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Development and Definition of Biquadratic Transformations Using Spheres and Two-Way Hyperboloids

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ABSTRACT

The insufficient use of quadratic transformations in applied geometry is explained by the fact that the methods of quadratic transformations are not developed much, although dozens of works by leading experts in applied geometry are devoted to the study of this problem, the development of graphic models, and their application in applied geometry. The research is devoted to the development of the theory of definition of biquadratic transformations of the plane. The essence of the proposed method for modeling biquadratic transformations of the plane, generated by a binary mapping of two surfaces of the second order, facilitation and solution of complex problems of applied geometries. And also, by means of graphical models of biquadratic transformations of the plane to facilitate the construction of curves of the second and fourfold orders. Considering a combination of non-linear surfaces of the second order, obtain subgroups of biquadratic transformations of the plane. The developed algorithm will make it possible to determine mathematical models of canonical biquadratic transformations of the plane, which is necessary for their practical application.

1. INTRODUCTION

In applied geometry, the method of quadratic transformations is effective in solving positional problems related to curves or surfaces. Quadratic transformations are very important because they have great practical significance in the engineering. Using it allows you to halve the order of the projected curve or surface, which greatly simplifies the solution of various positional problems [1, 2]. Multivalued dot geometric transformation can be one – two-digit, one – digit, etc. From them one – two-point conversion is rather well studied.

The work by Öztürk and Özdemir [3] provides a more complete overview of the main issues of the classical theory of nonlinear birational transformations of the plane of three-dimensional space. Dantas and Pan [4] considered Cremona quadratic transformations – the basics of the classical theory of nonlinear birational (cremonic) transformations of the plane of three-dimensional space. In the above work [5], the conditions of transformation of the second-order surface and the plane were studied, and the properties of transformations were studied: for example, one and two-digit quadratic transformation having a bundle of direct carriers of the corresponding points. Öztürk and Özdemir [3] in their work gives a more complete survey of the basic questions of the classical theory of nonlinear birational transformations of the plane and three-dimensional space. This work is of great value because it contains the resolution of singularities of curves and surfaces. The author reviews previously published materials on Cremona transformations' construction and use. Farouki et al. [6] proposed a method for obtaining special Cremona transformations of three-dimensional space. The mapping described in this study is constructive in nature, and it generates not only a model of projective spatial geometry on a plane, but allows us to practically attribute their images to specific spatial objects in the sense of descriptive geometry. This essential feature of the mapping, which has an independent theoretical significance, makes it valuable from the point of view of applications to non-elementary problems of descriptive geometry in Euclidean space. Research by Farouki et al. [6] is devoted to the study of two – two-valued correspondence T_{2-2} . This correspondence consists in constructing points A_2' and A_2'' from point A_1, which are projections on the frontal plane of projection Π_2 of intersection points perpendicular to the horizontal plane of projection Π_1.

In a rectangular system of two planes Π_1 and Π_2, the projection of a second-order surface was taken – a second-order curve can be specified by its traces. The curve of the second order b_1^2 in the horizontal plane of the projections Π_1 is given by the traces of the projection plane Γ, perpendicular to the plane of the projections Π_1. It is also given by an intersecting curve of the second order along a curve and a point belonging to the section of the curve of the second order, conjugate to the direction perpendicular to the plane of the projection Π_1 [6, 7].

The plane α is drawn through point A_1. The trace of the plane α on the horizontal projection plane intersects the contours of the curves a_1^2 at points B_1 and C_1, and the curve b_1^2 at points D_1 and L_1. Further, the points D_2 and L_2 belonging to the projection axis x are constructed. The rays drawn through points B_1 and C_1 intersect the curve a_2^2 on the frontal projection plane at points B_2 and C_2. Through these points D_2, L_2 and B_2, C_2, the contours of the second-order curve b_2^2 on the frontal projection plane were drawn. A straight line drawn through point A_1 on the projection axis x intersects curve b_2^2 at points A_2' and A_2''. This construction made it possible to solve the problem of a given point belonging to a surface of the second order. If point A_1 describes a straight line, then points A_2' and A_2'' describe a curve of the second order and vice versa, then the correspondence T_{2-2} is quadratic. If point A_1 describes a curve of the second order, then points A_2' and A_2'' describe two curves of the second order, then the correspondence T_{2-2} is biquadratic.

From the analysis and research of scientific works on geometry, it follows that quadratic transformations have been studied sufficiently and have found practical applications in science and technology, but little attention has been paid to the study of biquadratic transformations of the plane [8, 9]. The authors believe that this gap prevents the full use of biquadratic transformations of the plane in construction. Therefore, this article is devoted to the development of the theory of determining biquadratic transformations of the plane. As a result of the work, a scientific review of the history and ways of development of descriptive geometry in the Republic of Kazakhstan was conducted. In addition, the authors developed theoretical foundations for modeling biquadratic transformations of the plane, which allowed us to establish new patterns for obtaining four-four-digit correspondences between two non-displaced planes. The article consists of Introduction, Theoretical Overview, Materials and Methods, Results and Discussion and Conclusions. The degree of study of the topic, the importance of this research is presented in the Introduction. Theoretical Overview contains the main sources and theories on the selected topic. Results and Discussion comments and illustrates the obtained data. Conclusions contain a concise summary based on the results of the study.

2. THEORETICAL OVERVIEW

Alberich-Carramiñana et al. [10] considered a theory for constructing geometric models of quadratic Cremona transformations of a plane and three-dimensional space, as applied to solving problems of applied geometry. The work of Martín-Pastor [2] is devoted to proving a theorem that provides two-two-digit quadratic correspondences between points of combined fields. It describes geometric transformation tools that allow you to move from complex geometric shapes to simple geometric shapes that are described using a smaller number of parameters.

A graphical algorithm of two-two-digit quadratic correspondence Martín-Pastor [2] in which, depending on the type and relative position of the fundamental curves k^2, m^2, as well as the position of the poles $S=S'$, you can get a number of private algorithms for solving various applied problems. The polaritet is set on an irregular line, the double points of which are the points of intersection of the fundamental curve with the irregular line. The types of fundamental curves depend on the type of involutions. If the involution is infinite and has no double points, then the fundamental curve is an ellipse. If the involution is infinite and has two double points, then the fundamental curve is a hyperbola, in the special case, a pair of non-matching lines. If the involution is infinite and has one double point, then the fundamental curve is a parabola [2].

In the work of Baidabekov et al. [11] contains studies of the theory of central Cremona transformations of a plane and three-dimensional space, and also deals with the study of the relationship between the methods of descriptive and algebraic geometry in the design of surfaces. The presence of the relationship between the methods of two images and birational transformations in modeling and research of surfaces should contribute to the rational use of their advantages [12, 13]. It should be noted that the object of the research is the presentation of the theory of Cremona transformations of the plane, where much attention is paid to constructive issues of central transformations in solving problems of an applied nature. He also developed the theory of central involutional Cremona transformations of space.

Received by Zhang et al. [14] a constructive method for studying transformations of three-dimensional space, stratified in a bundle of planes into central quadratic involutions, was later used to construct the shape of surfaces such as pipelines of variable cross-section. Recommendations are developed for the transformation apparatus for quadratic involutions, as well as for the study of graphical and analytical methods of the properties of some surfaces of dependent sections of the fourth order, bearing the frame of conics.

In the work of Corcoran and Jones [15] by means of a double space-to-plane mapping in the classical method of two images, a cubic involutional transformation J_{3-3} was obtained, which splits in a bundle of planes into central quadratic involutions J_2, and possible special cases of specifying a cubic involution J_{3-3} in relation to the design of technical forms of surfaces were studied and a method of approximation was developed surfaces based on the properties of the involution J_{3-3}.

In the quadratic transformation Martín-Pastor [2] (1), the (1-2)-valued transformations given by the equations were considered:

$$\begin{cases} x = x' \\ y = \pm\sqrt{(y')^2 - (x')^2} \end{cases} \tag{1}$$

where, x and y are the Cartesian coordinates of the point – the image; x' and y' are the Cartesian coordinates of the preimage point.

This correspondence is mutually one – two-valued and is called a quadratic transformation.

With the help of a quadratic transformation, methods for constructing an ellipse, a hyperbola, as well as methods for solving positional problems were obtained, which made it possible to open a new direction in the study and application of geometric transformations [16]. Quadratic transformations do not have such broad capabilities, in the sense of deformability, which are inherent in topological transformations, therefore, using quadratic transformations, only some regular curves, in particular, conical sections, can be transformed into a straight or homothetic curve.

In the work of Liu et al. [17] analytical methods of geometric modeling and solving inverse problems of birational and rational transformations of the n-th order of the plane generated by immersion in the model of three-dimensional spatial transformation of algebraic surfaces of monoidal and General types are investigated and developed. A method is proposed for constructing instantaneous Cremona transformations of the n-th order of three-dimensional space using specified guide curves, the number of which can be equal to $n(n+3)/2+1$, where n is the order of the Cremona transformation under consideration.

Thus, in applied geometry, the theory of quadratic transformations is developed quite fully, so when creating theoretical positions, we take as the basis of the research being conducted. From the analysis and research of scientific works on geometry, it follows that quadratic transformations have been studied sufficiently and have found practical applications in science and technology, but little attention has been paid to the study of biquadratic transformations of the plane [17].

3. MATERIALS AND METHODS

The essence of the proposed method for modeling biquadratic transformations of the plane generated by binary mapping of two second-order surfaces is as follows.

In Euclidean three-dimensional space E_3, two intersecting algebraic surfaces of the second order Φ_1^0 and Φ_2^0 are given (2, 3), whose equations have the form:

$$\Phi_1^0\,(X_1, X_2, X_3) \tag{2}$$

$$\Phi_2^0\,(X_1, X_2, X_3) \tag{3}$$

where, X_1, X_2, X_3 are Cartesian coordinates; Φ_1^0 and Φ_2^0 are second-order continuous polynomials.

Two general position projection planes Π_1 and $\Pi_1^/$ are also specified. We also draw a projecting beam S_1, which intersects the specified surfaces Φ_1^0 and Φ_2^0, respectively, at points A_1^0 and A_2^0, A_3^0 and A_4^0. The surface of the second order Φ_1^0 is rotated around the axis of OX_2 by 90^0 so that the positive direction of the axis of OX_1 coincides with the negative direction of the axis of OX_3 following Figure 1.

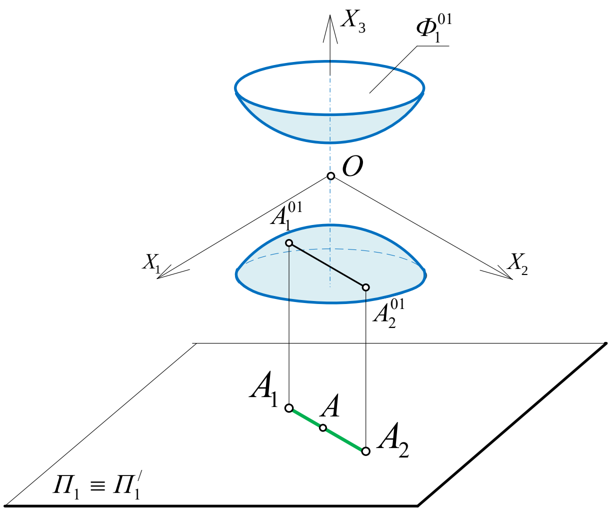


Figure 1. Projection of points A_1^{01} and A_2^{01} on the plane $\Pi_1^/ = \Pi_1$

In other words, the second-order surface Φ_1^0 undergoes a transformation q_1 (rotation around the ordinate axis at an angle of 90^0), whose matrix is given by the equation:

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{4}$$

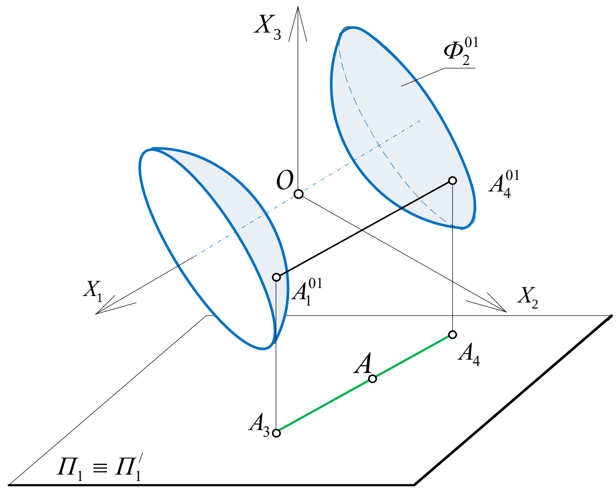


Figure 2. Projection of points A_3^{01} and A_4^{01} on the plane $\Pi_1^/ = \Pi_1$

Thus, we get a new position Φ_1^{01} of the second-order surface Φ_1^0 and points A_1^{01} and A_2^{01}, which correspond to points A_1^0 and A_2^0. The points A_1^{01} and A_2^{01} are projected by the projecting beam S_2 on the plane Π_2, we get the points A_1 and A_2 following Figure 1. Let's consider an example of modeling a biquadratic transformation of the plane when the first surface Φ_1^0 is a double-sided hyperboloid with a real axis OX_3, and the second surface Φ_2^0 is a double-sided hyperboloid with a real axis OX_1 following Figure 2.

According to formula (4), the surface Φ_1^0 and the surface Φ_2^0 are transformed (5):

$$\begin{pmatrix} X_1' \\ X_2' \\ X_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{5}$$

then it is orthogonally displayed on the projection plane $\Pi_1^/$ in accordance with Figure 1.

4. RESULTS AND DISCUSSION

As a result of the above method, each point of the Π_1 plane is converted to four points of the $\Pi_1^/$ plane. Given a two-parametric set of points of the combined plane $\Pi_1^/=\Pi_1$, we obtain a biquadratic transformation of the plane. Similarly, it can be shown that in the opposite direction, each point of the $\Pi_1^/$ plane is transformed into four points of the Π_1 plane.

As a result of the sequential execution of the above design apparatus, each point of the Π_1 plane is converted into four points of the $\Pi_1^/$ plane. Thus, in the two parametric sets of points of the plane $\Pi_1^/=\Pi_1$, we get a biquadratic transformation of the plane L. Similarly, we can show that in the opposite direction, each point $A^/$ of the plane $\Pi_1^/$ is transformed into four points of the plane Π_1, this will be the inverse transformation of $L^/$. Given a two-parameter set of points of the combined plane $\Pi_1^/=\Pi_1$, we obtain a biquadratic transformation of the plane indicated by the letter L. Similarly, we can show that in the opposite direction, each point $A^/$ of the plane $\Pi_1^/$ is transformed into four points of the plane Π_1. This transformation is indicated by the letter $L^/$ in accordance with Figure 3.

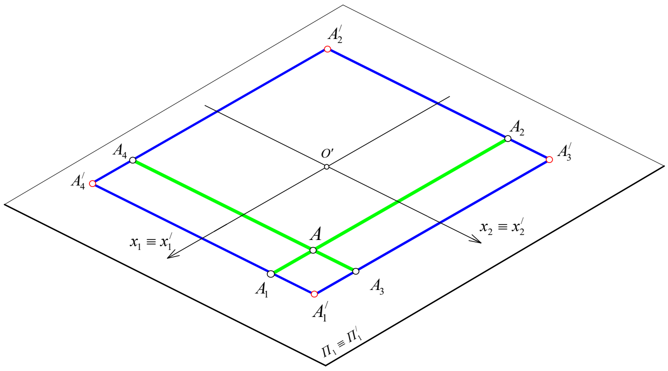


Figure 3. Getting points $A_1^/, A_2^/, A_3^/, A_4^/$

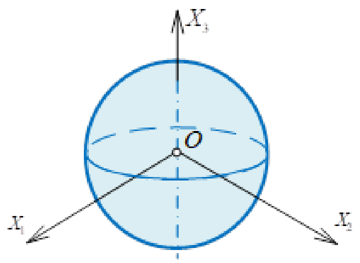
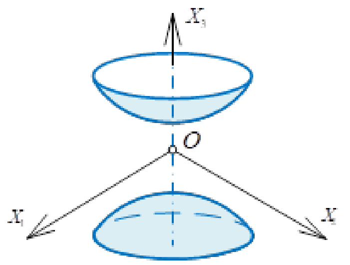
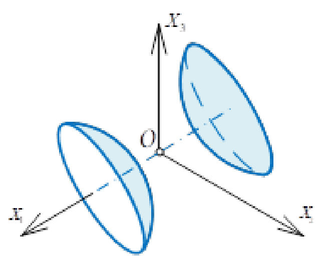
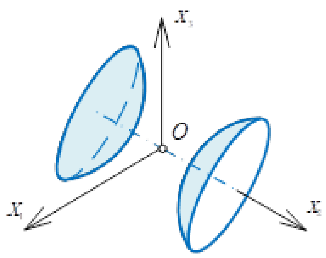
№	Surface	The equation
1		$X_1^2+X_2^2+X_3^2=R^2$
2		$X_3^2=X_1^2+X_2^2+R^2$
3		$X_1^2=X_2^2+X_3^2+R^2$
4		$X_2^2=X_1^2+X_3^2+R^2$

Figure 4. List of second-order non-linear surfaces

Using the above proposed spatial design scheme, various types of canonical biquadratic transformations of L, $L^/$ plane are obtained, which are considered below. To obtain biquadratic transformations of the plane, two second-order surfaces are mapped to the plane. In this case, we consider cases where combinations of non-linear surfaces of the second order as a result of the implementation of these cases have received subgroups of biquadratic transformations of the plane. To model a subgroup of biquadratic transformations of the plane, in the above spatial design scheme, we consider the case when the combination of binary mapped surfaces of the second order are non-linear surfaces of the 2nd order, such as a sphere and a double-walled hyperboloid.

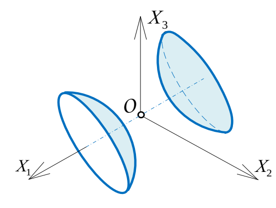
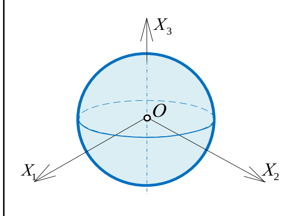
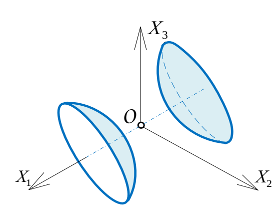
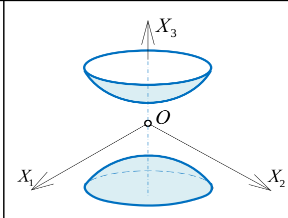
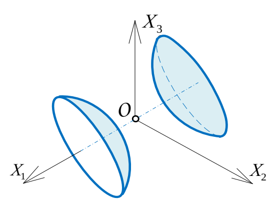
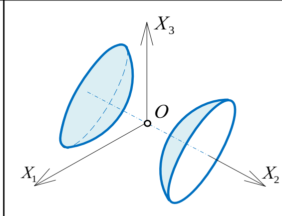
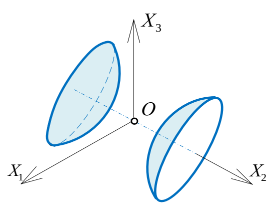
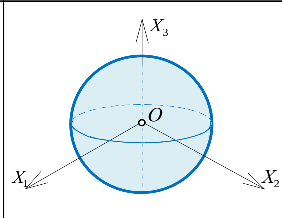
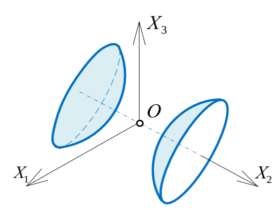
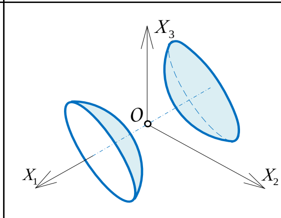
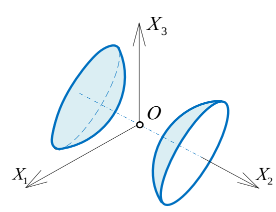
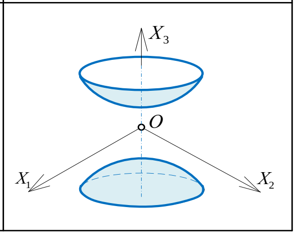
№	Surface	Surface
7		
8		
9		
10		
11		
12		

Figure 5. Combinations of non-linear surfaces of the 2nd order that participate in the spatial design scheme for

obtaining biquadratic transformations

Figure 4 contains a list of second-order non-linear surfaces that serve as paired elements of the spatial design scheme. From these four surfaces, twelve combinations of the displayed surfaces Φ_1^0 and Φ_2^0 have been created, which are shown in accordance with Figure 5.

Studies have shown that, in accordance with Figure 5, items with sequential numbers 1-4, 7, 9, 10, 12 they form degenerate (4-4)-valued correspondences of the plane, and points with ordinal numbers 5, 6, 8, 11 allowed us to obtain four types of canonical biquadratic transformations of the plane L and L' in accordance with Figure 6.

5. CONCLUSIONS

For the first time, the analysis of the research of scientists showed that there is a significant contribution of them to the development of theoretical provisions and methods of applied geometry. The data of this scientific review shows the history and ways of development of descriptive geometry in the Republic of Kazakhstan, and are also valuable for scientists, specialists in descriptive geometry.

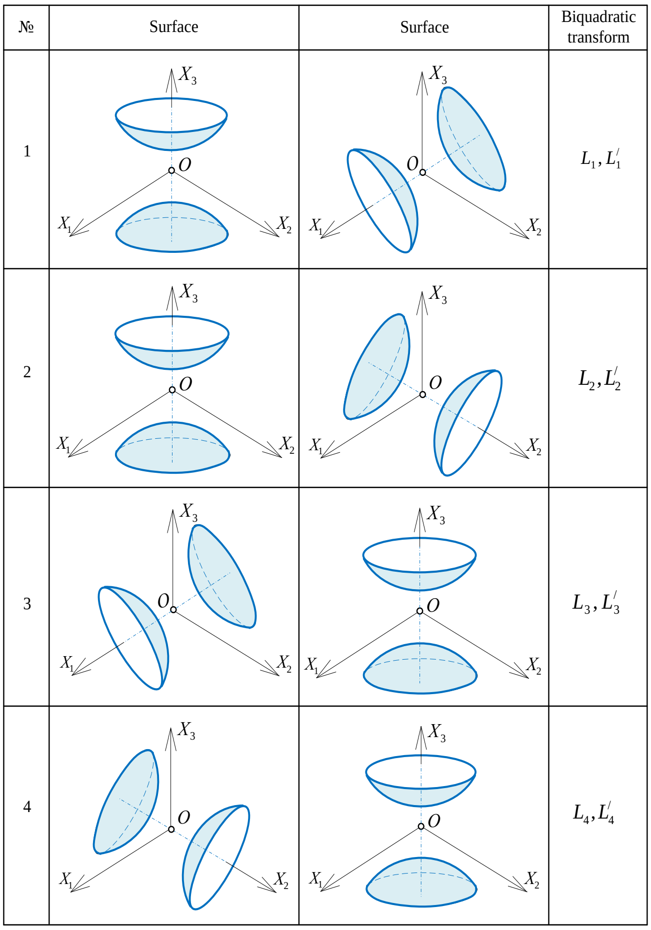


Figure 6. Biquadratic plane transformations generated by binary mapping of two non-linear surfaces

The main result is the development of theoretical foundations for modeling biquadratic transformations of the plane. Regularities of generating biquadratic transformations of the plane are developed. The performed theoretical and applied research allows us to draw the following conclusions: the developed spatial design scheme for displaying two second-order surfaces allowed us to establish new patterns for obtaining four-four-digit correspondences between two non-displaced planes. Theoretical provisions for modeling canonical biquadratic transformations of the plane have been created. A method for obtaining biquadratic transformations of the plane generated by binary mapping of two second-order surfaces to a combined plane is developed. This method allowed us to obtain four types of canonical biquadratic transformations of the plane. The developed algorithm would make it possible to determine mathematical models of canonical biquadratic transformations of the plane, which is necessary for their practical application. The data obtained can be implemented in the surfaces design, as well as in other types of architectural design. The results of the research are recommended to be validated with experimental data for high-confident in the empirical formulations and findings. The future research direction can involve the development of the theory of biquadratic transformations of the plane.

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NOMENCLATURE

T_{2-2}.	two-valued correspondence
A_1	point of the projection on the frontal plane
B_1	point of construction of rays 1
Π_1.	horizontal plane of projection
Π_2	frontal plane of projection
C_1	point of intersection of rays 1
C_2	point of intersection of rays 2
D_1	point of intersection of curves and a plane 1
D_2	constructing point 1 of the projection axis
J_2	central quadratic involution
J_{3-3}	cubic involutional transformation
L_1	point of intersection of curves and a plane 2
L_2	constructing point 2 of the projection axis
L_3	biquadratic transformation 1
L_4	biquadratic transformation 2
e	constant number, parameter
k^2	fundamental curve
L	plane
m^2	fundamental curve
n	set of natural numbers
O	constructing point of the plane
S	pole 1
$S^{/}$	pole 2
S_1	beam
x	Cartesian coordinate of the point 1
y	Cartesian coordinate of the point 2
A_2	constructing point 1
A_3	constructing point 2
A_4	constructing point 3
Γ	projection plane
OX_1	axis 1
OX_2	axis 2
OX_3	axis 3
Φ_1^0	second-order continuous polynomial 1
Φ_2^0	second-order continuous polynomial 2
X_1	Cartesian coordinate 1
X_2	Cartesian coordinate 2
X_3	Cartesian coordinate 3
a	curve
b	curve of the second order
α	plane