{-1}{2}$, Eq. (1) is used as a model in thermal expulsion of liquid helium for studying the behaviour and diffusion mechanisms of helium in nuclear ceramics [22]. For *n=-1*, Eq. (1) appeared in Carleman's model in the thermal limit approximation [23]. In addition, the isothermal Maxwellian distribution describes the growth of a thermalized electron cloud into a vacuum [24, 25].

2. The second type, which is concerned with the slow diffusion, where $D(u) = u^n, n > 0$. The case when *n=1* that arises as a mathematical model which discuss isothermal percolation for gas that is through a porous medium that is micro [26, 27]. While the case when *n=2*, Eq. (2) models the behaviour of metals after melting and evaporation [23, 25].

3. Other diffusion processes included the exponential case and its generalization, to include a diffusion function, that take one of the following forms:

$$D(u) = \frac{1}{1+u^2}, D(u) = \frac{1}{1-u^2},$$

for more details see [24, 28, 29], we also mention here some numerical methods similar to ours, in which mathematical models with physical and engineering applications were processed [30-33]. Our goal in this paper is to find approximate solutions, via the use of a promising technique called DTM, based on the Taylor series expansion, which generates an analytical solution in the form of a polynomial Symbolic computing is required for the classic high order Taylor series technique. DTM, on the other hand, derives a polynomial series solution by an iterative approach. The strategy decreases the size of the computational domain and is easily adaptable to a wide range of situations. Because its fundamental benefit is that it can be applied directly to nonlinear ordinary and partial differential equations without the need for linearization, discretization, or perturbation, it has been widely researched and utilized during the last two decades. DTM has been used to solve ordinary differential equations, partial differential equations, eigenvalue issues, differential algebraic equations, integral equations, and other problems. As an improvement on the method under consideration, we use the Laplace transform linked with the Pade' approximation in order of obtaining accurate solutions in a closed form. We used the proposed amendment after getting a series solution using the DTM, which can boost the convergence rate of the truncated series solution produced from the DTM. The phases of our method are summarized below:

1. First, we use the Laplace transform to convert the shortened series generated by DTM to the s-domain function, where s is the Laplace parameter.
2. Second, we approximate the obtained result in the above by a rational s-function using the Pad'e approximant.
3. Finally, we convert the output function back to the time-domain using the inverse Laplace transform.

The following is how the current paper is arranged. New theorems have been added to the theory in section 2, which is devoted to the description of the two-dimensional differential transform. Section 3 presents numerous numerical experiments as the application of DTM to several variants of the model Eq. (1), and Section 4 presents the conclusion.

2. TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

In 1986, Zhou [34] proposed a new method called the differential transformation method (DTM), which is considered to be one of the best numerical methods used to solve both ordinary and partial differential equations. Chen and others [35-37] they developed the method for solving partial differential equations. To review the previous studies in detail, and in order to avoid repetition, we draw the readers' attention to a comprehensive study on the use of DTM [38]. In this method, the solution is written in the form of a series of polynomials, with the assumption that the series of the imposed solution is differentiable as many times as needed. This makes it easy to calculate the approximate solution as a higher order series. The technique under discussion can be applied for solving more difficult differential equations such as nonlinear equations, as we will see in this paper. It is worth noting that Hassan [39] used the same method to solve partial differential equations, both linear and nonlinear. Below we present some necessary definitions and transformations resulting from the application of the differential transform method.

Definition 1 [34] To handle the $2-\mathrm{D}$ differential transformation of $\mathrm{w(x,y)}$ we use:

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(0,0)} \tag{2}$$

where, $w(x,y)$ is the original function, while the transformed function that result after applying the transformation is $W(k,h)$.

Definition 2 The inverse differential transform of *W(k,h)* can be defined as follows:

$$w(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k,h) x^k y^h. \tag{3}$$

From Eqns. (2) and (3) it can be concluded that:

$$w(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(0,0)} x^k y^h. \tag{4}$$

A differential equation in the domain of interest can be translated to an algebraic equation using the differential transformation. From Eqns. (2) and (3), the following theorems 1-12 may be inferred (3). The proofs are well-known in the literature [40] and will not be discussed further here.

Theorem 1 If $w(x,y) = c_1 u(x,y) + c_2 v(x,y)$, then $W(k,h) = c_1 U(k,h) + c_2 V(k,h)$, for some constants c_1, c_2.

Theorem 2 If $w(x,y) = \frac{dw(x,y)}{dx}$, then

$$W(k,h) = (k+1)W(k+1,h)$$

Theorem 3 If $w(x,y) = \frac{dw(x,y)}{dy}$, then

$$W(k,h) = (h+1)W(k,h+1)$$

Theorem 4 If $w(x,y) = \frac{d^{(r+s)}u(x,y)}{dx^r dy^s}$, then

$$W(k,h) = (k+1)(k+2)\ldots(k+r)(h+1)(h+2)\ldots(h+s)U(k+r,h+s)$$

Theorem 5 If *w(x,y)=u(x,y)v(x,y),* then

$$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} U(r,h-s)V(k-r,s).$$

Theorem 6 If $w(x,y) = x^m y^n$, then

$$W(k,h) = \delta(k-m,h-n) = \delta(k-m)\delta(h-n).$$

where,

$$\delta(k-m,h-n) = \begin{cases} 1, & k=m, h=n \\ 0, & otherwise \end{cases}$$

Proof: From the definition of differential transform, we have

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}(x^m y^n)}{\partial x^k \partial y^h}\right]_{(0,0)} = \frac{1}{k!h!}\left[\frac{d^h\left(y^n \frac{d^k x^m}{dx^k}\right)}{dy^h}\right]_{(0,0)}.$$

Substituting $\frac{d^k x^m}{dx^k} = \frac{m!}{(m-k)!} x^{m-k}$, we get

$$w(k,h) = \frac{1}{k!h!}\left[\frac{d^h}{dy^h}\left(y^n \frac{m!}{(m-k)!} x^{m-k}\right)\right]_{(0,0)}.$$

Also, substituting $\frac{d^h y^n}{dy^h} = \frac{n!}{(n-h)!} x^{n-h}$, we get

$$W(k,h) = \frac{1}{k!h!}\left[\frac{n!}{(n-h)!}\frac{m!}{(m-k)!}(y^{n-1} x^{m-k})\right]_{(0,0)} = \delta(k-m,h-n).$$

Theorem 7 If $w(x,y) = \frac{du(x,y)}{dx}\frac{dv(x,y)}{dx}$, then

$$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (r+1)(k-r+1)U(r+1,h-s)V(k-r+1,s).$$

Theorem 8 If $w(x,y) = \frac{du(x,y)}{dy}\frac{dv(x,y)}{dy}$, then

$$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (h-s+1)(s+1)U(r,h-s+1)V(k-r,s+1).$$

Theorem 9 If $w(x,y) = \frac{du(x,y)}{dx}\frac{dv(x,y)}{dy}$, then

$$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (h-s+1)(s+1)U(r,h-s+1)V(k-r,s+1).$$

Theorem 10 If $w(x,y) = u(x,y)v(x,y)\omega(x,y)$, then

$$W(k,h) = \sum_{r=0}^{k} \sum_{t=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{h-s} U(r,h-s-p) V(t,s)\Omega(k-r-t,p).$$

For the proof, see ([31]).

Theorem 11 If $w(x,y) = u(x,y)\frac{dv(x,y)}{dx}\frac{d\omega(x,y)}{dx}$, then

$$W(k,h) = \sum_{r=0}^{k} \sum_{t=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{h-s} (t+1)(k-r-t+1)U(r,h-s-p)V(t+1,s)\Omega(k-r-t+1,p).$$

Proof: The proof is just a simple applications for Theorems 10 and 2, where is Theorem 10, we take $u = u, v = v_x, \omega = \omega_x$, then apply Theorem 2 to find the differential transform of v_x, and ω_x to get the desired result.

Theorem 12 If $w(x,y) = u(x,y)v(x,y)\frac{d^2\omega(x,y)}{dx^2}$, then

$$W(k,h) = \sum_{r=0}^{k} \sum_{t=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{h-s} (k-r-t+2)(k-r-t+1)U(r,h-s-p)V(t,s)\Omega(k-r-t+2,p).$$

Proof: The proof is just a simple applications for Theorems 10 and 4, where is Theorem 10, we take $u = u, v = v_x, \omega = \omega_{xx}$, then apply Theorem 4 to find the differential transform of ω_{xx} to get the required result.

3. NUMERICAL APPLICATIONS AND THEIR OUTCOMES

In this section, we will use DTM linked with Laplace transform and Pad'e technique to approximate the solutions of some nonlinear diffusion equations of known exact solutions. Four important cases of the nonlinear term *D(u)* appeared in Eq. (1). Our technique will be used to explore several real-world physical processes. One of the cases $D(u) = u^n$ that plays a significant role in diffusion process applications. The fast and slow processes of the diffusion is based on *n>0* and *n<0* respectively.

Here, we will solve Eq. (1) with *D(u)* that has the form u^{-1}, u^{-2}, u^{-2} and $\frac{1}{1+u^2}$. The author illustrates the use of DTM to approximate the solution of (1), where the DTM was used for the diffusion term $D(u) = u^n$ for n positive [31]. Here, with the same analysis [31] we will use Laplace transform linked with Pad'e to improve the results [31] when n is positive. Also we extend the use of ADM to solve Eq. (1) for negative values of n. In the last example, we consider the case where $D(u) = \frac{1}{1+u^2}$, while the cases $D(u) = \frac{1}{u^2-1}, D(u) = \frac{1}{1-u^2}$ can be approached in a similar fashion.

Example 1 *Consider the nonlinear diffusion equation*

$$u_t = (uu_x)_x \tag{5}$$

subject to the initial condition

$$u(x,0) = \frac{x^2}{c} \tag{6}$$

where, *x>0* and the constant *c>0* is arbitrary. The exact solution [25, 41] is $u(x,t)=\frac{x^2}{c-6t}$. Following the discussion in section (10) we will solve this equation with *c=1*. Taking the two-dimensions differential transform of Eq. (5) we obtain

$$(h+1)U(k,h+1=\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)U(k-r+1,s)++\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)U(r,h-s)U(k-r+2,s).$$

The differential inverse transform of *U(k,h)* is defined by

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^kt^h=\sum_{k=1}^{\infty}\sum_{h=1}^{\infty}U(k,h)x^kt^h+\sum_{k=1}^{\infty}U(k,0)x^k+\sum_{h=1}^{\infty}U(0,h)x^h+U(0,0). \quad (7)$$

Applying the initial condition Eq. (6) into Eq. (7) gives

$$u(x,0)=\sum_{k=0}^{\infty}U(k,0)x^k=x^2 \quad (8)$$

Therefore, *U(k,0)=0* for *k=0,1,3,...* and *U(2,0)=1*. Plugging back and by recursive method, the following results are obtained *U(2,1)=6, U(2,2)=36, U(2,3)216, U(2,4)=1296*. Substituting the resulting values of *U(k,h)* into Eq. (7), we get the following approximate series solution

$$u_a(x,t)=x^2(1+6t+36t^2+216t^3+1296t^4+7776t^5+\ldots). \quad (9)$$

Which is the same solution obtained [31], and it is the partial sum of the Taylor series of the exact solution around *t=0*. To improve this result, apply Laplace transform to Eq. (9), with respect to the time variable *t*, gives

$$L[u_a(x,t)](s)=x^2\left(\frac{933120}{s^6}+\frac{31104}{s^5}+\frac{1296}{s^4}+\frac{72}{s^3}+\frac{6}{s^2}+\frac{1}{s}\right). \quad (10)$$

Now replace s by $1/\omega$, we arrive at

$$L[u_a(x,t)](\omega)=x^2(\omega+6\omega^2+72\omega^3+1296\omega^4+3110\omega^5+933120\omega^6) \quad (11)$$

The associated $[\frac{4}{2}]$ Pad'e approximation of Eq. (11) with respect to ω is

$$[\tfrac{4}{2}](x,\omega)=\frac{x^2\omega-42\,x^2\omega^2+216\;x^2\omega^3+432\,x^2\omega^4}{1-48\,\omega+432\,\omega^2} \quad (12)$$

Replacing back ω by $1/s$, we obtain the $[\frac{4}{2}]$ Pad'e approximation in terms of s as

$$[\tfrac{4}{2}](x,\tfrac{1}{s})=\frac{\frac{432\,x^2}{s^4}+\frac{216\,x^2}{s^3}-\frac{42\,x^2}{s^2}+\frac{x^2}{s}}{1+\frac{432}{s^2}-\frac{48}{s}}. \quad (13)$$

Taking the inverse Laplace transform of the above equation, we obtain the modified approximate solution $u_m(x,t)$, given by

$$u_m(x,t)=\frac{x^2}{72}(44+27e^{12t}+e^{36t}+72\,t)$$

A comparison between our modified solution $u_m(x,t)$, and the solution obtained [41], is depicted in Table 1. It can be easily seen from the results in Table 1 that our modified solution is better than the solution [41].

Table 1. A comparison between our modifiedDTM and DTM [41]

Point (x,t)	Absolute error $\lvert u(x,t)-u_a(x,t)\rvert$ [31]	Modified error $\lvert u(x,t)-u_m(x,t)\rvert$
(0.1,0.1)	1.16640×10^{-3}	3.55362×10^{-4}
(0.2,0.1)	4.66560×10^{-3}	1.42145×10^{-3}
(0.3,0.1)	1.04976×10^{-2}	3.19826×10^{-3}
(0.4,0.1)	1.86624×10^{-2}	5.68580×10^{-3}
(0.5,0.1)	2.91600×10^{-2}	8.88406×10^{-3}
(0.6,0.1)	4.19904×10^{-2}	1.27930×10^{-2}

Example 2 In this example, we consider Eq. (1), with $D(u)=\frac{1}{u}$. Differentiating the right hand-side with respect to x, we obtain the following nonlinear diffusion equation:

$$u_t=\frac{1}{u}u_{xx}-\frac{1}{u^2}u_xu_x.$$

Multiply both sides of the above equation by u^2, we obtain the following equation

$$u^2u_t=uu_{xx}-u_xu_x. \quad (14)$$

Apply DTM to both sides of Eq. (14), we obtain

$$\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(p+1)U(r,h-s-p)U(t,s)U(k-r-t,p+1)=\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)U(k-r+1,s)+\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)U(r,h-s)U(k-r,s).$$

We solve Eq. (14) subject to the initial condition

$$u(x,0)=\frac{2}{(1+x)^2}, \quad (15)$$

where the exact solution is given by

$$u(x,t)=\frac{2+2t}{(1+x)^2}. \quad (16)$$

The differential transform inverse of $U(k,h)$ is defined by

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^kt^h \quad (17)$$

Substituting $t=0$ into Eq. (17), we obtain $u(x,0)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k$. Note that the variable x appears in the denominator in the initial condition (15), while it appears in the numerator in the expression (17). To make sense of using the initial condition, expand $\frac{2}{(1+x)^2}$ into it's Taylor series abound $x=0$, we get

$$u(x,0)=\frac{2}{(1+x)^2}=\sum_{k=0}^{\infty}2(-1)^k(k+1)x^k. \quad (18)$$

Comparing the above equation with $u(x,0)=\sum_{k=0}^{\infty}U(k,0)x^k$, the following equations are obtained

$$U(k,0)=2(-1)^k(k+1)x^k,\qquad k=0,1,2,\ldots \quad (19)$$

Substituting Eq. (19) into Eq. (15), and by recursive method, the following results are obtained

$$U(0,0) = 2,\ U(1,0) = 4,\ U(0,1) = 2,\ U(1,1) = -4,$$
$$U(2,1) = 6,\ U(2,0) = 6.$$

Substituting the resulting values of *U(k,h)* into Eq. (17), we have the following approximate series solution

$$u_a(x,t) = 2 + 2t - 4x - 4xt + 6x^2 + 6x^2t - 8x^3 - 8x^3t. \quad (20)$$

To check the validity and accuracy of our approximate solutions, we consult Table 2, which shows that the approximate solution converges to the exact solution in the region near to the point (0,0), and the error increases as we get away from (0,0).

Table 2. The difference between the approximateand exact solutions for Example 2

Point (x,t)	$\lvert u_a(x,t) - u(x,t)\rvert$
(0.2,1.5)	3.222×10^{-02}
(0.2,1.0)	2.577×10^{-02}
(0.2,0.75)	2.225×10^{-02}
(0.2,0.5)	1.933×10^{-02}
(0.1,1.5)	2.231×10^{-03}
(0.1,1.0)	1.785×10^{-03}
(0.1,0.75)	1.561×10^{-03}
(0.1,0.5)	1.338×10^{-03}
(0.1,0.1)	9.818×10^{-04}
(0.01,0.01)	9.980×10^{-08}
(0.001,0.001)	9.998×10^{-12}

Example 3 Consider the nonlinear fast diffusion equation

$$u_t = (u^{-2}u_x)_x \quad (21)$$

subject to

$$u(x,0) = \frac{1}{\sqrt{x^2+1}} \quad (22)$$

The exact solution is $u(x,t) = (x^2 + e^{2t})^{-1/2}$. By differentiating the right hand-side of Eq. (21) with respect to x, we get $u_t = -2u^{-3}u_xu_x + u^{-2}u_{xx}$. Upon multiplying both sides by u^{-3}, we obtain

$$u^3u_t = -2u_xu_x + uu_{xx} \quad (23)$$

Now, take the differential transform to both sides of Eq. (23), we obtain

$$\sum_{r=0}^{k}\sum_{s=0}^{h}\left[\left(\sum_{n=0}^{r}\sum_{t=0}^{r-n}\sum_{m=0}^{h-s}\sum_{p=0}^{h-s-m} U(n,h-s-m-p)U(t,m)U(r-n-t,p)\right)\left((s+1)U(k-r,s+1)\right)\right] = -2\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)U(k-r+1,s) + \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)U(r,h-s)U(k-r+2,s).$$

The differential inverse transform of $U(k,h)$ is

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^kt^h. \quad (24)$$

Expanding the function appeared in Eq. (22) into its Taylor series, we get

$$u(x,0) = \frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{3x^2}{8} - \frac{5x^6}{16} + \cdots$$

Upon substituting $t = 0$ into Eq. (24) and comparing it with the above equation, we obtain

$$u(x,0) = \sum_{k=0}^{\infty} = U(k,0)x^k = \frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{3x^2}{8} - \frac{5x^6}{16} + \cdots$$

From the above, we get

$$U(0,0) = 1,\ U(1,0) = 0,\ U(2,0) = \frac{-1}{2},\ U(3,0) = 0,$$
$$U(4,0) = \frac{3}{8},\ U(5,0) = 0,\ U(6,0) = \frac{-5}{16}.$$

And so on, we may compute coefficients as much as we need. Substituting the above values into Eq. (24), and by recursive method, the first few results are obtained,

$$U(0,1) = -1,\ U(1,1) = 0,\ U(3,1) = 0,\ U(2,1) = \frac{9}{2},$$
$$U(0,2) = 1,\ U(1,2) = 0.$$

Upon using all these values of *U(k,h)*, we get the approximate solution, and after regrouping polynomials to their original function, our approximate solution has the form

$$u_a(x,t) = \frac{t^2(1-2x^2)}{2(1+x^2)^{5/2}} - \frac{t}{2(1+x^2)^{3/2}} + \frac{1}{(1+x^2)^{1/2}} - \frac{t^3(1-10x^2+4x^4)}{6(1+x^2)^{7/2}} + \frac{t^4(1-36x^2+60x^4-8x^6)}{24(1+x^2)^{9/2}} \quad (25)$$

Table 3 presents the absolute error, which again show that our approximate solution is good enough for values close to the origin.

Table 3. For varying values of *t*, the error for the approximate solution in Example 3

t_i	$\lvert u_a(5,t_i) - u(5,t_i)\rvert$	$\lvert u_a(3,t_i) - u(3,t_i)\rvert$
0.1	2.55015×10^{-07}	$1{,}99652 \times 10^{-08}$
0.2	7.67731×10^{-08}	6.89257×10^{-07}
0.3	5.39100×10^{-07}	5.62998×10^{-06}
0.4	2.05292×10^{-06}	2.54295×10^{-05}
0.5	5.48313×10^{-06}	8.28379×10^{-05}
0.6	1.13898×10^{-05}	2.18984×10^{-04}
0.7	1.90335×10^{-05}	5.00138×10^{-04}
0.8	2.47680×10^{-05}	1.02425×10^{-03}
0.9	1.97262×10^{-05}	1.92627×10^{-03}
1.0	1.32231×10^{-05}	3.38110×10^{-03}

Example 4 Consider the nonlinear diffusion Eq. (1) with $D(u) = \frac{1}{1+u^2}$, which can be written in the form:

$$u_t = \frac{-2uu_xu_x}{(1+u^2)^2} + \frac{u_{xx}}{(1+u^2)}.$$

Multiplying both sides by $(1+u^2)^2$ and simplify more, we obtain

$$u_t + 2u^2u_t + u^4u_t = -2uu_xu_x + u_{xx} + u^2u_{xx} \quad (26)$$

We solve Eq. (26) subject to the initial condition

$$u(x,0) = \tan(x). \quad (27)$$

Taking the DTM to both sides of Eq. (26), we get

$$(h+1)U(k,h+1) + 2\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(p+1)U(r,h-s-p)U(t,s)U(k-r-t,p+1) + \sum_{r=0}^{k}\sum_{s=0}^{h}[(\sum_{n=0}^{r}\sum_{t=0}^{r-n}\sum_{m=0}^{h-s}\sum_{p=0}^{h-s-m}U(n,h-s-m-p)U(t,m)U(r-n-t,p)) \times (\sum_{n=0}^{k-r}\sum_{m=0}^{s}(m+1)U(n,s-m)U(k-r-n,m+1))] = -2\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(t+1)(k-r-t+1)U(r,h-s-p)U(t+1,s)U(k-r-t+1,p) + (k+2)(k+1)U(k+2,h) + \sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(k-r-t+2)(k-r-t+1)U(r,h-s-p)U(t,s)U(k-r-t+2,p)$$

The differential inverse transform of $U(k,h)$ is

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h. \quad (28)$$

Substitute $t=0$ into Eq. (28), we obtain $u(x,0) = \sum_{k=0}^{\infty}U(k,0)x^k$. Comparing with the above equation with the initial condition $u(x,0) = tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$, we obtain

$$U(0,0)=0,\ U(1,0)=1,\ U(2,0)=0,\ U(3,0)=\frac{1}{3},\ U(4,0)=0,\ U(5,0)=\frac{2}{15}.$$

Substituting the obtained values of *U(k,h)*, we obtain *U(k,h)=0, k=0,1,2,...*, and *h=1,2,....* From Eq. (28), we have

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h = \sum_{k=0}^{\infty}U(k,0)x^k + \sum_{k=0}^{\infty}\sum_{h=1}^{\infty}U(k,h)x^k t^h = u(x,0) + \sum_{k=0}^{\infty}\sum_{h=1}^{\infty}U(k,h)x^k t^h.$$

Now, substituting the results that *U(k,h)=0, k=0,1,2,...*, and *h=1,2,...* in the above equation, we obtain *u(x,t)=tan(x)*, which is the exact solution.

4. CONCLUSIONS

DTM was effectively employed in this research to get approximate and accurate solutions to nonlinear diffusion equations, which has various applications in engineering and physical sciences. For acceptable beginning conditions, the solution given by differential transform technique is an infinite power series, which may be used to describe the precise solutions in a closed form. The Laplace transform was utilized, along with the DTM and Pad'e approximation, and the results were compared to findings from earlier studies, revealing that our technique is quicker and better. The reliability of the differential transform technique, as well as the reduction in the size of the computing domain, increase its use. DTM is definitely a powerful and efficient method for solving a wide range of nonlinear problems analytically. It is worth mentioning that this method produces rapid convergence of responses. As a result, we think that the suggested method may be extended to solve a broad variety of PDEs with variable coefficients encountered in physical and engineering applications, and that DTM provides extremely accurate numerical solutions for nonlinear problems. As a result, without the requirement for linearization, discretization, or perturbation, this technique may be used to a wide range of complicated linear and nonlinear PDEs.

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