IIETA International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 9, No. 4, August, 2022, pp. 989-993
Journal homepage: http://iieta.org/journals/mmep

Differential Sandwich Theorems Involving Linear Operator

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https://doi.org/10.18280/mmep.090416

Received: 3 January 2022
Accepted: 7 June 2022

Keywords:
integral operator, univalent function, convex function, linear operator, sandwich theorem

ABSTRACT

The current study is fully devoted to studying differential subordination and superordination theorems of analytic functions with some sandwich results involving linear operators $I^{\lambda}_{s,a,\mu}$. This operator was obtained by the Hadamard product with the family of integral operators and the Hurwitlz-Lerch Zeta function. The current results demonstrate the possibility and capability of extracting the Sandwich theorem, and the conclusion includes differential subordination and differential superordination.

1. INTRODUCTION

Through the integral operator, Cotirlă [1], obtained a sandwich theorem by different method. Recently, Shammugam et al. [2] and Goyal et al. [3] and Atshan [4], Atshan & HadiAbd [5], Ibrahim et al. [6], Atshan & Hussain [7], and Atshan & Jawad [8], studied sandwich theorems for another condition. Previous research established those differential subordinations and superordinations can be used to obtain adequate conditions to meet the sandwich implication of a large number of well-known sandwich theorems.

Let me use and called H=$H(\nabla)$ is the class of functions (analytic) in the disk $\nabla = \{z \in \mathbb{C} : |z| < 1\}$. $\forall\, n \in (+integer)$. In addition, $a \in \mathbb{C}$. Now, we define $H\,[a,\,n]$ is the subclass of H. $H\,[a,\,n]$ include the shape:

$$\mathcal{F}(z) = a + a_n z^n + a_{n+1} z^{n+1} + (a\ belong\ to\ \mathbb{C}) \tag{1}$$

Too, suppose φ subclass of H include of the functions in the shape:

$$\mathcal{F}(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

Let F, g belong to φ. The function f is called subordinate to g, or g is called superordinate to $\mathcal{F}$, if $\exists$ schwarz analytic function in ∇ is w, with $zero$=$w(zero)$ and the absolute value of $w(z)$ less than 1, $(z \in \nabla)$ s.t $\mathcal{F}(z)$ equal $g(w(z))$. we shall write $\mathcal{F} \prec g$.

If g is univalent in ∇, then $\mathcal{F} \prec g$ iff $F(zero)$=$g(zero)$, $\mathcal{F}(\nabla)$ is subset of $g(\nabla)$.

Now: Suppose $h, p \in H$ and $\psi(r, \lambda, t; z): \mathbb{C}^3 \times \nabla \longrightarrow \mathbb{C}$. If the variable p and the set of concepts $\psi(p(z)\ and\ zp'(z)\ plus\ z^2p''(z); z)$ are function univalent in the disk ∇ in addition if p in the shape the 2nd-order superordinations.

$$h(z) \prec \psi(p(z), zp'(z) and\ z^2p''(z): z) \tag{3}$$

So, p gives title a solution of the superordination of (3). (If f is subordinate to g, then g is superordinate to f). Now An analytic $\mathfrak{q}$ is said to be a subordinant of (3), if $\mathfrak{q} \prec p\forall$ the functions p satisfies (3). Now $\tilde{q}$ univalent subordinant have the property $\mathfrak{q} \prec \tilde{\mathfrak{q}}$, $\forall$ differential subordinants $\mathfrak{q}$ of (3) is known the best subordinant. The two researchers Miller & Mocanu [9] they own set on the analytic h; $\mathfrak{q}$ and ψ by the set of concepts:

$$h(z) \prec \psi\left(p(z), z\frac{dp}{dz}, z^2\frac{d^2p}{dz^2}(z); z\right) \Rightarrow \mathfrak{q}(z) \prec p(z) \tag{4}$$

Komatu [10] gave the family of integral operator:

$$J^{\lambda}_{\mu}: \Sigma \to \Sigma.$$

Define as follows:

$$J^{\lambda}_{\mu}\mathcal{F}(z) = z + \sum_{n=1}^{\infty} (\mu/n + \mu - 1)^{\lambda} a_n z^n\,, \quad (z\ belong\ to\ \nabla, n > 1, \lambda \geq 0)$$

By function Hurwitlz-Lerch Zeta:

$$\phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^k}{(k+a)^s}\,, a \in \mathbb{C}/z_0{}^-, s \in \mathbb{C}\ when\ 0 < |z| \text{ greater than 1}$$

Using convolution where $G_{s,a(z)}$ by:

$$G_{s,a(z)} = (1+a)^s[\phi(z,s,a) - a^{-s}], (z \in \nabla)$$

Then a linear operator $I^{\lambda}_{s,a,\mu}\mathcal{F}(z): \Sigma \to \Sigma$[11] is defined:

$$I^{\lambda}_{s,a,\mu}\mathcal{F}(z) = G_{s,a(z)} * J^{\lambda}_{\mu} f(z) \tag{5}$$

$$= z + \sum_{n=2}^{\infty} \left(\frac{1+a}{k+a}\right)^{s} \left(\frac{\mu}{\mu+n-1}\right)^{\lambda} a_n z^n. \tag{6}$$

From Eq. (6) get:

$$z\left(I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z)\right)' = \mu I^{\lambda}_{s,a,\mu}\mathcal{F}(z) - (\mu - 1) I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z) \tag{7}$$

AL-Ameedee et al. [12] got conditions for the concepts certain analytic.

$$\mathbb{q}_1(z) \prec z\mathcal{F}'(z)/\mathcal{F}(z) \prec \mathbb{q}_2(z)$$

where, $\mathbb{q}_1$ and $\mathbb{q}_2$ are univalent in ∇ in addition $\mathbb{q}_1(zero) = \mathbb{q}_2(zero) = 1$.

The fundamental objective our research is to find some properties of normalized analytic functions $\mathcal{F}$ like sufficient condition.

$$\mathbb{q}_1(z) \prec \left(\frac{I^{\lambda}_{s,a,\mu}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \mathbb{q}_2(z)$$

And:

$$\mathbb{q}_1(z) \prec \left(\frac{tI^{\lambda+1}_{s,a,\mu}\mathcal{F}(z) + (1-t)I^{\lambda}_{s,a,\mu}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \mathbb{q}_2(z)$$

We got univalent functions q_1 and q_2 in ∇ with, $1 = \mathbb{q}_2(0) = \mathbb{q}_1(0)$.

In the next step, we need some basic information [13-20].

2. DEFINITIONS AND PRELIMINARIES

Definition (1) [13]. The set functions $\mathcal{F}$ Denote by Q, is analytic and one to one $\overline{\nabla} \setminus E(\mathcal{F})$, where $\overline{\nabla} = \nabla \cup \{z \in \partial\nabla\}$ and $E(\mathcal{F}) = \left\{\zeta \in \partial\nabla: \lim_{z\to\zeta} \mathcal{F}(z) = \infty\right\}$ s.t
$\mathcal{F}'(\zeta) \neq 0$ for $\zeta \in \partial\nabla \setminus E(\mathcal{F})$.
$\mathcal{F}(z) = a$, then: Q(zero)= Q_0 and Q(one)= $Q_1 = \{\mathcal{F} \epsilon Q : \mathcal{F}(0) = 1\}$.

Lemma (1)[13]. In ∇ suppose be univalent and suppose θ in addition ϕ analytic function in D consists of $q(\nabla)$ and $0 \neq \phi(w)$ such that $w \in q(\nabla)$. $z\mathbb{q}'(z)\phi(\mathbb{q}(z)) = Q(z)$ and $h(z) = \theta(\mathbb{q}(z)) + Q(z)$. let:
i. ∇ contains the starlike univalent function $Q(z)$.
ii. $Re\{h'(z)z/Q(z)\}$ greater than 0, $\forall$ z belong to ∇.
If the analytic function p in ∇, too p(zero) $= \mathbb{q}$ (zero), $p(\nabla)$ sub set D,

$$\theta(p(z)) plus\ zp'(z)\phi(p(z)) \prec \theta(\mathbb{q}(z)) plus\ z\mathbb{q}'(z)\phi(\mathbb{q}(z)),$$

So $\mathbb{q}$ is the best dominant in addition $p \prec \mathbb{q}$."

Lemma (2)[14]. Suppose the convex univalent function in ∇ is $\mathbb{q}$ in addition let α belong to $\mathbb{C}$, β belong to $\mathbb{C} \setminus \{zero\}$ and:

$$Re\{(z\mathbb{q}''(z)/\mathbb{q}'(z)) + 1\}$$
greater than $\max\left\{zero, -Re\left(\frac{\alpha}{\beta}\right)\right\}$

then:

$$\alpha p(z) + \beta z p'(z) \prec \alpha\mathbb{q}(z) + \beta z\mathbb{q}'(z),$$

So, the function is the best dominant in addition $p \prec \mathbb{q}$.

Lemma (3) [14]. Let the convex univalent function in ∇ is $\mathbb{q}$ in addition suppose β belong to $\mathbb{C}$. In addition, suppose that $Re(\beta)$>$zero$. If the function $p \in H\ [\mathbb{q}\ (zero), 1]$ intersection with Q and the univalent in ∇ is $p(z)$ plus $\beta zp'(z)$ when:

$$\mathbb{q}(z) + \beta z\mathbb{q}'(z) \prec p(z)\ plus\ \beta z p'(z)$$

i.e., the function q is the best subordinant and $\mathbb{q} \prec p$.

Lemma (4) [8]. Suppose the convex univalent function in ∇ is $\mathbb{q}$ in addition suppose ϕ and θ are analytic function in D contains $q(\nabla)$. Then, suppose:

(1) Real $\{\theta'(\mathbb{q}(z)) \div \phi(\mathbb{q}(z))\}$ greator than zero, $\forall$ complex variable z belong to ∇.

(2) the function $Q(z)$ equal $z\mathbb{q}'(z)$ product $\phi(q(z))$ is function univalent star like in ∇.

If p belong to $Q \cap H[\mathbb{q}(zero), 1]$, in addition $p(\nabla)$ subset D, $\theta(p(z))$ plus $\lambda p'(z)\ \phi(p(z))$ is univalent function in unit disk.

$$\theta(\mathbb{q}(z)) plus\ z\mathbb{q}'(z)\phi(q(z)) \prec \theta(p(z)) plus\ zp'(z)\ product\ \phi(p(z))$$

then, $\mathbb{q} \prec p$ and the function q satisfy the best subordinate.

Theorem (1). Let the convex univalent in ∇ is $\mathbb{q}$ with $1 = \mathbb{q}\ (zero)$, $0 \neq \varepsilon \in \mathbb{C}, \gamma$ greater than to zero and let $\mathbb{q}$ satisfies

$$Real\{[z\mathbb{q}''(z)/\mathbb{q}'(z)] + 1\} > \text{max}immum\left\{Zero, -Real\left(\frac{\gamma}{\varepsilon}\right)\right\} \tag{8}$$

when, $\mathcal{F} \in \varphi$ satisfies the subordination,

$$(1-\varepsilon\mu)\left(\frac{I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I^{\lambda}_{s,a,\mu}\mathcal{F}(z)}{I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z)}\right) \prec \mathbb{q}(z) plus \frac{\varepsilon}{\gamma} z\mathbb{q}'(z) \tag{9}$$

then,

$$\left(\frac{I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \mathbb{q}(z) \tag{10}$$

and the "best dominant of (9) is $\mathbb{q}$.

Proof. p defined as follows:

$$p(z) = \left(\frac{I^{\lambda+1}_{s,a,\mu}\mathcal{F}(z)}{z}\right)^{\gamma} \tag{11}$$

By purely mathematical operations for condition (11) with respect to z.

$$zp'(z) \big/ p(z) = \gamma\left(-1+\frac{z\left(I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)\right)'}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right) \tag{12}$$

Then, obtain the following subordination in view of (7).

$$p'(z) \big/ p(z) = \gamma\left(\mu\left(-1+\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)+\left(-1+\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)\right)$$

Therefore,

$$\frac{zp'(z)}{\gamma} = \left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\mu\left(-1+\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)+\left(-1+\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)\right)$$

from the hypothesis the subordination (9) becomes:

$$p(z) plus \frac{\varepsilon}{\gamma} zp'(z) \prec \mathbb{q}(z) plus \frac{\varepsilon}{\gamma} z\mathbb{q}'(z)$$

we obtain (10) by application of (Lemma (2)) with $\beta = \frac{\varepsilon}{\gamma}$ and α=1,

Corollary (1). Let $0 \neq \varepsilon \in \mathbb{C}$, $\gamma>0$ and:

$$Re\left\{\frac{2z}{-z+1}+1\right\} > \max\left\{zero, -Real\left(\frac{\gamma}{\varepsilon}\right)\right\}.$$

If $\mathcal{F} \in \varphi$ got the subordination:

$$(1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right) \prec \left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z^2)}\right)$$

So, $\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}$ subordinat to (z plus 1⁄z minus 1).

In addition, $\mathbb{q}$(z) equal (z+1⁄z-1) is the perfect dominant.

Theorem (2). Suppose the convex univalent in ∇ is $\mathbb{q}$ in addition 1=$\mathbb{q}$(zero), zero not equal $\mathbb{q}$(z), (z belong to ∇) and suppose:

$$Re\left\{1-\frac{\gamma}{\varepsilon}+\frac{zq''(z)}{q'(z)}\right\} > 0 \tag{13}$$

where, $\varepsilon>0$, $\varepsilon \in \mathbb{C}\setminus\{zero\}$ and $z \in \nabla$.

Let the starlike univalent in ∇ is $(-\varepsilon z\mathbb{q}'(z))$. If $\mathcal{F} \in \varphi$ then:

$$\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \varepsilon; z) \prec \gamma\mathbb{q}(z) - \varepsilon z\mathbb{q}'(z) \tag{14}$$

where,

$$\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \varepsilon; z) = \gamma\left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+(1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} - \gamma\varepsilon\left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+(1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z))}{z}\right)^{\gamma}\left(-1+\frac{\mathrm{t}I_{s,a,\mu}^{\lambda}\mathcal{F}(z))+(1-\mathrm{t})I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+I_{s,a,\mu}^{\lambda}\mathcal{F}(z))}\right) \tag{15}$$

then,

$$\left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+(1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \mathbb{q}(z) \tag{16}$$

and the best dominant of (14) is $\mathbb{q}(z)$.

Proof: Now we will start defined *p*:

$$p(z) = \left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+(1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} \tag{17}$$

Setting: $\emptyset(w) = -\varepsilon$ in addition $\theta(w)=\gamma w$, *w* not equal to zero. Then, $\theta(w)$ and $\emptyset(w)$ is analytic in $\mathbb{C}$ with $\mathbb{C}/\{zero\}$ respectively.

So, $\emptyset(w)$ not equal to zero, *w* belong to $\mathbb{C}/\{0\}$, then;

$$z\mathbb{q}'(z).\emptyset\mathbb{q}(z) = -\varepsilon z\mathbb{q}'(z) = \mathrm{Q}(z)$$

and

$$\theta\mathbb{q}(z) \text{ plus } Q(z) \text{ equal } \gamma\mathbb{q}(z) - \varepsilon z\mathbb{q}'(z) = \mathrm{h}(z)$$

Q(*z*) is a starlike univalent in ∇,

$$\text{Real}\{\frac{z\mathrm{h}'(z)}{\mathrm{Q}(z)}\} = \text{Real}\{1-\frac{\gamma}{\varepsilon}+\frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)}\} > 0.$$

and, obtain:

$$\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z) = \gamma\mathrm{p}(z) - \varepsilon z\mathrm{p}'(z) \tag{18}$$

So, $\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z)$ is given by (15).

By (14) with (18), we own:

$$\gamma\mathrm{p}(z) - \varepsilon z\,\mathrm{p}'(z) \prec \gamma\mathbb{q}(z) - \varepsilon z\,\mathbb{q}'(z) \tag{19}$$

Therefore, got $\mathrm{p}(z) \prec \mathbb{q}(z)$ by (Lemma (1)), and by using (17).

Corollary (2):

Let, Real $\left\{1-\frac{\gamma}{\varepsilon}+\frac{z2\mathrm{B}}{(1+\mathrm{B}z)}\right\} > 0$, and $-1 \leq \mathrm{B} < \mathrm{A} \leq 1$. where ε belong to $\mathbb{C}/\{zero\}$ in addition z be long to ∇, if $\mathcal{F} \in \varphi$ such that:

$$\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z) \prec \left(\gamma\left(\frac{1+\mathrm{A}z}{1+\mathrm{B}z}\right) - \varepsilon z\frac{\mathrm{A}-\mathrm{B}}{(1+\mathrm{B}z)^2}\right)$$

and $\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z)$ is given by condition (8),

$$\left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+(1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \frac{\mathrm{A}z+1}{\mathrm{B}z+1}$$

and the perfect dominant is $\mathbb{q}(z) = \frac{1+\mathrm{A}z}{1+\mathrm{B}z}$

Theorem (3): suppose the convex univalent in ∇ is $\mathbb{q}$ in addition γ greator than zero, $\mathbb{q}$(0)=1 and Re $\{\varepsilon\}>0$.

Suppose that $\mathcal{F} \in \varphi$ satisfies: $\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z))}{z}\right)^{\gamma}$ $\in$Q intersect *H* [$\mathbb{q}$(0),1], so,

$$(1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)$$

is univalent function in ∇.

$$\text{if } \mathbb{q}(z) + \frac{\varepsilon}{\gamma}z\mathbb{q}'(z) \prec (1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right) \tag{20}$$

then,

$$\mathbb{q}\,(z) \prec \left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z))}{z}\right)^{\gamma} \quad (21)$$

the best subordinant of (20) is $\mathbb{q}$.

Proof: *p* defined as follows:

$$\mathrm{p}(z) = \left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} \quad (22)$$

Now we get (22) by differentiating:

$$zp'(z)\big/p(z) = \delta\left(\frac{z(I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z))'}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)} - 1\right) \quad (23)$$

Using (7) in (23), we obtain:

$$(1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right) = \mathbb{q}(z) + \frac{\varepsilon}{\gamma}z\mathbb{q}'(z)$$

we get the result by using (Lemma (3)).

Corollary (3): If $\mathcal{F} \in \varphi$ satisfies: $\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z))}{z}\right)^{\gamma} \in \mathrm{Q} \cap \mathrm{H}[\mathbb{q}(0), 1]$ and let $\gamma>0$ and Re$\{\varepsilon\}>0$.

and,

$$(1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)$$

be univalent in ∇. If

$$\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z^2)} \prec (1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}\right)$$

then,

$$\left(\frac{1+z}{1-z}\right) \prec \left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z))}{z}\right)^{\gamma}$$

and the best subordinant is $\mathbb{q}$(z) equal $\frac{1+z}{1-z}$.

Theorem (4): Let the convex univalent in ∇ is $\mathbb{q}$ in addition 1=$\mathbb{q}$(zero), and let *q* satisfies:

$$\mathrm{Re}\,\{\frac{-\gamma \mathrm{q}'(z)}{\varepsilon}\} > 0 \quad (24)$$

where, $z \in \nabla$ and $\varepsilon \in \mathbb{C}/\{0\}$.

"Let-γzq'(z)" is a starlike univalent function "in ∇ and let $\mathcal{F} \in$ φ satisfies: " $\left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)+(1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right) \in \mathrm{Q} \cap \mathrm{H}\,[\mathbb{q}\,(\text{zero}), 1]$, "and $\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; \mathrm{z})$ is a univalent function in ∇,where $\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; \mathrm{z})$ is given by (15). If:

$$\gamma\mathbb{q}\,(z) - \varepsilon z\mathbb{q}\,'(z) \prec \emptyset(\gamma, \mathrm{n}, \lambda, \mathrm{m}, \varepsilon; z) \quad (25)$$

Then,

$$\mathbb{q}\,(z) \prec \left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z) + (1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} \quad (26)$$

and the best subordinant of (25) that is $\mathbb{q}$.

Proof: Defined *p* in the shape,

$$\mathrm{p}(z) = \left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z) + (1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} \quad (27)$$

by $\emptyset(w)=-\varepsilon$ and $\theta(w)=\gamma w$, $0\neq w$.

So, $\theta(w)$ and $\emptyset(w)$ are analytic in $\mathbb{C}$ in addition $\mathbb{C}\backslash\{\text{zero}\}$ respectively and $\emptyset(w)$ does not equal zero, $w\in \mathbb{C}\backslash\{zero\}$. Then:

$$z\mathbb{q}'(z)\emptyset\mathbb{q}(z) = -\varepsilon z\mathbb{q}'(z) = \mathrm{Q}(z)$$

We got starlike univalent" in ∇ is $Q(z)$.

$$\mathrm{Real}\left\{\frac{\theta'(\mathbb{q}(z))}{\emptyset(\mathbb{q}(z))}\right\} = \mathrm{Re}\left\{\frac{-\gamma\mathbb{q}'(z)}{\varepsilon}\right\} > 0$$

Now, obtain:

$$\gamma\mathrm{p}(z) - \varepsilon z\mathrm{p}'(z) = \emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z) \quad (28)$$

where, $\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z)$ is get by (15).

By (25) and (28);

$$\gamma\mathbb{q}(z) - \varepsilon z\mathbb{q}'(z) \prec \gamma\mathrm{p}(z) - \varepsilon\mathrm{p}'(z) \quad (29)$$

where, $\mathbb{q}(z) \prec \mathrm{p}(z)$ by (Lemma (3)). By using (27), will get to the desired result.

The concept of Sandwich represented by (Theorems (5) and (6)).

3. SUBORDINATION AND SUPERORDINATION

Theorem (5): Let the convex univalent in ∇ is $\mathbb{q}_1$(zero)=1, Real {ε}>0 and let the univalent in ∇ is $\mathbb{q}_2$(zero)=1 and realize (8), let $\mathcal{F} \in \varphi$ such that.

$$\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} \in \mathrm{Q} \cap \mathrm{H}\,[1,1],$$

and,

$$(1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\ \mathcal{F}(z)}\right),$$

is univalent function in ∇.

If,

$$\mathbb{q}_1(z) + \frac{\varepsilon}{\gamma}\mathrm{z}\mathbb{q}'_1(z) \text{ subordinant to } (1-\varepsilon\mu)\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} + \varepsilon\mu\left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma}\left(\frac{I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{I_{s,a,\mu}^{\lambda+1}\ \mathcal{F}(z)}\right) \prec \mathbb{q}_2(z) + \frac{\varepsilon}{\gamma}\mathrm{z}\mathbb{q}'_2(z)$$

$$\mathbb{q}_1(z) \prec \left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \mathbb{q}_2(z)$$

We got concepts best subordinant in addition best dominant $\mathbb{q}_1$ and $\mathbb{q}_2$ are respectively.

Theorem (6): Suppose the univalent convex in ∇ is $\mathbb{q}_1$, $\mathbb{q}_1(\text{zero}) = 1$ and satisfies (24), let the univalent function in ∇ is $\mathbb{q}_2$, $\mathbb{q}_2$(zero)=1, realize (13), let $\mathcal{F} \in \varphi$ satisfies:

$$\left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z) + (1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right)^{\gamma} \in \mathrm{Q} \cap \mathrm{H}\,[1,1],$$

and $\emptyset(\gamma, s, \lambda, a, \mu, \varepsilon; z)$ is univalent in ∇, where $\emptyset(\gamma, s, \lambda, a, \mu, \varepsilon; z)$ we got by (15).

If $\gamma\mathbb{q}_1\ (z) - \varepsilon z\mathbb{q}'_1(z)\emptyset(\gamma, \mathrm{s}, \lambda, \mathrm{a}, \mu, \varepsilon; z) \prec \gamma\mathbb{q}_2\ (z) - \varepsilon z\mathbb{q}'_2(z)$,

Then,

$$\mathbb{q}_1(z) \prec \left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z) + (1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right) \prec \mathbb{q}_2(z)$$

We got the concepts of best subordinant and best dominant $\mathbb{q}_1$ and $\mathbb{q}_2$, respectively.

4. CONCLUSIONS

The conclusion of this research gained subordination and superordination results by using the linear operator $I_{s,a,\mu}^{\lambda+1}$ for example for these results:

$$1 - \mathbb{q}_1(z) \prec \left(\frac{I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z)}{z}\right)^{\gamma} \prec \mathbb{q}_2(z)$$

$$2 - \mathbb{q}_1(z) \prec \left(\frac{\mathrm{t}I_{s,a,\mu}^{\lambda+1}\mathcal{F}(z) + (1-\mathrm{t})I_{s,a,\mu}^{\lambda}\mathcal{F}(z)}{z}\right) \prec \mathbb{q}_2(z)$$

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