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# Analysis of Solution for the Stochastic Model Representing Water Scarcity in the Society

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**ABSTRACT**

The main aim of this paper is to study the stochastic effect of water scarcity in the society through our model. It has been shown that there is a unique global positive solution to the proposed stochastic epidemic model with boundedness and permanence. We have selected some effective Lyapunov functions to provide sufficient conditions for investigating water scarcity persistence and extinction. The theoretical results of this work have been verified based on numerical experiments.

## 1. INTRODUCTION

Water scarcity is described as a lack of available water or the failure to access safe drinking water. In many parts of the world, water is in short supply. Water is becoming increasingly scarce as it is required to raise and prepare food, make electricity, and support industry for an ever-increasing population. The human population has increased by more than 50% in the last 50 years. Water ecosystems all around the world have been transformed by fast growth, which has been accompanied by economic development and industrialization.

India's yearly rainfall is unevenly distributed throughout the country and at various times of the year. As a result, despite adequate yearly rainfall, certain river basins are classified as water limited or water stressed. India is facing a catastrophic water deficit as a result of government mishandling, pollution, and groundwater depletion. India's growing water consumption, along with its economic expansion, is a big concern [1].

Climate change is certainly a global problem, but it also has serious domestic consequences that are sometimes neglected. Climate change is having a significant impact on Iran's social and economic environment, especially given the country's existing dry geographic position in the Middle East [2]. Water pollution has a significant influence in generating water shortage in the river basin, as indicated by the large value of a quality related water shortage index. As a result, concentrating just on lowering water usage may not be sufficient to considerably relieve the water scarcity problem [3].

The paper concludes the major causes of anticipated water shortage in the future decades are India's exponential population growth and an unbalance in the recharging and use of ground water. Renewable solar energy, wind, and tidal waves, along with seawater desalination, are environmentally benign and realistic solutions for overcoming this predicament [4, 5]. The effects of climate change-induced variations in irrigated and rainfed agricultural yields on water consumption were investigated. Crop yields are affected differently by climate change depending on the crop and ecological zone.

It's important to note that a lot of mathematical models are deterministic. Random population fluctuations, individual death rates, immigration rates, and other difficulties are not considered. We can't predict more accurately with the deterministic model since the systems have few restrictions. As a result, using randomness in deterministic models leads to stochastic differential equations, which provides another level of realism to the real-world problem.

Many researchers have recently studied the impacts of stochasticity on epidemic models in exploring the effects of environmental noises on population dynamics. Stochastic differential equations have been a popular topic in applied science, mathematical biology, environmental science, and ecology in recent years [6-8]. Many authors have researched at the dynamics of epidemic models that include randomness [9-11].

The environment elements change randomly over time and should be treated as stochastic [12, 13]. The theory of stochastic differential equations was used to study classic epidemic models such as SIS [14, 15], SIR [16], SEIR [17], [18] and SVIR [19]. Furthermore, Mishra and Tripathi presented the stochastic version of artificial rain [20-22].

As mentioned above they all have analyzed and proved that water scarcity is one of the major problems but they have proved through statistically and theoretically. We are here giving a new try to prove the same on using Mathematical model.

We examine our model using the concept of ordinary differential equation. In section 2 we extend our model to stochastic differential equation model. We also demonstrated the existence and uniqueness of global positive solutions, the stochastic boundedness and permanence of the model (2). In section 3 pth-moment exponential stability are analyzed. In section 4, the persistent of the proposed stochastic model is presented. In section 5 demonstrates the simulation results for both deterministic and stochastic models. Finally, our results are summarized in section 6 as conclusion.

In this paper, we will study the analysis of solution for the stochastic model representing water scarcity in the society:

$$\frac{dW}{dt} = \Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r,$$
$$\frac{dH}{dt} = \alpha_2 WH - k_1 H,$$
$$\frac{dW_s}{dt} = \alpha_1 W + \beta H - \delta_1 W_s, \tag{1}$$
$$\frac{dW_r}{dt} = \delta_1 W_s - \delta_2 W_r.$$

where, $k_1 = \beta + \mu + \mu_1$. $\Lambda$ is the recruitment rate, $\alpha_1$ is the water draining rate, $\alpha_2$ is the rate of human consumption of water, $\delta_1$ is the rate of water Recovery, $\delta_2$ is the rate of water going to Normal water, $\beta$ is the rate of human population affected water scarcity, $\mu$ is the natural death $\mu_1$ is the rate of due to scarcity death and W(t), H(t), $W_s(t)$, $W_r(t)$ denote total usage of water, human, water scarcity, water recover respectively.

## 2. STOCHASTIC MODEL

We assume that stochastic perturbations are of the white noise type, with W(t), H(t), $Ws(t)$, and $Wr(t)$ are directly proportional. Then the deterministic system (1) will bw extended to the following system to stochastic differential equations of the form:

$$\begin{aligned} dW &= [\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r]dt + \sigma_1 W dW_1(t) \\ dH &= [\alpha_2 WH - k_1 H]dt + \sigma_2 H dW_2(t) \\ dW_s &= [\alpha_1 W + \beta H - \delta_1 W_s]dt + \sigma_3 W_s dW_3(t), \\ dW_r &= \delta_1 W_s - \delta_2 W_r]dt + \sigma_4 W_r dW_4(t). \end{aligned} \tag{2}$$

where, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ are the intensities of standard Gaussian white noise and $W_1(t)$, $W_2(t)$, $W_3(t)$, $W_4(t)$ are independent standard Brownian motions.

### 2.1 Preliminaries

We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ be the complete probability space with a filtration $(\mathcal{F})_{t\geq 0}$ satisfying the usual conditions. (i.e., it is right continuous and increasing while $\mathcal{F}$ contains all $\mathbb{P}$-null sets).

$$X(t) = \big(W(t), H(t), W_s(t), W_r(t)\big) = x_1(t), x_2(t), x_3(t), x_4(t))$$

And the form $|X(t)| = \sqrt{W^2(t) + H^2(t) + W_s^2(t) + W_r^2(t)}$. And denote $C^{2,1}(\mathbb{R}^4 \times (0,\infty); \mathbb{R}_+)$ as the family of all non- negative functions V(X,t) denote on $\mathbb{R}^4 \times (0,\infty)$ such that they are twice differentiable in X and once in t.

We consider the differential operator $\mathcal{L}$ associated with four dimensional SDE.

$$dX(t) = f(X,t)dt + g(Z,t)dB(t)$$

as

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{4} f_i(X,t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{4} (gT(X,t)g(X,t))_{i,j}\frac{\partial^2}{\partial X_i \partial X_j}$$

where,

$$f = \begin{pmatrix} \Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r \\ \alpha_2 WH - \beta H - \mu H - \mu_1 H \\ \alpha_1 W + \beta H - \delta_1 W_s \\ \delta_1 W_s - \delta_2 W_r \end{pmatrix}$$

$$g = \text{diag}(\,(\sigma_1(W - W^*), \sigma_2(H - H^*), \sigma_3(W_s - W_s^*), \sigma_4(W_r W_r^*))$$

If $\mathcal{L}$ acts ona function $V \in C^{2,1}(\mathbb{R}^4 \times (0,\infty))$, then

$$\mathcal{L}V(X,t) = V_t(X,t) + V_X(X,t)f(X,t) + \frac{1}{2}\text{trace}(g^T(X,t)V_{XX}(X,t)g(X,t))$$

where, T means transposition. In view of Ito's formula, if X(t) $\in \mathbb{R}^4$

$$dV(X,t) = LV(X(t),t) + V_X(X(t),t)g(X(t),t)dB(t). \tag{3}$$

**Theorem 2.1**
Let [0,∞] × D = U be the domain containing the line $x = x^\star$ and assume there exists a function V(t,X) twice continuously differentiable in U which is a positive define in Lyapunov function and satisfies $\mathcal{L}V \leq 0$ for $x \neq x^\star$. Then the solution $X(t) = x^\star$ of SDE is stable in probability [23].

### 2.2 Analysis of solution

**Theorem 2.2**
System (2) is said to be stochastically ultimately bounded if for any $\epsilon \in (0,1)$ there exists a positive constant $H = H(\epsilon)$ s uch that for any initial value $\big(W(0), H(0), W_s(0), W_r(0)\big) \in \Omega$ the solution $X(t) = \big(W(0), H(0), W_s(0), W_r(0)\big)$ of (2) has the property $\lim_{n\to\infty} sup\ \mathbb{p}\{|X(t)| \leq H(\epsilon)\} \geq 1 - \epsilon$ [24].

In this section, we exhibit the existence of a unique positive global solution, stochastic ultimate boundedness, and stochastic permanence of the solution of the model 2. We will use some ideas from [16, 25, 26] for our proof.

2.2.1 Positive and global solution

**Theorem 2.3**
If the initial value $(W(0), H(0), W_s(0), W_r(0)) \in \mathbb{R}_+^4$ of the solution of the stochastic model (2) is given there exits a unique solution $(W(t), H(t), W_s(t), W_r(t))$ in $\mathbb{R}_+^4$ for $t \geq 0$ with probability one. That is $(W, H, W_s(t), W_r(t)) \in \mathbb{R}_+^4$, $\forall\, t \geq 0$ almost surely.

**Proof.** The given initial value is $(W(0), H(0), W_s(0), W_r(0)) \in \mathbb{R}_+^4$, there is a unique local solution $(W(t), H(t), W_s(t), W_r(t))$ on $[0, \tau_\epsilon)$, where $\tau_e$ is the explosion time, because the stochastic epidemic model (2) satisfies the locally Lipschitz continuous conditions. The solution of the stochastic model (2) must be shown to be global. We simply need to show that $\tau_e = \infty$ almost surely.

Choose a sufficiently large positive constant $k_0$ such that ($W(0), H(0), W_s(0)$ and $W_r(0)$) belong to $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time $\tau_k = \inf\{t \in [0, \tau_e]$: $W(t) \notin (\frac{1}{k}, k), H(t) \notin (\frac{1}{k}, k), W_s(t) \notin (\frac{1}{k}, k), W_r(t) \notin (\frac{1}{k}, k)\}$.

For the empty set θ, we set inf θ = ∞. Since $\tau_k$ is non-decreasing as $k \to \infty$, we have

$$\tau_\infty = \lim_{t\to\infty} \tau_k$$

Then $\tau_\infty \leq \tau_e$ a.s. Now, we have to show that $\tau_\infty = \infty$ a.s. If not, then there exist $T > 0$ and $\epsilon \notin (0, 1)$ such that $P[\epsilon_\infty \leq T] > \epsilon$.

There is an integer $k_1 \geq k_0$ such that

$$P[\epsilon_\infty \leq T] > \epsilon \text{ for all } k \geq k_1. \tag{4}$$

We denote a function $V_1: \mathbb{R}_+^4 \to \mathbb{R}_+$ by

$$V_1(W, H, W_s, W_r) = (W - 1 - \log W) + (H - 1 - \log H) + (W_s - 1 - \log W_s) + (W_s - 1 - \log W_s).$$

by applying Ito's formula, we get

$$\begin{aligned} dV_1 = {} & \left(1 - \frac{1}{W}\right)(\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r)dt \\ & + \left(1 - \frac{1}{W}\right)\sigma_1 W dW_1 + \frac{1}{2}\sigma_1^2 dt \\ & + \left(1 - \frac{1}{H}\right)(\alpha_2 WH - \beta H - \mu H - \mu_1 H)dt \\ & + \left(1 - \frac{1}{H}\right)\sigma_2 H dW_2 + \frac{1}{2}\sigma_2^2 dt \\ & + \left(1 - \frac{1}{W_s}\right)(\alpha_1 W + \beta H - \delta_1 W_s)dt \\ & + \left(1 - \frac{1}{W_s}\right)\sigma_3 W_s dW_3 + \frac{1}{2}\sigma_3^2 dt \\ & + \left(1 - \frac{1}{W_r}\right)(\delta_1 W_s - \delta_2 W_r)dt \\ & + \left(1 - \frac{1}{W_r}\right)\sigma_4 W_r dW_4 + \frac{1}{2}\sigma_4^2 dt \end{aligned}$$

which can be simplified to

$$\begin{aligned} = {} & \left[\left(1 - \frac{1}{W}\right)(\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r) + \frac{1}{2}\sigma_1^2 \right. \\ & + \left(1 - \frac{1}{H}\right)(\alpha_2 WH - \beta H - \mu H - \mu_1 H) \\ & + \frac{1}{2}\sigma_2^2 + \left(1 - \frac{1}{W_s}\right)(\alpha_1 W + \beta H - \delta_1 W_s) \\ & + \frac{1}{2}\sigma_3^2 + \left(1 - \frac{1}{W_r}\right)(\delta_1 W_s - \delta_2 W_r) \\ & \left. + \frac{1}{2}\sigma_4^2\right] \\ & + [\sigma_1(W - 1)dW_1 \\ & + \sigma_2(H - 1)dW_2 + \sigma_3(W_s - 1)dW_3 \\ & + \sigma_4(W_r - 1)dW_4] \end{aligned}$$

$$dV_1 = \mathcal{L}V_1 dt + [\sigma_1(W - 1)dW_1 + \sigma_2(H - 1)dW_2 + \sigma_3(W_s - 1)dW_3 + \sigma_4(W_r - 1)dW_4]$$

where,

$$\begin{aligned} \mathcal{L}V_1 = {} & \left(1 - \frac{1}{W}\right)(\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r) + \frac{1}{2}\sigma_1^2 \\ & + \left(1 - \frac{1}{H}\right)(\alpha_2 WH - \beta H - \mu H - \mu_1 H) \\ & + \frac{1}{2}\sigma_2^2 + \left(1 - \frac{1}{W_s}\right)(\alpha_1 W + \beta H - \delta_1 W_s) \\ & + \frac{1}{2}\sigma_3^2 + \left(1 - \frac{1}{W_r}\right)(\delta_1 W_s - \delta_2 W_r) + \frac{1}{2}\sigma_4^2 \end{aligned}$$

$$\begin{aligned} \mathcal{L}V_1 = {} & \Lambda + \alpha_1 + \alpha_2 H + \beta + \mu + \mu_1 + \delta_1 + \delta_2 \\ & - \left[-\frac{\Lambda}{W} + \frac{\delta_2 W_r}{W} + \mu H + \mu_1 H + \alpha_2 W \frac{\alpha_1 W}{W_s} \right. \\ & \left. + \frac{\beta H}{W_s} + \frac{\delta_1 W_s}{W_r}\right] + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 \\ & + \frac{1}{2}\sigma_4^2 \end{aligned}$$

$$\mathcal{L}V_1 \leq \Lambda + \alpha_1 + \alpha_2 + \beta + \mu + \mu_1 + \delta_1 + \delta_2 + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 + \frac{1}{2}\sigma_4^2 = K$$

Then, we have

$$dV_1 = Kdt + [\sigma_1(W - 1)dW_1 + \sigma_2(H - 1)dW_2 + \sigma_3(W_s - 1)dW_3 + \sigma_4(W_r - 1)dW_4] \tag{5}$$

Now integrating both sides of (5) from 0 to $\tau_k \wedge T$,

$$\int_0^{\tau_k \wedge T} dV_1(W, H. W_s, W_r) \leq \int_0^{\tau_k \wedge T} Kdr + \int_0^{\tau_k \wedge T} [\sigma_1(W - 1)dW_1 + \sigma_2(H - 1)dW_2 + \sigma_3(W_s - 1)dW_3 + \sigma_4(W_r - 1)dW_4]. \tag{6}$$

Taking expectation on both sides of the above equation, we get

$$\mathbb{E}V_1(W(\tau_k \wedge T), H(\tau_k \wedge T), W_s(\tau_k \wedge T), W_r(\tau_k \wedge T)) \leq V_1(W(0), H(0), W_s(0), W_r(0)) + \int_0^{\tau_k \wedge T} Kdr.$$

$$\mathbb{E}V_1(W(\tau_k \wedge T), H(\tau_k \wedge T), W_s(\tau_k \wedge T), W_r(\tau_k \wedge T)) \leq V_1(W(0), H(0), W_s(0), W_r(0)) + KT.$$

Let $\Omega_k = \{\tau_k \leq T\}$ for all $k \geq k_1$ and from 4, we have $\mathbb{P}(\Omega_k) \geq \epsilon$, Note that for every $\omega \in \Omega_k$, there is atleast $W(\tau_k, \omega)$ or $H(\tau_k,\omega)$ or $W_s(\tau_k,\omega)$ or $W_r(\tau_k,\omega)$ equals either k or $\frac{1}{k}$, since $k-1-\log k$ or $\frac{1}{k}-1-\log\frac{1}{k} = \frac{1}{k}-1+\log k$.

Hence

$W(\tau_k, \omega)$, $H(\tau_k,\omega)$, $W_s(\tau_k,\omega)$,$W_r(\tau_k,\omega) \geq (k - 1 - \log k) \wedge (\frac{1}{k}-1+\log k)$.

Then it follows as

$$
\begin{aligned}
&V_1\big(W(0),H(0),W_s(0),W_r(0)\big)+\mathrm{KT} \\
&\geq \mathbb{E}(\mathbb{I}_{\Omega k}(\omega)V_1(W(\tau_k\wedge T),H(\tau_k \wedge T),W_s(\tau_k\wedge T), W_r(\tau_k\wedge T))\big) = \mathbb{E}(\mathbb{I}_{\Omega k}(\omega)V_1(W(\tau_k\wedge T),H(\tau_k \wedge T),W_s(\tau_k\wedge T), W_r(\tau_k\wedge T)) \\
&\geq \mathbb{E}(\mathbb{I}_{\Omega k}(\omega)(k-1-\log k) \wedge \left(\frac{1}{k}-1+\log k\right), \\
&= (k-1-\log k) \wedge \left(\frac{1}{k}-1+\log k\right)\mathbb{E}(\mathbb{I}_{\Omega k}(\omega)), \\
&\leq \epsilon(k-1-\log k) \wedge \left(\frac{1}{k}-1+\log k\right)
\end{aligned} \tag{7}
$$

where, $\mathbb{I}_{\Omega k}(\omega)$ is the function of $\Omega_k(\omega)$. Letting $k\to\infty$, we get $\infty \leq V_1(W(0),H(0),W_s(0),W_r(0)) + \mathrm{KT} = \infty$ therefore, we have the contradiction. The proof is completed.

2.2.2 Stochastic boundedness

**Definition 2.4**

The model (2) is said to be stochastically ultimately bounded if $\mathrm{X(t)} = (\mathrm{W(0)},\mathrm{H(0)},\mathrm{Ws(0)},\mathrm{Wr(0)})$. If for any $\epsilon \in (0,1)$, there exists a positive constant $\theta > 0$ such that the solution X(t) to the epidemic model has the property [27]

$$
\lim_{t\to\infty}\mathbb{P}(|\mathrm{X(t)}>\theta|)<\epsilon \tag{8}
$$

**Theorem 2.5**

*Any positive initial value is* $(W(0),H(0),W_s(0),W_r(0)) \in \mathbb{R}_+^4$, *the solutions of stochastic epidemic model (2) are stochastically ultimately bounded.*

**Proof.** *Define*

$$
U(W(t),H(t),W_s(t),W_r(t)) = W^v + H^v + W_s^v + W_r^v
$$

for $U(W(t),H(t),W_s(t),W_r(t)) \in \mathbb{R}_+^4$ and $v>1$. By using Ito's formula $e^tU(W(t),H(t),W_s(t),W_r(t))$.

$$
\begin{aligned}
&\left[e^tU\big(W(t),H(t),W_s(t),W_r(t)\big)\right] \\
&= e^tU(W(t),H(t),W_s(t),W_r(t)) + e^t dU(W(t),H(t),W_s(t),W_r(t)) = e^t[W^v + H^v + W_s^v \\
&+ W_r^v + vW^{v-1}(\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r) \\
&+ vH^{v-1}(\alpha_2 WH - \beta H - \mu H - \mu_1 H) + vW_s^{v-1}(\alpha_1 W + \beta H \\
&- \delta_1 W_s) + vW_r^{v-1}(\delta_1 W_s - \delta_2 W_r) + \frac{v(v-1)}{2}(\sigma_1^2 W^v \\
&+ \sigma_2^2 H^v + \sigma_3^2 W_s^v + \sigma_4^2 W_r^v)]dt + e^t v[\sigma_1 W^v dW_1 \\
&+ \sigma_2 H^v dW_2 + \sigma_3 W_s^v dW_3 + \sigma_4 W_r^v dW_4], \\
&\leq Me^t dt + e^t v[\sigma_1 W^v dW_1 + \sigma_2 H^v dW_2 + \sigma_3 W_s^v dW_3 \\
&+ \sigma_4 W_r^v dW_4].
\end{aligned}
$$

where, $M>0$ is a constant.

Taking expectation and integrating the above equality from 0 to t

$$
\begin{aligned}
&\mathbb{E}[e^t\mathrm{U}(\mathrm{W(t)},\mathrm{H(t)},\mathrm{W_s(t)},\mathrm{W_r(t)})] \\
&\leq \mathrm{U}(\mathrm{W(0)},\mathrm{H(0)},\mathrm{W_s(0)},\mathrm{W_r(0)}) + \mathrm{M}\mathbb{E}\int_0^t e^s ds
\end{aligned}
$$

$$
\begin{aligned}
&e^t\mathbb{E}\mathrm{U}(\mathrm{W(t)},\mathrm{H(t)},\mathrm{W_s(t)},\mathrm{W_r(t)}) \\
&\leq \mathrm{U}(\mathrm{W(0)},\mathrm{H(0)},\mathrm{W_s(0)},\mathrm{W_r(0)}) + \mathrm{M}(e^t - 1)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\mathbb{E}U(W(t),H(t),W_s(t),W_r(t)) \\
&\leq e^{-t}U(W(0),H(0),W_s(0),W_r(0)) + M
\end{aligned}
$$

Since

$$
\begin{aligned}
&|X(t)|^v = (W^2(t) + H^2(t) + W_s^2(t) + W_r^2(t))^{\frac{v}{2}} \\
&\leq 4^{\frac{v}{2}}\max\{W^v(t),H^v(t),W_s^v(t),W_r^v(t)\} \\
&\leq 4^{\frac{v}{2}}(W^v + H^v + W_s^v + W_r^v)
\end{aligned} \tag{9}
$$

we get

$$
\mathbb{E}|X(t)|^v \leq 4^{\frac{v}{2}}(e^{-t}U(W(0),H(0),W_s(0),W_r(0)) + M).
$$

which means

$$
\limsup_{t\to\infty}\mathbb{E}|X(t)|^v \leq 4^{\frac{v}{2}}M < \infty.
$$

This implies that there is a constant $\theta_1$, such that

$$
\limsup_{t\to\infty}\mathbb{E}|\sqrt{X(t)}| \leq \theta_1.
$$

Then, given any $\epsilon > 0$, choose $\theta = \frac{\theta_1^2}{\epsilon^2}$, applying Chebyshev's inequality, we have

$$
\mathbb{P}[|X(t)>\theta|] \leq \frac{\mathbb{E}|\sqrt{X(t)}|}{\sqrt{\theta}}
$$

Hence

$$
\limsup_{t\to\infty}\mathbb{P}[|X(t) \;\; > 0|] \leq \frac{\theta_1}{\sqrt{\theta}}. = \epsilon.
$$

This completes the proof.

2.2.3 Stochastically permanent

**Definition 2.6**

The solution $\mathrm{X(t)} = (\mathrm{W(t)},\mathrm{H(t)},\mathrm{W_s(t)},\mathrm{W_r(t)})$ of system 2 is said to be stochastically permanent. if any $\epsilon \in (0,1)$, there exists a pair of positive constant $\theta$ and $\chi$ such that for any initial value $(\mathrm{W(0)},\mathrm{H(0)},\mathrm{W_s(0)},\mathrm{W_r(0)}) \in \mathbb{R}_+^4$, the solution X(t) to model (2) has the properties [28].

$$
\liminf_{t\to\infty}\mathbb{P}(|X(t)\leq\theta|) \geq 1-\epsilon \tag{10}
$$

$$
\liminf_{t\to\infty}\mathbb{P}(|X(t)\geq\chi|) \geq 1-\epsilon \tag{11}
$$

**Theorem 2.7** *Assume* $\mu+\mu_1<\Lambda$ *and for any positive initial value* $(W(0),H(0),W_s(0),W_r(o)) \in \mathbb{R}_+^4$ *the solution* $(W(t),H(t),W_s(t),W_r(t))$ *satisfies*

$$
\limsup_{\to\infty}\mathbb{E}(|X(t)|^{-p}) \leq Q \tag{12}
$$

where, p is a positive constant satisfying

$$\frac{p+1}{2}\max[\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2] < \Lambda-(\mu+\mu_1) \quad (13)$$

$$Q=\frac{4^p(4\eta B_1+B_2)}{4\eta B_1}\max\left[1,\left(\frac{2B_1+B_2+\sqrt{B_2^2+4B_1B_2}}{2B_1}\right)^{p-2}\right] \quad (14)$$

where, $\eta>0$ is a positive constant satisfying

$$\begin{aligned}\eta &< \Lambda-(\mu+\mu_1)-\frac{p+1}{2}\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)\\ B_1 &= \Lambda-(\mu+\mu_1)-\frac{p+1}{2}\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)-\eta\\ B_2 &= (\mu+\mu_1)+\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)+2\eta\end{aligned} \quad (15)$$

**Proof.** Let

$$V(W,H,W_s,W_r)=\frac{1}{W+H+W_s+W_r} \quad (16)$$

for $(W(t)+H(t)+W_s(t)+W_r(t))\in\mathbb{R}_+^4$, applying Ito's formula, we get

$$\begin{aligned}&dV(W+H+W_s+W_r)\\ &=-V^2[\Lambda-(\mu+\mu_1)H]dt\\ &+V^3[\sigma_1^2W^2,\sigma_2^2H^2,\sigma_3^2W_s^2,\sigma_4^2W_r^2]dt\\ &-V^2[\sigma_1WdW_1,\sigma_2HdW_2,\sigma_3W_sdW_3,\sigma_4W_rdW_4]\end{aligned}$$

Choosing a positive constant p satisfy (13) and using Ito's formula, we obtain

$$\begin{aligned}\mathcal{L}[(1+V)^p]&=p(1+V)^{p-1}\{-V^2[\Lambda-(\mu+\mu_1)H]+V^3[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]\}\end{aligned}$$

$$\begin{aligned}\frac{p(p-1)}{2}V^4(1+V)^{p-2}[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]\\ =p(1+V)^{p-2}\{-V^2[\Lambda-(\mu+\mu_1)H]\\ -V^3[\Lambda-(\mu+\mu_1)H]\\ +V^3[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]\end{aligned}$$

$$\begin{aligned}\frac{(p-1)}{2}V^4[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]\}\\ =p(1+V)^{p-2}Q\end{aligned}$$

where

$$Q=-V^2[\Lambda-(\mu+\mu_1)H]-V^3[\Lambda-(\mu+\mu_1)H]$$

$$V^3[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]+\frac{(p-1)}{2}V^4[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]$$

Using the facts

$$V^3[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]<\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)V$$

$$V^4[\sigma_1^2W^2+\sigma_2^2H^2+\sigma_3^2W_s^2+\sigma_4^2W_r^2]<\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)V^2$$

$$Q\le(\mu+\mu_1)+\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)V-[\Lambda-(\mu+\mu_1)-\frac{p+1}{2}\max(\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2)]V^2$$

Let $\eta>0$ be sufficiently small positive such that it satisfies (15) by Ito's formula, we get

$$\begin{aligned}\mathcal{L}[e^{\eta t}(1+V)^p]&=\eta e^{\eta t}(1+V)^p+e^{\eta t}\mathcal{L}[(1+V)^p]\\ &=e^{\eta t}(1+V)^{p-2}[\eta(1+V)^2+Q]\\ &=e^{\eta t}(1+V)^{p-2}[\eta-B_1V^2+B_2V]\\ &\le Q_0e^{\eta t}\end{aligned}$$

where

$$Q=\frac{(4\eta B_1+B_2)}{4\eta B_1}\max\left[1,\left(\frac{2B_1+B_2+\sqrt{B_2^2+4B_1B_2}}{2B_1}\right)^{p-2}\right]$$

$B_1,B_2$ are already defined in the theorem, Thus,

$$\mathbb{E}[e^{\eta t}(1+V)^p]\le[1+V(0)]^p+\frac{Q_0}{\eta}e^{\eta t}$$

Hence

$$\limsup_{t\to\infty}\mathbb{E}[V(t)^p]\le\limsup_{t\to\infty}\mathbb{E}[1+V]^p\le\frac{Q_0}{\eta}$$

$$\begin{aligned}(W+H+W_s+W_r)P&\le4^p[W^2+H^2+W_s^2+W_r^2]^{\frac{p}{2}}\\ &\le4^p|X(t)|^p\end{aligned}$$

Consequently,

$$\limsup_{t\to\infty}\mathbb{E}[\frac{1}{|X(t)|^p}]\le\limsup_{t\to\infty}\mathbb{E}[V(t)^p]\le\frac{4^pQ_0}{\eta}=Q$$

which completes the proof.

**Theorem 2.8** *Assume $max[\sigma_1^2,\sigma_2^2,\sigma_3^2,\sigma_4^2]<2(\Lambda-(\mu+\mu_1))$, then the solutions of systems (2) are stochastically permanent.*

**Proof.** Theorem (2.4), we have

$$\mathbb{P}\{|X(t)|>\theta\}\le\epsilon,$$

$$\mathbb{P}\{|X(t)|\le\theta\}\ge1-\epsilon,$$

This follows that

$$\liminf_{t\to\infty}\mathbb{P}\{|X(t)|\le\theta\}\ge1-\epsilon,$$

using theorem (2.6), we get

$$\limsup_{t\to\infty}\mathbb{P}[\frac{1}{|X(t)|^p}]\le Q$$

For any $\epsilon>0$, Let $\chi=\frac{\epsilon^p}{Q^p}$, then

$$\mathbb{P}[X(t) < \chi] = \mathbb{P}[\frac{1}{|X(t)|} > \frac{1}{\chi}] \leq \chi^{\frac{1}{p}} \mathbb{E}(|X(t)|^{-p})$$

hence,

$$\limsup_{t\to\infty} \mathbb{P}[X(t) < \chi] \leq \chi^{\frac{1}{p}} Q = \epsilon$$

which gives

$$\limsup_{t\to\infty} \mathbb{P}[X(t) \geq \chi] \geq 1 - \epsilon$$

The proof is complete.

## 3. P-TH MOMENT

**Lemma 3.1** *Set $p \geq 2$ and $\epsilon$, $x$, $y > 0$ then*

$$x^{p-1} y \leq \frac{(p-1)\epsilon}{p} \chi^p + \frac{1}{p\epsilon^{p-1}} y^p$$
$$x^{p-2} y^2 \leq \frac{(p-2)\epsilon}{p} \chi^p + \frac{2}{p\epsilon^{\frac{p-2}{2}}} y^p \quad (17)$$

using the Lemma (17) to prove the following Theorem (3.2).

**Theorem 3.2** Let $p \geq 2$. If the condition $\beta + \mu + \mu_1 - \frac{1}{2}(p-1)\sigma_2^2 - \alpha_2 \epsilon^{1-p} > 0$, $\delta_1 - \frac{1}{2}(p-1)\sigma_3^2 > 0$ and $p\delta_2 - p(p-1)\frac{\sigma_4^2}{2} > 0$ hold the equilibrium of stochastic model (2) is pth-moment exponentially stable.

**Proof.**

$$V_3 = (1-W)^p + \frac{1}{p} H^p + \frac{1}{p} W_s^p + W_r^p \quad (18)$$

By virtue of Ito's formula, we have

$$\begin{aligned}\mathcal{L}V_3 = &-p(1-W)^{p-1}[\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r] \\ &+ H^{p-1}[\alpha_2 WH - \beta H - \mu H - \mu_1 H] \\ &+ W_s^{p-1}[\alpha_1 W + \beta H - \delta_1 W_s] \\ &+ pW_r^{p-1}[\delta_1 W_s - \delta_2 W_r] + \frac{1}{2}[p(p-1)(1 \\ &- W)^{p-2}\sigma_1^2 W^2] + \frac{1}{2}[(p-1)H^{p-2}\sigma_2^2 H^2] \\ &+ \frac{1}{2}[(p-1)W_s^{p-2}\sigma_3^2 W_s^2] + \frac{1}{2}[p(p \\ &- 1)W_r^{p-2}\sigma_4^2 W_r^2]\end{aligned}$$

After a little algebra, we have

$$\begin{aligned}\mathcal{L}V_3 = &-p(1-W)^{p-1}\Lambda + p(1-W)^{p-1}\alpha_1 W + p(1 \\ &- W)^{p-1}\alpha_2 WH - p(1-W)^{p-1}\delta_2 W_r \\ &+ \alpha_2 WH^p - \beta H^p - \mu H^p - \mu_1 H^p \\ &+ \alpha_1 WW_s^{p-1} + \beta HW_s^{p-1} - \delta_1 W_s^p \\ &+ p\delta_1 W_r^{p-1} W_s - p\delta_2 W_r^p + \frac{1}{2}[p(p-1)(1 \\ &- W)^{p-2}\sigma_1^2 W^2] + \frac{1}{2}[(p-1)H^p\sigma_2^2] \\ &+ \frac{1}{2}[(p-1)W_s^p\sigma_3^2] + \frac{1}{2}[p(p-1)W_r^p\sigma_4^2]\end{aligned}$$

In $\mathbb{D}$, we have $\max\{W, H, W_s, W_r\} \leq 1$, hence

$$\begin{aligned}\mathcal{L}V_3 \leq &-p(1-W)^{p-1}\Lambda + p(1-W)^{p-1}\alpha_1 + p(1 \\ &- W)^{p-1}\alpha_2 H - p(1 \\ &- W)^{p-1}\delta_2 W_r + \alpha_2 H^p - \beta H^p \\ &- \mu H^p - \mu_1 H^p + \alpha_1 W_s^{p-1} \\ &+ \beta H W_s^{p-1} - \delta_1 W_s^p \\ &+ p\delta_1 W_r^{p-1} W_s - p\delta_2 W_r^p + \frac{1}{2}[p(p \\ &- 1)(1-W)^{p-2}\sigma_1^2 W^2] + \frac{1}{2}[(p \\ &- 1)H^p\sigma_2^2] + \frac{1}{2}[(p-1)W_s^p\sigma_3^2] \\ &+ \frac{1}{2}[p(p-1)W_r^p\sigma_4^2]\end{aligned} \quad (19)$$

Now, apply the Lemma (17) for any $\epsilon \geq 0$, we obtain

$$(1-W)^{p-1}H \leq \frac{(p-1)\epsilon}{p}(1-W)^p + \frac{1}{p\epsilon^{p-1}} H^p$$
$$(1-W)^{p-1}W_r \leq \frac{(p-1)\epsilon}{p}(1-W)^p + \frac{1}{p\epsilon^{p-1}} W_r^p$$
$$W_s^{p-1}H \leq \frac{(p-1)\epsilon}{p} W_s^p + \frac{1}{p\epsilon^{p-1}} H^p$$
$$W_r^{p-1}W_s \leq \frac{(p-1)\epsilon}{p} W_r^p + \frac{1}{p\epsilon^{p-1}} W_s^p$$
$$(1-W)^{p-2}W^2 \leq \frac{(p-2)\epsilon}{p}(1-W)^p + \frac{2}{p\epsilon^{\frac{p-2}{2}}} W^p$$

substituting the above inequalities in (19), we have

$$\begin{aligned}\mathcal{L}V_3 \leq &-p(1-W)^{p-1}\Lambda + p(1-W)^{p-1}\alpha_1 \\ &+ p\alpha_2\left[\frac{(p-1)\epsilon}{p}(1-W)^p + \frac{1}{p\epsilon^{p-1}} H^p\right] \\ &- p\delta_2\left[\frac{(p-1)\epsilon}{p}(1-W)^p + \frac{1}{p\epsilon^{p-1}} W_r^p\right] \\ &+ \alpha_2 H^p - \beta H^p - \mu H^p - \mu_1 H^p + \alpha_1 W_s^{p-1} \\ &+ \beta\left[\frac{(p-1)\epsilon}{p} W_s^p + \frac{1}{p\epsilon^{p-1}} H^p\right] - \delta_1 W_s^p \\ &+ p\delta_1\left[\frac{(p-1)\epsilon}{p} W_r^p + \frac{1}{p\epsilon^{p-1}} W_s^p\right] \\ &- p\delta_2 W_r^p \\ &+ \frac{1}{2}p(p-1)\sigma_1^2\left[\frac{(p-2)\epsilon}{p}(1-W)^p \right. \\ &\left. + \frac{2}{p\epsilon^{\frac{p-2}{2}}} W^p\right] + \frac{1}{2}[(p-1)H^p\sigma_2^2] + \frac{1}{2}[(p \\ &- 1)W_s^p\sigma_3^2] + \frac{1}{2}[p(p-1)W_r^p\sigma_4^2]\end{aligned}$$

Simplifying, we get

$$\mathcal{L}V_3 \le (1-W)^p[\alpha_2(p-1)\epsilon - \delta_2(p-1)\epsilon + \frac{1}{2}(p-1)(p-2)\epsilon\sigma_1^2](1-W)^{p-1}[-p\Lambda + p\alpha_1] + (p-1)\epsilon^{\frac{p-2}{2}}\sigma_1^2 W^p - H^p\left[-\alpha_2 + \beta + \mu + \mu_1 - \frac{1}{2}(p-1)\sigma_2^2 - \alpha_2\epsilon^{1-p} - \frac{\beta}{p}\epsilon^{1-p}\right] - W_s^p\left[\delta_1 - \frac{1}{2}(p-1)\sigma_3^2 - \delta_1\epsilon^{1-p} - \beta\frac{(p-1)\epsilon}{p}\right] - W_r^p\left[p\delta_2 - p(p-1)\frac{\sigma_4^2}{2} + \delta_2\epsilon^{1-p} - \delta_1\epsilon^{1-p}\right] \quad (20)$$

We choose $\epsilon$ sufficiently small such that the co-efficient of $(1-W)^p$ be negative and as $\beta + \mu + \mu_1 - \frac{1}{2}(p-1)\sigma_2^2 - \alpha_2\epsilon^{1-p} > 0$, $\delta_1 - \frac{1}{2}(p-1)\sigma_3^2 > 0$ and $p\delta_2 - p(p-1)\frac{\sigma_4^2}{2} > 0$ the co-efficients of W, $W_s$ and $W_r$ be negative.

## 4. PERSISTENCE

The endemic equilibrium $E = (W^*, H^*, W_s^*, W_r^*)$ where $W^* = \frac{k_1}{\alpha_2}$, $H^* = \frac{\Lambda}{k_1 - \beta}$, $W_s^* = \frac{\Lambda\alpha_2\beta - \alpha_1 k_1\beta + \alpha_1 k_1^2}{\delta_1\alpha_2(k_1-\beta)}$, $W_r^* = \frac{\Lambda\alpha_2\beta - \alpha_1 k_1\beta + \alpha_1 k_1^2}{\delta_2\alpha_2(k_1-\beta)}$ for the deterministic model (1) is globally stable but there is no endemic equilibrium is stochastic model (2). In this part, we will demonstrate the persistence of the stochastic model (2).

**Theorem 4.1** let $(W(t), H(t), W_s(t), W_r(t)) \in \mathbb{R}_+^4$ be the solution of stochastic model (2) with any initial value $(W(0), H(0), W_s(0), W_r(0))$. If $\sigma_1 > 0, \sigma_2 > 0, \sigma_3 > 0, \sigma_4 > 0$, then the stochastic model (2) is globally asymptotically stable at endemic equilibrium point if

$$\lim_{t\to\infty}\frac{1}{2}E\int_0^t \{l_1(W(s)-W^*)^2 + l_2(H(s)-H^*)^2 + l_3(W_s(s)-W_s^*)^2 + l_4(W_r(s)-W^*)^2\}ds \le l. \quad (21)$$

where,

$$l_1 = \left[\alpha_1 - \left(1 + \frac{(\alpha_1+\delta_2)}{2\delta_1}\right)\sigma_1^2\right] > 0,$$

$$l_2 = \left[(\beta+\mu+\mu_1) - \left(1 + \frac{\beta^2}{2\delta_2}\right)\sigma_2^2\right] > 0,$$

$$l_3 = (\delta_1 - \sigma_3^2) > 0, l_4 = (\delta_2 - \sigma_4^2) > 0$$

$$l = \left[W^{*2} + \frac{(\alpha_1+\delta_2)W^{*2}}{2\delta_1} + \frac{W^*}{2}\right]\sigma_1^2 + \left[H^{*2} + \frac{\beta^2}{2\delta_2} + \frac{H^*}{2}\right]\sigma_2^2 + \left[W_s^{*2} + \frac{(\alpha_1+\delta_2)W^*W_s^*}{2\delta_1} + \frac{W_s^*}{2}\right]\sigma_3^2 + \left[W_r^{*2} + \frac{\beta^2 H^* W_r^*}{2\delta_2} + \frac{W_r^*}{2}\right]\sigma_4^2$$

**Proof.** For our proof we will use some concepts from [23, 29, 30]. there is a unique endemic equilibrium point $E = (W^*, H^*, W_s^*, W_r^*)$ of the model (1)

$$\Lambda = \alpha_1 W^* + \alpha_2 W^* H^* - \delta_2 W_r^*; \quad \alpha_2 = \frac{(\beta+\mu+\mu_1)}{W^*}$$

$$\alpha_1 W^* = \delta_1 W_s^* - \beta H^*; \quad \delta_2 = \frac{\delta_1 W_s^*}{W_r^*}$$

Define the function $V_{41}, V_{42}, V_{43}, V_{44}, V_{45}, V_{46}$ and $V_{47}$ are defined for $(W, H, W_s, W_r) \in \mathbb{R}_+^4$

$$V_{41} = \frac{1}{2}[(W - W^*) + (W_s - W_s^*)]^2;$$

$$V_{42} = \frac{1}{2}[W - W^*]^2;$$

$$V_{43} = W_s - W_s^* - W_s^*\log\frac{W_s}{W_s^*}$$

$$V_{44} = \frac{1}{2}[(H - H^*) + (W_r - W_r^*)]^2;$$

$$V_{45} = \frac{1}{2}[H - H^*]^2;$$

$$V_{46} = W_r - W_r^* - W_r^*\log\frac{W_r}{W_r^*}$$

$$V_{47} = \left(W - W^* - W^*\log\frac{W}{W^*}\right) + \left(W_s - W - s^* - W_s^*\log\frac{W_s}{W_s^*}\right) + \left(H - H^* - H^*\log\frac{H}{H^*}\right) + \left(W_r - W_r^* - W_r^*\log\frac{W_r}{W_r^*}\right)$$

By using Ito's formula, we have

$$\mathcal{L}V_{41} = [(W - W^*) + (W_s - W_s^*)][\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r + \alpha_1 W + \beta H - \delta_1 W_s] + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_3^2 W_s^2] = [(W - W^*) + (W_s - W_s^*)]$$

$$[\alpha_1 W^* + \alpha_2 W^* H^* - \delta_2 W_r^* - \alpha_1 W - \alpha_2 WH + \delta_2 W_r + \frac{\delta_1 W_s^*}{W^*} - \frac{\beta H^*}{W^*} + \beta H - \delta_1 W_s] + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_3^2 W_s^2] = [(W - W^*) + (W_s - W_s^*)]$$

$$
\begin{aligned}
&[-\alpha_1(W-W^*)-\alpha_2(WH-W^*H^*)-\delta_2(W_r-W_r^*)-\delta_1(W_s-W_s^*)-\beta(H-H^*)]+\frac{1}{2}[\sigma_1^2W^2+\sigma_3^2W_s^2] \\
&=-\alpha_1(W-W^*)^2-\delta_1(W_s-W_s^*)^2-\delta_2(W_s-W_s^*)(W_r-W_r^*)-\alpha_1(W_s-W_s^*)(W-W^*)+\frac{1}{2}[\sigma_1^2W^2+\sigma_3^2W_s^2] \\
&\leq -\alpha_1(W-W^*)^2-\delta_1(W_s-W_s^*)^2+\frac{\alpha_1(\alpha_1+\delta_2)}{\delta_1}(W-W^*)^2+\frac{1}{2}[\sigma_1^2W^2+\sigma_3^2W_s^2]
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\mathcal{L}V_{42}&=[(W-W^*)][\Lambda-\alpha_1W-\alpha_2WH+\delta_2W_r]+\frac{1}{2}[\sigma_1^2W^2] \\
&=[(W-W^*)][\alpha_1W^*+\alpha_2W^*H^*-\delta_2W_r^*-\alpha_1W-\alpha_2WH+\delta_2W_r]+\frac{1}{2}[\sigma_1^2W^2] \\
&=[(W-W^*)][-\alpha_1(W-W^*)+\delta_2(W_r-W_r^*)-\alpha_2H(W-W^*)-\alpha_2W^*(H-H^*)]+\frac{1}{2}[\sigma_1^2W^2] \\
&=-\alpha_1(W-W^*)^2+\delta_2(W-W^*)(W_r-W_r^*)-\alpha_2H(W-W^*)^2-\alpha_2W^*(W-W^*)(H-H^*)+\frac{1}{2}[\sigma_1^2W^2] \\
&\leq -\alpha_1(W-W^*)^2+\delta_2(W-W^*)(W_r-W_r^*)-\alpha_2W^*(W-W^*)(H-H^*)+\frac{1}{2}[\sigma_1^2W^2]
\end{aligned} \tag{23}
$$

where the inequality in (23) is derived by $-\alpha_2H(W-W^*)^2\leq 0$.

$$
\begin{aligned}
\mathcal{L}V_{43}&=[(1-\frac{W_s^*}{W_s})][\alpha_1W-\beta H-\delta_1W_s]+\frac{1}{2}[\sigma_3^2W_s^*] \\
&=[(1-\frac{W_s^*}{W_s})][\beta H-\delta_1W_s+\frac{\delta_1W_s^*-\beta H}{W^*}W]+\frac{1}{2}[\sigma_3^2W_s^*] \\
&=\beta(H-H^*)-\delta_1(W_s-W_s^*)+(\delta_1W_s^*-\beta H^*)\left[\frac{W}{W^*}+\frac{H}{H^*}-\frac{W_s}{W_s^*}-\frac{(\delta_1W_s-\beta H)W^*}{(\delta_1W_s^*-\beta H^*)W}\right]+\frac{1}{2}[\sigma_3^2W_s^*] \\
&\leq \beta(H-H^*)-\delta_1(W_s-W_s^*)+\frac{1}{2}[\sigma_3^2W_s^*]
\end{aligned} \tag{24}
$$

The second Inequality is derived from the fact.
$\log x \leq$ x-1, ∀ x ≥0 and last inequality implied by

$$\left[\frac{W}{W^*}+\frac{H}{H^*}-\frac{W_s}{W_s^*}-\frac{(\delta_1W_s-\beta H)W^*}{(\delta_1W_s^*-\beta H^*)W}\right]\leq 0.$$

$$
\begin{aligned}
\mathcal{L}V_{44}&=[(H-H^*)+(W_r-W_r^*)][\alpha_2WH-(\beta+\mu+\mu_1)H+\delta_1W_s-\delta_2W_r]+\frac{1}{2}[\sigma_2^2H^2+\sigma_4^2W_r^2] \\
&=[(H-H^*)+(W_r-W_r^*)][(\beta+\mu+\mu_1)H^*-(\beta+\mu+\mu_1)H+\delta_2W_r^*-\delta_2W_r]+\frac{1}{2}[\sigma_2^2H^2+\sigma_4^2W_r^2] \\
&=[(H-H^*)+(W_r-W_r^*)][-(\beta+\mu+\mu_1)(H-H^*)-\delta_2(W_r-W_r^*)]+\frac{1}{2}[\sigma_2^2H^2+\sigma_4^2W_r^2] \\
&=-(\beta+\mu+\mu_1)(H-H^*)^2-\delta_2(W_r-W_r^*)^2-(\beta+\mu+\mu_1+\delta_2)(W_r-W_r^*)(H-H^*)+\frac{1}{2}[\sigma_2^2H^2+\sigma_4^2W_r^2] \\
&\leq -(\beta+\mu+\mu_1)(H-H^*)^2-\delta_2(W_r-W_r^*)^2+\frac{\beta^2(\beta+\mu+\mu_1+\delta_2)}{\delta_2}(H-H^*)+\frac{1}{2}[\sigma_2^2H^2+\sigma_4^2W_r^2]
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\mathcal{L}V_{45}&=[(H-H^*)][\alpha_2WH-(\beta+\mu+\mu_1)H]+\frac{1}{2}[\sigma_2^2H^2] \\
&=[(H-H^*)][(\beta+\mu+\mu_1)H^*-(\beta+\mu+\mu_1)H]+\frac{1}{2}[\sigma_2^2H^2] \\
&=[(H-H^*)][-(\beta+\mu+\mu_1)(H-H^*)]+\frac{1}{2}[\sigma_2^2H^2] \\
&\leq -(\beta+\mu+\mu_1)(H-H^*)^2+\frac{1}{2}[\sigma_2^2H^2]
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
V_{46}&=\left(1-\frac{W_r^*}{W_r}\right)[\delta_1W_s-\delta_2W_r]+\frac{1}{2}[\sigma_4^2W_r^*] \\
&=\left(1-\frac{W_r^*}{W_r}\right)\left[\delta_1W_s-\frac{\delta_1W_s^*}{W_r^*}W_r\right]+\frac{1}{2}[\sigma_4^2W_r^*] \\
&=\delta_1(W_s-W_s^*)-\delta_1W_s^*\left[\frac{W_r}{W_r^*}-\frac{W_s}{W_s^*}-\frac{W_sW_r^*}{W_s^*W_r}\right]+\frac{1}{2}[\sigma_4^2W_r^*]=\delta_1(W_s-W_s^*)+\frac{1}{2}[\sigma_4^2W_r^*]
\end{aligned} \tag{27}
$$

$$\mathcal{L}V_{47} = \left(1 - \frac{W^*}{W}\right)[\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r] + \left(1 - \frac{H^*}{H}\right)[\alpha_2 WH - (\beta + \mu + \mu_1)H] + \left(1 - \frac{W_s^*}{W_s}\right)[\alpha_1 W + \beta H - \delta_1 W_s] + \left(1 - \frac{W_r^*}{W_r}\right)[\delta_1 W_s - \delta_2 W_r] + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_2 H^2 + \sigma_3^2 W_s^2 + \sigma_4^2 W_r^2]$$
$$= \left(1 - \frac{W^*}{W}\right)[\alpha_1 W^* + \alpha_2 W^* H^* - \delta_2 W_r^* - \alpha_1 W - \alpha_2 WH + \delta_2 W_r] + \left(1 - \frac{H^*}{H}\right)[\alpha_2 WH - \alpha_2 W^* H] + \left(1 - \frac{W_s^*}{W_s}\right)\left[\frac{\delta_1 W_s^* - \beta H^*}{W^*} W + \beta H - \delta_1 W_s\right] + \left(1 - \frac{W_r^*}{W_r}\right)\left[\delta_1 W_s - \frac{\delta_1 W_s^*}{W_r^*} W_r\right] + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_2 H^2 + \sigma_3^2 W_s^2 + \sigma_4^2 W_r^2]$$
$$= \alpha_1 W^*\left[2 - \frac{W}{W^*} - \frac{W^*}{W}\right] + \alpha_2 W^* H^*\left[2 - \frac{H}{H^*} - \frac{H^*}{H}\right] + (\delta_1 W_s^* - \beta H^*)\left[2 - \frac{W}{W^*} - \frac{H}{H^*} - \frac{W_s}{W_s^*} - \frac{(\delta_1 W_s - \beta H)W^*}{(\delta_1 W^* - \beta H^*)W}\right] + \delta_1 W_s^*\left[2 - \frac{W_r}{W_r^*} - \frac{W_s}{W_s^*} - \frac{W_s W_r^*}{W_s^* W_r}\right] + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_2 H^2 + \sigma_3^2 W_s^2 + \sigma_4^2 W_r^2]$$
$$\leq + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_2 H^2 + \sigma_3^2 W_s^2 + \sigma_4^2 W_r^2] \tag{28}$$

The arithmetic mean is greater than or equal to the geometric mean, it follows that

$$2 - \frac{W}{W^*} - \frac{W^*}{W} \leq 0, 2 - \frac{H}{H^*} - \frac{H^*}{H} \leq 0,$$
$$2 - \frac{W_r}{W_r^*} - \frac{W_s}{W_s^*} - \frac{W_s W_r^*}{W_s^* W_r} \leq 0, 2 - \frac{W_r}{W_r^*} - \frac{W_s}{W_s^*} - \frac{W_s W_r^*}{W_s^* W_r} \leq 0$$

From (23) and (24)

$$\mathcal{L}V_{42} + W^*\mathcal{L}V_{43} \leq -\alpha_1 (W - W^*)^2 + \frac{1}{2}\sigma_1^2 W^2 + \frac{1}{2}\sigma_3^2 W^* W_s^* \tag{29}$$

Taking (22) and (29), we have

$$\mathcal{L}V_{41} + \frac{\alpha_1 + \delta_2}{\delta_1}(\mathcal{L}V_{42} + W^*\mathcal{L}V_{43}) \leq -\alpha_1 (W - W^*)^2 - \delta_1 (W_s - W_s^*)^2 + \frac{\alpha_1(\alpha_1 + \delta_2)}{\delta_1}(W - W^*)^2 + \frac{1}{2}[\sigma_1^2 W^2 + \sigma_3^2 W_s^2] + \frac{(\alpha_1 + \delta_2)}{\delta_1}\left[-\alpha_1 (W - W^*)^2 + \frac{1}{2}\sigma_1^2 W^2 + \frac{1}{2}\sigma_3^2 W^* W_s^*\right]$$
$$\leq -\alpha_1 (W - W^*)^2 - \delta_1 (W_s - W_s^*)^2 + \frac{1}{2}\sigma_1^2 W^2 + \frac{1}{2}\sigma_3^2 W_s^2 + \frac{(\alpha_1 + \delta_2)}{2\delta_1}[\sigma_1^2 W^2 + \sigma_3^2 W^* W_s^*] \tag{30}$$

From (26) and (27),

$$\mathcal{L}V_{45} + H^*\mathcal{L}V_{46} \leq -(\beta + \mu + \mu_1)(H - H^*)^2 + \frac{1}{2}\sigma_2^2 H^2 + \frac{1}{2}\sigma_4^2 H^* W_r^* \tag{31}$$

By (25) and (31),

$$\mathcal{L}V_{44} + \frac{\beta^2}{\delta_2}(\mathcal{L}V_{45} + H^*\mathcal{L}V_{46}) \leq -(\beta + \mu + \mu_1)(H - H^*)^2 - \delta_2 (W_r - W_r^*)^2 + \frac{\beta^2(\beta + \mu + \mu_1)}{\delta_2}(H - H^*) + \frac{1}{2}[\sigma_2^2 H^2 + \sigma_4^2 W_r^2] + \frac{\beta^2}{\delta_2}\left[-(\beta + \mu + \mu_1)(H - H^*)^2 + \frac{1}{2}\sigma_2^2 H^2 + + \frac{1}{2}\sigma_4^2 H^* W_r^*\right]$$
$$\leq -(\beta + \mu + \mu_1)(H - H^*)^2 - \delta_2 (W_r - W_r^*)^2 + \frac{1}{2}\sigma_2^2 H^2 + \frac{1}{2}\sigma_4^2 W_r^2 + \frac{\beta^2}{2\delta_2}[\sigma_2^2 H^2 + \sigma_4^2 H^* W_r^*] \tag{32}$$

$$\mathcal{L}V_4 = \mathcal{L}V_{41} + \frac{\alpha_1 + \delta_2}{\delta_1}(\mathcal{L}V_{42} + W^*\mathcal{L}V_{43}) + \mathcal{L}V_{44} + \frac{\beta^2}{\delta_2}(\mathcal{L}V_{45} + H^*\mathcal{L}V_{46}) + \mathcal{L}V_{47}$$
$$\leq -\alpha_1 (W - W^*)^2 - \delta_1 (W_s - W_s^*)^2 + \frac{1}{2}\sigma_1^2 W^2 + \frac{1}{2}\sigma_3^2 W_s^2 + \frac{(\alpha_1 + \delta_2)}{2\delta_1}[\sigma_1^2 W^2 + \sigma_3^2 W^* W_s^*] - (\beta + \mu + \mu_1)(H - H^*)^2 - \delta_2 (W_r - W_r^*)^2 + \frac{1}{2}\sigma_2^2 H^2 + \frac{1}{2}\sigma_4^2 W_r^2 + \frac{\beta^2}{2\delta_2}[\sigma_2^2 H^2 + \sigma_4^2 H^* W_r^*] \tag{33}$$

Using the inequality $a^2 = 2(a-b)^2 + 2b^2$, ∀ a, b ∈ ℝ. we have

$$\begin{aligned}\mathcal{L}V_4 \leq & -\alpha_1(W-W^*)^2 - \delta_1(W_s-W_s^*)^2 + \frac{1}{2}\sigma_1^2[2(W-W^*)^2+2W^{*2}] + \frac{1}{2}\sigma_3^2[2(W_s-W_s^*)^2+2W_s^{*2}] \\ & + \frac{(\alpha_1+\delta_2)}{2\delta_1}[\sigma_1^2(2(W-W^*)^2 + 2W^{*2}) + \sigma_3^2 W^*W_s^*] - (\beta+\mu+\mu_1)(H-H^*)^2 - \delta_2(W_r-W_r^*)^2 \\ & + \frac{1}{2}\sigma_2^2[2(H-H^*)^2+2H^{*2}] + \frac{1}{2}\sigma_4^2[2(W_r-W_r^*)^2+2W_r^{*2}] + \frac{\beta^2}{2\delta_2}[\sigma_2^2(2(H-H^*)^2+2H^{*2}) + \sigma_4^2 H^*W_r^*] \\ \leq & -\left[\alpha_1 - \left(1+\frac{(\alpha_1+\delta_2)}{2\delta_1}\right)\sigma_1^2\right](W-W^*)^2 - \left[(\beta+\mu+\mu_1) - (1+\frac{\beta^2}{2\delta_2})\sigma_2^2\right](H-H^*)^2 - (\delta_1-\sigma_3^2)(W_s-W_s^*)^2 - (\delta_2-\sigma_4^2)(W_r-W_r^*)^2 \\ & + \left[W^{*2}+\frac{(\alpha_1+\delta_2)W^{*2}}{2\delta_1}+\frac{W^*}{2}\right]\sigma_1^2 + \left[H^{*2}+\frac{\beta^2}{2\delta_2}+\frac{H^*}{2}\right]\sigma_2^2 + \left[W_s^{*2}+\frac{(\alpha_1+\delta_2)W^*W_s^*}{2\delta_1}+\frac{W_s^*}{2}\right]\sigma_3^2 + \left[W_r^{*2}+\frac{\beta^2H^*W_r^*}{2\delta_2}+\frac{W_r^*}{2}\right]\sigma_4^2\end{aligned} \tag{34}$$

Simplifying, we get

$$\mathcal{L}V_4 \leq -l_1(W-W^*)^2 - l_2(H-H^*)^2 - l_3(W_s-W_s^*)^2 - l_4(W_r-W_r^*)^2 + l \tag{35}$$

Integrating (35) from 0 to t and taking expectation, we get

$$\begin{aligned}0 &\leq E\big[V_4\big(W(t),H(t),W_s(t),W_r(t)\big)\big] \\ &\leq E[V_4(W(0),H(0),W_s(0),W_r(0))] \\ &\leq -E\int_0^t \{-l_1(W-W^*)^2 - l_2(H-H^*)^2 - l_3(W_s-W_s^*)^2 - l_4(W_r-W_r^*)^2\}ds + l\end{aligned} \tag{36}$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}E\int_0^t \{-l_1(W-W^*)^2 - l_2(H-H^*)^2 - l_3(W_s-W_s^*)^2 - l_4(W_r-W_r^*)^2\}ds \leq l \tag{37}$$

## 5. NUMERICAL SIMULATION

We provide some numerical results to shows the analytical result of stochastic model (2). The Eq. (2) can be rewritten as the following discretization equations

$$\begin{aligned}W(i+1) &= W(i) + [\Lambda - \alpha_1 W(i) - \alpha_2 W(i)H(i) + \delta_2 W_r(i)]\Delta t + \sigma_1 W(i)\sqrt{\Delta t}\chi(i) + \frac{\sigma_1^2}{2}W(i)(\chi(i)^2-1)\Delta t H(i+1) \\ &= H(i) + [\alpha_2 W(i)H(i) - \beta H(i) - \mu H(i) - \mu_1 H(i)]\Delta t + \sigma_1 H(i)\sqrt{\Delta t}\chi(i) + \frac{\sigma_1^2}{2}H(i)(\chi(i)^2-1)\Delta t \\ W_s(i+1) &= W_s(i) + [\alpha_1 W(i) + \beta H(i) - \delta_1 W_s(i)]\Delta t + \sigma_1 W_s(i)\sqrt{\Delta t}\chi(i) + \frac{\sigma_1^2}{2}W_s(i)(\chi(i)^2-1)\Delta t \\ W_r(i+1) &= W_r(i) + [\delta_1 W_s(i) - \delta_2 W_r(i)]\Delta t + \sigma_1 W_r(i)\sqrt{\Delta t}\chi(i) + \frac{\sigma_1^2}{2}W_r(i)(\chi(i)^2-1)\Delta t\end{aligned} \tag{38}$$

where, $\chi(i)$, $i$=1,2,...$n$ is the Gaussian random variable N(0,1).

We choose the intensities of the noise $\sigma_1 = 0.05$, $\sigma_2 = 0.01$, $\sigma_3 = 0.03$, $\sigma_4 = 0.04$ and the other parameter values of the stochastic model (2) are chosen as: Λ=200, $\alpha_1 = 0.00067$, $\beta$=0.0199, $\mu$=0.0167, $\mu_1 = 0.0143$, $\delta_1 = 0.085$, $\delta_2 = 0.067$ and the initial values are W=1.065, H=2.3478, $W_s$=1.5652, $W_r$= 1.067.

Note that,

$$\begin{aligned}l_1 &= \left[\alpha_1 - \left(1+\frac{(\alpha_1+\delta_2)}{2\delta_1}\right)\sigma_1^2\right] = 0.0072 > 0, \\ l_2 &= \left[(\beta+\mu+\mu_1) - \left(1+\frac{\beta^2}{2\delta_2}\right)\sigma_2^2\right] = 0.0531 > 0, \\ l_3 &= (\delta_1 - \sigma_3^2) = 0.0550 > 0, \\ l_4 &= (\delta_2 - \sigma_4^2) = 0.0270 > 0,\end{aligned}$$

Theorem 4.1 conditions are satisfied. The stochastic model (2) solutions fluctuate for a long time around the positive unique endemic equilibrium of the deterministic model 1 (see Figures 1 - 4).

We show the variation between the deterministic and stochastic simulation result for exhibiting the actual fluctuation/noise effect on each compartment. In all these Figures we observe that the scarcity level at each point of time is fluctuating (increasing/decreasing) depending upon the current time availability or source of the particular compartment.

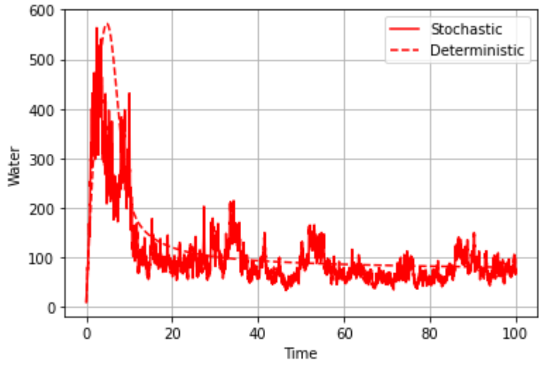


**Figure 1.** Numerical simulations of the path W(t) for the deterministic model 1 and stochastic model 2

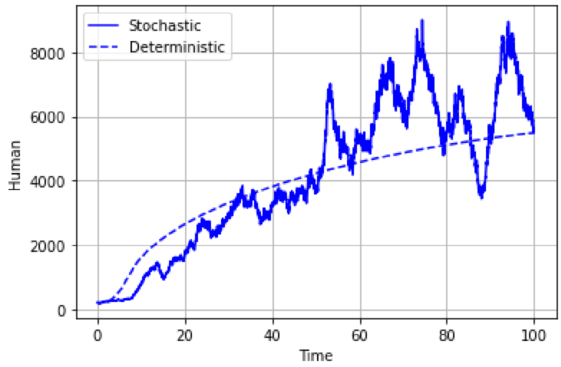


**Figure 2.** Numerical simulations of the path H(t) for the deterministic model 1 and stochastic model 2

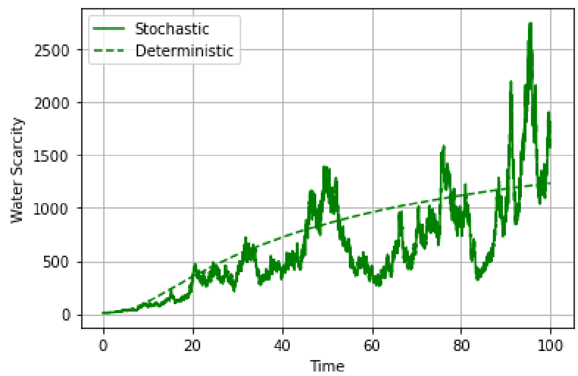


**Figure 3.** Numerical simulations of the path $W_s(t)$ for the deterministic model 1 and stochastic model 2

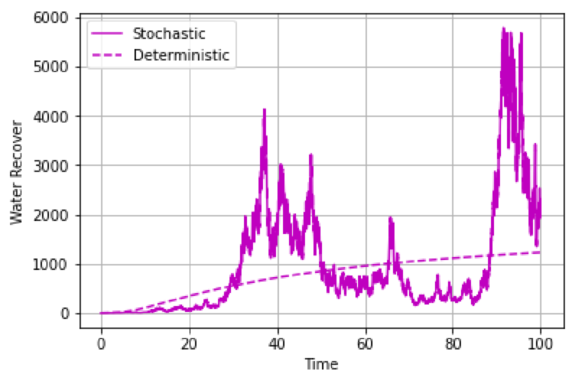


**Figure 4.** Numerical simulations of the path $W_r(t)$ for the deterministic model 1 and stochastic model 2

## 6. CONCLUSION

The most real world problems are not deterministic. The stochastic effects that take place in the deterministic model give us a more practical way to create epidemic models. In this paper, we have studied the stochastic model representing water scarcity in the society. We have formulating Lyapunov functions and applying Ito's formula then proved some fundamental qualitative properties, such as the existence of global positive solutions, boundedness and permanence solution of the proposed model (2). Persistence refers here about the equilibrium position of the model. It says that our stochastic model also exhibits the equilibrium points in and around stable equilibrium position of deterministic model.

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