



Applications of Numerical Integrations on the Trapezoidal and Simpson's Methods to Analytical and MATLAB Solutions

Ali Jalal Ali^{1*}, Ali Fahem Abbas²

¹ Department of Mathematics, Ministry of Education, Open Education College, Kirkuk University, Kirkuk 36001, Iraq

² Department of Mathematics, Ministry of Education, Babylon 51001, Iraq

Corresponding Author Email: alijalal@uokirkuk.edu.iq

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ABSTRACT

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Applied mathematics has become widely used among the computer engineering community and various sciences. Not only that, but accuracy and speed are now required. Numerical methods have been submitted to demonstrate the efficiency and accuracy of these methods by assigning them to a computer that previously been corrected in computer formats. This key feature makes the solution perfect and easy to used Simpson 1/3 and Simpson 3/8 are applied to the proposed equation. A trapezoidal method of fractional differential equations, solved using different numerical methods to demonstrate the accuracy of properties, was used. We dealt with many numerical algorithms. Below are decisions with written and unwritten differential equations. Generate error analysis and stability analysis for a high-resolution digital system. Research objectives: Turns out numerical solutions are very accurate near the exact solution. This paper aims to provide numerical calculations for different methods. The graph is compared with the computer literary graphs for clarity. Effectiveness of numerical algorithms used with exact solutions and profitable MATLAB solutions.

1. INTRODUCTION

Most numerical methods for understanding nonlinear conditions are still important, because arithmetic is a link in many sciences, where science and innovation are linked to a reasonable and clear communication so we resort to numerical solutions because they can find integrations that may be difficult to find in analytical solutions, although numerical solutions are approximate [1]. Analytical solutions cannot be closed securely due to various difficulties that may guide us in the solution. It is difficult to find a solution that may be complex or long. Therefore, it requires that we resort to numerical computational techniques, and master the test in a number of related numerical tasks [2]. The feature of different answer to the request is based on Simpson's basic principle. Recently, several revised methods for understanding the nonlinear state that used the two Simpson types have appeared. At the moment, it is convenient for the scientist to develop a reasonable model of use, which can already be compared to the correct user application before we solved in Simpson [3]. The description of this delay can be considered due to the fact that intermediate solutions do not work when they are created, or that physical approximations are usually a guarantee when independence in differential and integral variance with integers is evident on the way to a better approximation [4]. More recently, if so, subdivisions have been evaluated according to Newton's laws and false deviations, just like Simpson's. Differential conditions have Acquired an impressive level of unpredictability compared to the hypothesis which has strong consistency with segmented computations. Many applications will be developed in different sciences that are used in applications for all purposes and tasks in solving complex micro-problems [5, 6]. The

integer coefficient of variation as shown in the case type is a relative parameter that can be memorized in the same way as the start time also as shown in its format as there are many incomplete sections that go against the old usage patterns depicting the properties of the land for different materials and procedures Numerical solutions are also used in solving Boolean equations [7]. One of the reasons for the gradual spread of mathematics into logical and specialized fields of life is that it is used in solving many problems. In addition, there are some inconsistencies in reports about the correct separation with mathematical solutions. The subsequent influence of small parts of the course may be gradually appropriate in approximating numerical linear solutions [8]. Where the typical subprime, incomplete as a rule leads to many inconsistencies with numerical strategies that are difficult to implement, and measures are taken for them in solving high-order fractional equations [9]. Analytical solutions for most of the differential equations are complex and longer than the exact value as they used their polynomial expansion [10]. Accurate analytical solutions are often obtained in a few simple steps that lead to special functions. However, there are difficulties in finding exact solutions to most problems of Partial differential equations in applied sciences in agriculture, medicine and technology. Therefore, calculating numerical solutions using these equations becomes more important [11, 12]. It should also be noted that hypothetical studies of numerical methods and error estimates of the quarterly differential equation are very limited because theoretical analysis using numerical solutions of fraction methods is often long and complex. We mean derivatives, which are often used [13]. When using the Riemann method for contracting numerical models, the prerequisites must be converted into exact equations to find the parts not covered by

any known physical studies, and thus the approach, as a general rule, is not well suited to the real applications of solving various numerical integrals [14]. Recently, impressive attempts have been made to validate a digital device, indicating that reading drugs in numerical solutions is close to microcomputer methods for device discovery, although it takes a lot of time to get proper solutions. Attempts to improve it with the help of MATLAB [15]. Who used the following system elements to access the computer architecture with Simpson [16]. Numerical integration methods with the standard PCF query are a suitable set of numerical and logical options for subsequent tests in applied systems using MATLAB [17]. This document is sorted differently by different digital solutions. The state of the different species in the solutions was understood and summarized in five numerical ways, and one of the models was the amplitude of the parts. To emphasize the use of exponential computing and numerical algorithms, a large extended classification that is close to the ideal solution will be provided, which can be developed or improved, because the results of the current study are shown to exceed the numerical methods. The purpose of this study is to obtain an effective method through irreplaceable numerical solutions, that is, its solution is easy and short with fewer errors. We are always looking for it, but a computer using MATLAB gives useful and direct results depending on the solving of the numerical methods used in this study of various numerical and analytic solutions problems that distinguish the trapezoid method from numerical methods [18]. It also adds speed to computer solutions and eliminates truncation errors. The change of reliability in a high-level advanced state is manifested in agreements with many models, for example, numerical methods and various strategies, to demonstrate the skill and accuracy of utilitarian agreements. In addition, many equations cannot find their integrals as we mentioned.

Through our research and our knowledge of many studies in the field of mathematics in the solutions of numerical analysis methods and numerical integration methods that always help us in finding many integrations, including some functions that are difficult to find solutions, we were able to look in a study of several types of numerical methods for different problems and then compare them with The exact solutions and the MATLAB solution, although we cannot cover the whole topic, as we have chosen some diverse examples to illustrate, which we will consider as a measure of the degree of accuracy of the methods that we will choose, in addition to the MATLAB solution, which is a distinctive solution and gives useful results. In addition to that the above numerical solution methods, today's more advanced methods are limited in terms of various numerical approximation methods variable in order [19, 20]. However, we cannot dispense with numerical methods in finding solutions to problems Different types to cover the largest area of numerical solutions, which are Simpson of its two types and Trapezoidal, which is one of the distinctive numerical integration methods that overcame Simpson through solutions. General for these arithmetic operations, which we seek and hope to develop and benefit from.

2. IMPROVE METHODS OF SOLUTIONS

In this study, we will present selected methods of numerical solution which focus on the performance of good

approximation methods to benefit from approximation solutions aimed at trapezoidal method, Simpson's methods, accurate analytical solution and MATLAB. In this regard it suffices to compare performance in terms of numerical accuracy and calculation error. Let us offer solutions to solve various problems.

2.1 Trapezoidal base

Task $n=1$ in the general formula:

$$\int_{x_0}^{x_n} y dx = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

All the top first differences will be zero, and we'll get them:

$$\int_{x_0}^{x_1} y dx = h \left[y_0 + \frac{1}{2} \Delta y_0 \right] = h \left[y_0 + \frac{1}{2} (y_1 + y_0) \right] = \frac{h}{2} (y_0 + y_1)$$

For the next interval $[x_1, x_2]$ we conclude in a similar way:

$$\int_{x_1}^{x_2} y dx = \frac{h}{2} (y_1 + y_2)$$

And so on. For our last interval $[x_{n-1}, x_n]$ we have:

$$\int_{x_{n-1}}^{x_n} y dx = \frac{h}{2} (y_{n-1} + y_n)$$

Combining the express, we develop law:

$$\int_{x_0}^{x_n} y dx = \frac{h}{2} (y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n)$$

which is known as a trapezoid law.

2.2 Simpson 1/3 rule

This rule is obtained if we put $n=2$, as in the formula

$$\int_{x_0}^{x_n} y dx = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

Replace the curve with $n/2$ quadratic or equivalent polynomial arcs.

We have what:

$$\int_{x_0}^{x_2} y dx = 2h \left[y_0 + \frac{1}{2} \Delta y_0 + \frac{1}{6} \Delta^2 y_0 \right] = \frac{h}{3} [y_0 + 4y_1 + y_2]$$

Similarity

$$\int_{x_2}^{x_4} y dx = \frac{h}{3} [y_2 + 4y_3 + y_4]$$

Ultimately

$$\int_{x_{n-2}}^{x_n} y dx = \frac{h}{3} [y_{n-2} + 4y_{n-1} + y_n]$$

Summarize, we get it:

$$\int_{x_0}^{x_n} y dx = \frac{h}{3} (y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n)$$

which is commonly known as Simpson 1/3.

2.3 Simpson 3/8 rule

Setting n=3 in equation:

$$\int_{x_0}^{x_n} y dx = nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

Notice that all variations above the third become equal to zero, and we get them:

$$\begin{aligned} \int_{x_0}^{x_3} y dx &= 3h \left[y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right] \\ &= 3h \left[y_0 + \frac{3}{2} (y_1 - y_0) + \frac{3}{4} (y_2 - 2y_1 + y_0) + \frac{1}{8} (y_3 - 3y_2 + 3y_1 - y_0) \right] \\ &= 3h/8 (y_0 + 3y_1 + 3y_2 + y_3) \end{aligned}$$

In the same way:

$$\int_{x_0}^{x_3} y dx = \frac{3h}{8} (y_3 + 3y_4 + 3y_5 + y_6)$$

And so on. Summing up all this, we get it:

$$\begin{aligned} \int_{x_0}^{x_n} y dx &= \frac{3h}{8} [(y_0 + 3y_1 + 3y_2 + y_3) \\ &+ (y_3 + 3y_4 + 3y_5 + y_6) + \dots \\ &+ (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)] \\ &= 3h/8 (y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 \\ &+ 3y_5 + 2y_6 + \dots + 2y_{n-3} + 3y_{n-2} \\ &+ 3y_{n-1} + y_n) \end{aligned}$$

This rule is called Simpson 3/8.

2.4 Exacting solutions

This section demonstrates the ability to solve overlapping generations of different algorithms using an accurate solution. Rather than converge in developing a multifaceted and detailed model for many practical reasons for real systems and provide a comprehensive analysis, we intend to describe the procedure to implement the solution of these algorithms and present its computational behavior. Not all functions have an analytical solution, with the help of this motivation we resort to approximate solutions.

2.5 MATLAB method

The MATLAB function attempts to approximate the integration of numerical function pleasure from the use of high-precision global adaptive quadrature and default error tolerance.

Example 1. Evaluate $\int_0^1 \frac{1}{1+x} dx$ by trapezoidal, Simpson 1/3 Rule and Simpson 3/8 Rule, also exciting and MATLAB solutions with $h=0.25$, and 0.125 , respectively its shown in the results of Table 1 and graph 1.

When $h=0.25$, $y = f(x) = \frac{1}{1+x}$, $x_0=0, y_0=1, x_1=0.25, y_1=0.8, x_2=0.50, y_2=0.6667, x_3=0.7, y_3=0.5714, x_4=1, y_4=0.5$.

By using trapezoidal rule

$$\begin{aligned} \int_0^1 \frac{1}{1+x} &= \frac{h}{2} [y_0 + y_4 + 2(y_1 + y_2 + y_3)] \\ &= \frac{0.25}{2} [1 + 0.5 \\ &+ 2(0.8 + 0.6667 + 0.5714)] = 0.6970 \end{aligned}$$

Using Simpson rule as 1/3

$$\begin{aligned} \int_0^1 \frac{1}{1+x} &= \frac{h}{3} [y_0 + y_4 + 2(y_2) + 4(y_1 + y_3)] \\ &= \frac{0.25}{3} [1 + 0.5 + 2(0.6667) \\ &+ 4(0.8 + 0.5714)] = 0.6932 \end{aligned}$$

Using Simpson rule as 3/8

$$\begin{aligned} \int_0^1 \frac{1}{1+x} &= \frac{3h}{8} [y_0 + y_4 + 2(y_3) + 3(y_1 + y_2)] \\ &= \frac{3(0.25)}{8} [1 + 0.5 + 2(0.5714) \\ &+ 3(0.8 + 0.6667)] = 0.6602 \end{aligned}$$

Now $h=0.125$, $y = f(x) = \frac{1}{1+x}$, $x_0=0, y_0=1, x_1=0.125, y_1=0.8889, x_2=0.250, y_2=0.8, x_3=0.375, y_3=0.7273, x_4=0.5, y_4=0.6667, y_5=0.6154, x_5=0.625, x_6=0.750, y_6=0.5714, x_7=0.875, y_7=0.5333, x_8=1, y_8=0.5$.

Trapezoidal results with Example 1

$$\begin{aligned} \int_0^1 \frac{1}{1+x} &= \frac{h}{2} [y_0 + y_8 \\ &+ 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)] \\ &= \frac{0.125}{2} [1 + 0.5 + 2(0.8889 + 0.8 \\ &+ 0.7273 + 0.6667 + 0.6154 + 0.5714)] \\ &= 0.6941 \end{aligned}$$

Simpson Rule 1/3 results

$$\begin{aligned} \int_0^1 \frac{1}{1+x} &= \frac{h}{3} [y_0 + y_8 \\ &+ 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)] \\ &= \frac{0.125}{3} [1 + 0.5 + 2(0.8 + 0.6667 + 0.5714) \\ &+ 4(0.8889 + 0.7273 + 0.6154 + 0.5333)] \\ &= 0.6932 \end{aligned}$$

Simpson Rule 3/8 solutions

$$\int_0^1 \frac{1}{1+x} dx = \frac{3h}{8} [y_0 + y_8 + 2(y_3 + y_6) + 3(y_1 + y_2 + y_4 + y_5 + y_7)]$$

$$= \frac{3(0.125)}{8} [1 + 0.5 + 2(0.7273 + 0.5714)] + 3(0.8889 + 0.8 + 0.6667 + 0.6154 + 0.5333)] = 0.6848$$

By Exacting solution for first example

$$\int_0^1 \frac{1}{1+x} dx$$

Let $u=1+x, du=dx$

$$\int_0^1 \frac{1}{u} du = \ln(u)_0^1 = \ln(1+x)_0^1 = \ln(1+1) - \ln(1+0)$$

$$= \ln 2 - \ln 1 = 0.6931 - 0 = 0.6931$$

By using script will receiving the results:

```
clc
Clear
Close all
Fun = @(x) 1./ (1 + x);
q = integral (fun, 0,1);
q = 0.6931
```

Table 1. Numerical results with Simpson 1/3, Simpson3/8 and exacting

Trapezoidal	Simpson 1/3	Simpson 3/8	Exacta solution	Matlab global
0.6941	0.6932	0.6848	0.6931	0.6931

Discussion of three numerical outcomes with demanding arrangement and MATLAB on partial errors we understood Simpson 1/3 its best estimate to correct and MATLAB so he is the, winner with fragmentary capacity in Figure 1.

```
x = 0: 0.001: 1
y = 1./ (1 + x);
Plot(x,y, 'r', 'linewidth', 2);
Grid on
xlabel ('x - axis')
ylabel ('y - axis')
```

Example 2. Find the approximation value of $\int_0^\pi \sin x dx$ using three numerical methods trendy matlab solutions with $h = \frac{\pi}{6}$ and Simpson 1/3, Simpson 3/8 rules it will be represented in the Table 2 and Figure 2.

Solution of Trapezoidal rule

$$y = f(x) = \int_0^\pi \sin x dx$$

$$x_0 = 0, y_0 = 0, x_1 = \frac{\pi}{6}, y_1 = 0.5, x_2 = \frac{\pi}{3}, y_2 = 0.8660,$$

$$x_3 = \frac{\pi}{2}, y_3 = 1, x_4 = \frac{2\pi}{3}, y_4 = 0.8660, x_5 = \frac{5\pi}{6}, y_5 = 0.5,$$

$$x_6 = \pi, y_6 = 0.$$

By Trapezoidal Rule getting the results from example second

$$\int_0^\pi \sin x dx = \frac{h}{2} [y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12} [0 + 0 + 2(0.5 + 0.8660 + 1 + 0.8660 + 0.5)]$$

$$= 1.9540$$

Simpson 1/3 rule results

$$\int_0^\pi \sin x dx = \frac{h}{3} [y_0 + y_6 + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{\pi}{18} [0 + 0 + 2(0.8660 + 0.8660) + 4(0.5 + 1 + 0.5)] = 2.0008$$

Simpson 3/8 rule solution

$$\int_0^\pi \sin x dx = \frac{3h}{8} [y_0 + y_6 + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{\pi}{16} [0 + 0 + 2(1) + 3(0.5 + 0.8660 + 0.8660 + 0.5)] = 2.0019$$

Exacting solution of example two

$$\int_0^\pi \sin x dx = [-\cos x]_0^\pi = [-\cos\pi] - [-\cos 0] = 2$$

Using MATLAB script system as shown in Figure 1.

```
clc
Clear
Close all
Fun = @(x) sin(x);
q = integral(fun, 0, 22/7);
q = 2.0000
```

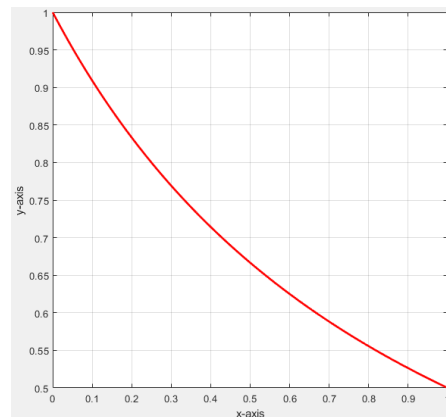


Figure 1. Trail of numerical solutions with the exact and MATLAB on distances of $h=0.25$

Table 2. The results of the second example are numerically solved with accurate and MATLAB solution

Trapezoidal	Simpson 1/3	Simpson 3/8	Exact solution	MATLAB (global adaptive quadrature)
1.9540	2.0008	2.0019	2	2

Exchange the estimates of numerical arrangements with a trapezoid on the sin function, include many disturbing errors but both are approaching strict and confused conformity, we note that Simpson's best digital technique from trapezoid to correct them in Figure 2.

```
x = 0:0.001:22/7;
y = sin(x);
Plot(x,y,'r','linewidth',2);
Grid on
```

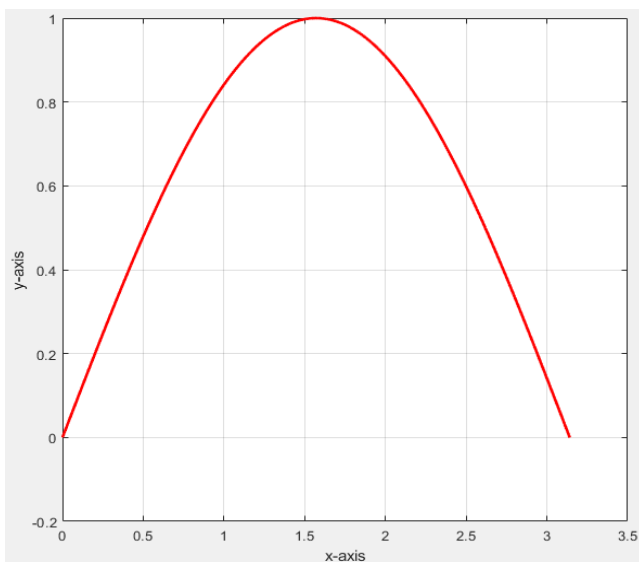


Figure 2. Paths of label ('y-axis') and ('x-axis') responses the function thru MATLAB

Example 3. Find the approximation value of $\int_1^5 \log x dx$ using Trapezoidal, Simpson 1/3, Simpson 3/8 rules in addition to the exact solution and MATLAB solution, which will be explained in the result of Table 3 and will draw in Figure 3.

With $h=0.5$, $y = f(x) = \int_1^5 \log x dx$, $x_0 = 1$, $y_0 = 0$, $x_1 = 1.5$, $y_1 = 0.4055$, $x_2 = 2$, $y_2 = 0.6931$, $x_3 = 2.5$, $y_3 = 0.9163$, $x_4 = 3$, $y_4 = 1.0986$, $x_5 = 3.5$, $y_5 = 1.2528$, $x_6 = 4$, $y_6 = 1.3863$, $x_7 = 4.5$, $y_7 = 1.5041$, $x_8 = 5$, $y_8 = 1.6094$.

Trapezoidal

$$\int_1^5 \log x dx = \frac{h}{2} [y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.5}{2} [0 + 1.6094 + 2(0.4055 + 0.6931 + 0.9163 + 1.0986 + 1.2528 + 1.3863 + 1.5041)] = 4.0307$$

Simpson 1/3

$$\int_1^5 \log x dx = \frac{h}{3} [y_0 + y_8 + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)]$$

$$= \frac{0.5}{3} [0 + 1.6094 + 2(0.6931 + 1.0986 + 1.3863) + 4(0.4055 + 0.9163 + 1.2528 + 1.5041)]$$

Simpson 3/8

$$\int_1^5 \log x dx = \frac{3h}{8} [y_0 + y_8 + 2(y_3 + y_6) + 3(y_1 + y_2 + y_4 + y_5 + y_7)]$$

$$= \frac{3(0.5)}{8} [0 + 1.6094 + 2(0.9163 + 1.3863) + 3(0.4055 + 0.6931 + 1.0986 + 1.2528 + 1.5041)] = 3.9519$$

Exacting solution

$$\int_1^5 \log x dx = [x \log x - x]_1^5 = (5 \log 5 - 5) - (1 \log 1 - 1) = 4.0471$$

MATLAB scrip results

```
clc
Clear
Close all
Fun = @(x) log(x)
q = integral(fun,1,5)
q = 4.0472
```

Table 3. The results from logarithmic function with numerical solutions, exact and MATLAB

Trapezoidal	Simpson 1/3	Simpson 3/8	Exact solution	MATLAB (global adaptive quadrature)
4.0307	4.0467	3.9519	4.0471	4.0472

Examine the logarithmic arrangement that we note from numerical outcomes, just Simpson 3/8 being adjusted away from numerical arrangements but other estimation techniques are great have about 0.160 decimal mistake, MATLAB in every case superior to anything and numerical errors has only 0.1 as apparently in Table 3.

Now, using MATLAB to explain the values through Figure 3.

```
x = 1:0.001:5;
y = log(x);
Plot(x,y,'r','linewidth',2);
Grid on
xlabel('x - axis')
ylabel('y - axis')
```

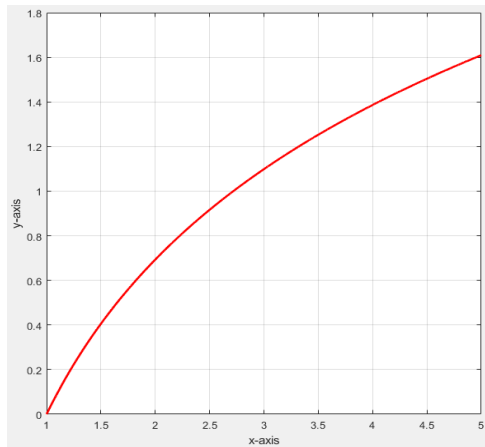


Figure 3. Answer of the function with x-axes and y-axes by MATLAB

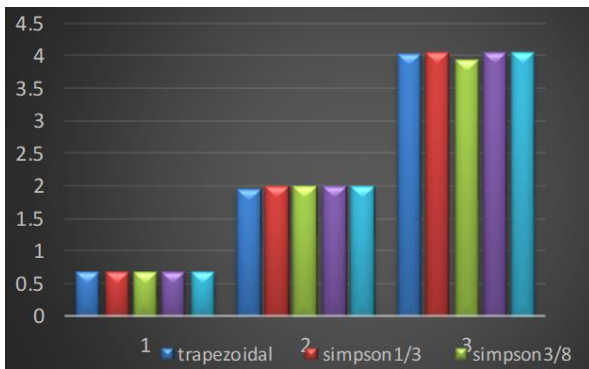


Figure 4. Comparing between Trapezoidal, Simpson 1/3, Simpson3/8, Exacting and MATLAB

Discussions of numerical methods it's come near mtalab solutions in exact order, we see that each of the formulas is settled through the different methods as shown in the models. The different outlines start from dark blue trapezoid to the light blue MATLAB, arrangement in three stages with five solutions. In the first step when $h=0.25$ the main model on all methods have a small error ratio less than one, in the second step of second formula with Simpson at $h=0.52$ We estimate the best numerical arrangements near specified order, in stage 3 of the model third $h=0.5$ trapezoid and Simpson 1/3 move towards a small number of errors but less than accurate in solutions as Simpson 3/8, column shows the green color is less accurate number, MATLAB recorded the best digital union where it contains an error very simple with the ability of the algorithmic equation a single decimal error number.

3. CONCLUSIONS

From the schematic diagram of the high-precision numerical solution of the numerical methods used in the solution, it is shown in the first stage that all methods give useful results at the initial values. The trapezoidal method represents close-range results with an exponential function of the exact solution and MATLAB. Simpson's rule 1/3 is used to solve all numerical formulas, as it gives less accurate results than the 1/8 trapezoid and Simpson's 8/3. Computer results remain better with MATLAB solutions that are close to the exact solution most of the time. We analyzed the discussed error to illustrate the higher order accuracy of the methods and

are illustrated in Figure 4. Stability analysis is presented in numerical examples for comparison with the results corresponding to the various functions used, such as exponential solutions, analytic formulas, and partial formulas. Arithmetic errors are generally accepted in technology. Through these results, which were represented in different solutions for several types of arithmetic problems, and that MATLAB records the best from numerical results for the function that distinguish it from other methods of speed and accuracy in solutions to save time and effort, so we hope that we have given an idea of the aforementioned methods, as we did not find a studies that combines these Solutions Although we cannot cover the whole topic, so we seek to develop the MATLAB solution and the numerical solution to give more accurate for many solutions.

REFERENCES

- [1] Weerakoon, S., Fernando, T. (2000). A variant of Newton's method with accelerated third-order convergence. *Applied Mathematics Letters*, 13(8): 87-99. [https://doi.org/10.1016/S0893-9659\(04\)90119-X](https://doi.org/10.1016/S0893-9659(04)90119-X)
- [2] Chen, J.Y., Kincaid, D.R., Lin, B.R. (2018). A variant of Newton's method based on Simpson's three-eighths rule for nonlinear equations. *Applied Mathematics Letters*, 79: 1-5. <https://doi.org/10.1016/j.aml.2017.11.014>
- [3] Toropov, V.V. (2001). Modelling and approximation strategies in optimization—global and mid-range approximations, response surface methods, genetic programming, low/high fidelity models. In *Emerging methods for multidisciplinary optimization*, Vienna, pp. 205-256. https://doi.org/10.1007/978-3-7091-2756-8_5
- [4] Kincaid, D., Kincaid, D.R., Cheney, E.W. (2009). *Numerical analysis: mathematics of scientific computing*. American Mathematical Soc, 2.
- [5] Roohi, M., Aghababa, M.P., Haghghi, A.R. (2015). Switching adaptive controllers to control fractional-order complex systems with unknown structure and input nonlinearities. *Complexity*, 21(2): 211-223. <https://doi.org/10.1002/cplx.21598>
- [6] Liu, W., Chen, K. (2015). Chaotic behavior in a new fractional-order love triangle system with competition. *Journal of Applied Analysis and Computation*, 5(1): 103-113.
- [7] Ahmad, B., Ntouyas, S.K., Alsaedi, A. (2016). On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. *Chaos, Solutions & Fractals*, 83: 234-241. <https://doi.org/10.1016/j.chaos.2015.12.014>
- [8] Taarav, D., Almeida, R., Torres, D.F. (2016). Caputo derivatives of fractional variable order numerical approximations. *Communications in Nonlinear Science and Numerical Simulation*, 35: 69-87. <https://doi.org/10.1016/j.cnsns.2015.10.027>
- [9] Yan, Y., Pal, K., Ford, N.J. (2014). Higher order numerical methods for solving fractional differential equations. *BIT Numerical Mathematics*, 54(2): 555-584. <https://doi.org/10.1007/s10543-013-0443-3>
- [10] Sudret, B. (2008). Global sensitivity analysis using polynomial chaos expansions. *Reliability Engineering & System Safety*, 93(7): 964-979. <https://doi.org/10.1016/j.ress.2007.04.002>
- [11] Liao, S. (2003). *Beyond perturbation: Introduction to the*

- homotopy analysis method. Chapman and Hall/CRC. <https://doi.org/10.1201/9780203491164>
- [12] Gao, G.H., Sun, Z.Z., Zhang, H.W. (2014). A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications. *Journal of Computational Physics*, 11: 017. <https://doi.org/10.1016/j.jcp>
- [13] Cao, J., Xu, C. (2013). A high order schema for the numerical solution of the fractional ordinary differential equations. *Journal of Computational Physics*, 238: 154-168. <https://doi.org/10.1016/j.jcp.2012.12.013>
- [14] Freeden, W., Reuter, R. (1982). Remainder terms in numerical integration formulas of the sphere. In *Multivariate Approximation Theory*, Birkhäuser Basel, pp. 151-170. https://doi.org/10.1007/978-3-0348-7189-1_13
- [15] Hansen, P.C. (1994). Regularization tools: A Matlab package for analysis and solution of discrete ill-posed problems. *Numerical Algorithms*, 6(1): 1-35. <https://doi.org/10.1007/BF02149761>
- [16] Tošner, Z., Andersen, R., Stevansson, B., Edén, M., Nielsen, N. (2014). Computer-intensive simulation of solid-state NMR experiments using SIMPSON. *Journal of Magnetic Resonance*, 246: 79-93. <https://doi.org/10.1016/j.jmr.2014.07.002>
- [17] De Cat, B., Bogaerts, B., Bruynooghe, M., Janssens, G., Denecker, M. (2018). Predicate logic as a modeling language: The IDP system. In *Declarative Logic Programming: Theory, Systems, and Applications*, pp. 279-323. <https://doi.org/10.1145/3191315.3191321>
- [18] Riccardi, F., Kishta, E., Richard, B. (2016). A step-by-step global crack-tracking approach in E-FEM simulations of quasi-brittle material. *Engineering Fracture Mechanics*, 170: 44-58. <https://doi.org/10.1016/j.engfracmech.2016.11.032>
- [19] Iman, G. (2021). A numerical method for solving the mobile/immobile diffusion equation with non-local conditions. *Mathematical Modelling of Engineering Problems*, 8(4): 557-565. <https://doi.org/10.18280/mmep.080408>
- [20] Hasan, Q., Abbas, F., Ammar, M., Karrar, A., Ali, B. (2022). Numerical investigated to improve heat transfer in a pipe using nanofluid. *Mathematical Modelling of Engineering Problems*, 9(4): 1073-1078. <https://doi.org/10.18280/mmep.090425>