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Features of Constructing a Solution Heterogeneous Equation and Clausen-Type Systems

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ABSTRACT

The possibilities of constructing a solution to an inhomogeneous the third-order equation have been studied, in particular the Clausen equation in the vicinity of special points *x*=0 and *x*=∞. The construction of the method for indeterminate coefficients to the construction of partial solutions of the Clausen equation is shown. These ideas are extended to constructing private solutions of the simple Clausen system near a regular feature with solutions in the form of a product of two hypergeometric Clausen functions, each of which depends on one variable. A number of properties of the product of Clausen functions constructed near these features have been proved. The construction of a common solution of the main heterogeneous system of Clausen is investigated. Four new functions have been created, representing the private solutions of the heterogeneous equation and systems like Clausen.

1. INTRODUCTION

The monograph by Appel and de Feriet [1] are the main literature where the basic properties of systems on differential equations in partial derivatives of the second and the third orders are studied. The monograph pays more attention to the second-order systems whose solutions are generalized hypergeometric functions of two variables and their properties have been studied in detail, as well as the relation to orthogonal polynomials of two variables. This level has not been achieved with the third-order systems research. Although some information is given about the Clausen system and its decisions in the form of the Clausen function of two variables. So far, the problems of constructing solutions to inhomogeneous systems of the third order, in particular the system and the Clausen equations have been learned.

Constructions of partial solutions on heterogeneous linear differential equations, which study heterogeneous special equations with elementary functions in the right-hand parts are considered in the works of various authors [2-4]. Usually the second-order differential equations and solutions in the form of different special functions of a single variable are usually studied. The Lommel, Struve, Anger and Weber functions are used in the applications depending on the type of the right and the ratios between the index and the degree. They are partial solutions to Bessel's heterogeneous equation.

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = x^{\mu-1} \tag{1}$$

where, μ and ν permanent.

The partial solution of the heterogeneous Bessel Eq. (1) is represented by [4] as the Lommel function:

$$\begin{aligned} S_{\mu,\nu}(x) &= x^{\mu+1}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{[(\mu+1)^2-\nu^2][(\mu+3)^2-\nu^2]\ldots[(\mu+2m+1)^2-\nu^2]} = \\ &= \frac{1}{4}x^{\mu+1}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{2}\right)^{2m}\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+m+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+m+\frac{3}{2}\right)} = \\ &= \frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}{}_1F_2\left(1;\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{3}{2};\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{3}{2};-\frac{x^2}{4}\right). \end{aligned} \tag{2}$$

The functions of Struve, Anger and Weber are special cases of this function. The relationship between them and their property is studied in the monograph [2]. It should be noted that the problems are limited to the integration of inhomogeneous Bessel equations where there are sources distributed by volume. These and other non-homogeneous hypergeometric equations have been studied in the Babister [4] monograph. This monograph is the only fundamental work dedicated to research of this nature. Sikorski and Tereshchenko investigated more general linear differential equations, a with an inhomogeneous right part in the form of a normal series by method of null coefficients [5].

Recently, in connection with the study of multidimensional degenerate equations, the properties of generalized hypergeometric functions of the third and the higher orders have often been used [6, 7].

The first example of a generalized hypergeometric function is the Clausen function [8] with five parameters.

Definition 1. View function.

$$\begin{aligned} {}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;x) &= {}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right) = \\ &= \sum_{n=0}^{\infty}\frac{(\alpha_1,n)(\alpha_2,n)(\alpha_3,n)}{(\beta_1,n)(\beta_2,n)(1,n)}\cdot x^n, \end{aligned} \tag{3}$$

where, Pochhammer notation (α, 0)=1, (α, n)=α(α+1)...(α+n-1), n>0, (1,n)=1 · 2 · ... · n=n!, is used with five parameters α_j (j=1, 2, 3) and β_l(l=1, 2) is called Clausen function.

Define a generalized hypergeometric function.

Definition 2. A generalized hypergeometric function is a function of the species.

$$ {}_pF_q\left(\alpha_1,...,\alpha_p;\beta_1,...\beta_q;x\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1,n)...(\alpha_n,n)}{(\beta_1)_n...(\beta_n)_n(1,n)}\cdot x^n \quad (4) $$

it is assumed that none of them β is an integer negative. The series (4) is reduced to a polynomial if at least one α is negative number.

Generalized hypergeometric functions of many variables have also been defined and studied.

Definition 3. Generalized hypergeometric functions of two variables are defined by double series.

$$ F(x,y)=\sum_{m,n=0}^{\infty}a_{m,n}x^m y^n \quad (5) $$

coefficients satisfying the following ratios:

$$ \frac{a_{m+1,n}}{a_{m,n}}=\frac{P(m,n)}{R(m,n)},\frac{a_{m,n+1}}{a_{m,n}}=\frac{Q(m,n)}{S(m,n)} \quad (6) $$

where, P, Q, R and S polynomials of m, n. They are subject to three conditions. Of these, the common condition is:

$$ \frac{P(m,n+1)\cdot Q(m,n)}{R(m,n+1)\cdot S(m,n)}=\frac{P(m,n)\cdot Q(m+1,n)}{R(m,n)\cdot S(m+1,n)} \quad (7) $$

provides for the unambiguous determination of the coefficients $a_{m,n}$ in the series (5).

The purpose of this work is to explore the possibility of constructing solutions to the heterogeneous Clausen equation near regular special points x=0 and x=∞, as well as Clausen-type systems near a regular feature (x=0, y=0). The study of their properties, constructed solutions in the form of the work of Clausen functions.

The work consists of three parts. The introduction summarizes the research conducted in this direction and the generalized hypergeometric functions of one and two variables. The second part shows the features of constructing solutions to Clausen's heterogeneous equation in the vicinity of regular special points x=0 and x=∞. Here, by analogy of the Lommel function, partial solutions of the Clausen equation are constructed according to the features x=0 and x=∞. The third part extends these ideas to the simple Clausen system consisting of two ordinary differential equations near regular features (0, 0). A number of properties on the product of Clausen functions constructed near these features have been proved. Then the general and private solutions of the main system of Clausen are built and a number of properties are considered.

2. SOLUTION OF THE INHOMOGENEOUS CLAUSEN EQUATION BY THE FROBENIUS- LATYSHEVA METHOD IN THE VICINITY OF REGULAR SPECIAL POINTS

Formulation of the problem. Showfeatures of constructing solutions to the inhomogeneous Clausen equation:

$$ \begin{aligned} &x^2(1-x)\frac{d^3y}{dx^3}+\left[1+\beta_1+\beta_2-(3+\alpha_1+\alpha_2+\alpha_3)x\right]x\frac{d^2y}{dx^2}+\\ &+\left[\beta_1\beta_2-(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)x\right]\frac{dy}{dx}-\\ &-\alpha_1\alpha_2\alpha_3 y=f(x), \end{aligned} \quad (8) $$

subject to the right side.

Situation where an ordinary differential equation has a right:

$$ f(x)=x^{\rho}\sum_{j=0}^{\infty}\alpha_j x^j,\alpha_0\neq 0 \quad (9) $$

has been studied in work [4].

In this case, it is efficient to use the method of indeterminate coefficients, where the partial solution is found in a generalized power series:

$$ Y(x)=x^{\lambda}\sum_{j=0}^{\infty}A_j x^j,(j=0,1,...) \quad (10) $$

and get a ratio:

$$ \begin{aligned} &x^{\lambda}\begin{Bmatrix} A_0 f_0(\lambda)+[A_1 f_0(\lambda+1)+A_0 f_1(\lambda)]x+ \\ +[A_2 f_0(\lambda+2)+A_1 f_1(\lambda+1)+A_0 f_2(\lambda)]x^2+... \end{Bmatrix}=\\ &=x^{\rho}\sum_{j=0}^{\infty}\alpha_j x^j \end{aligned} \quad (11) $$

in (11) $f_0(\lambda)$ coincides with the left part of the defining equation:

$$ \begin{aligned} &A_0 f_0(\lambda)=A_0[\lambda(\lambda-1)(\lambda-2)+\\ &+(1+\beta_1+\beta_2)\lambda(\lambda-1)]+\beta_1\beta_2\lambda=0 \end{aligned} \quad (12) $$

the corresponding uniform equation. Put λ=ρ, then (11) implies that the generalized power series (9) will be the formal solution of Eq. (8) with the right side (9) when A_j (j=0, 1, 2, ...) is satisfying a particular system:

$$ \begin{aligned} &A_0 f_0(\lambda)=\alpha_0,\\ &A_1 f_0(\lambda+1)+A_0 f_1(\lambda)=\alpha_1,\\ &A_2 f_0(\lambda+2)+A_1 f_1(\lambda+1)+A_0 f_2(\lambda)=\alpha_2, \end{aligned} \quad (13) $$

The unknown A_j (j=0, 1, 2, ...) can be determined sequentially from the system (13) provided that λ+k, where k-any natural number is not an indicator of the solution for Eq. (8). The convergence of the series in the expression (9) will lead to the convergence of the series in the expression (10).

As is known [9] the construction of a general solution of the inhomogeneous equation (2.1) consists of two parts.

Theorem 2.1. If the partial solution $Y(x)$ of heterogeneous Eq. (8) is known, the general solution is the sum of that particular solution and the general solution of the corresponding homogeneous equation.

Start constructing a partial solution $Y(x)$ (10) of the heterogeneous Eq. (8) by the method of undefined coefficients. The type of general solution of the corresponding uniform equation is established by the following theorem [9].

Theorem 2.2. The Clausen function (3) is a partial solution of the differential equation of the third-order hypergeometric type:

$$x^2(1-x)\frac{d^3y}{dx^3}+[1+\beta_1+\beta_2-(3+\alpha_1+\alpha_2+\alpha_3)x]x\frac{d^2y}{dx^2}+$$
$$+[\beta_1\beta_2-(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)x]\frac{dy}{dx}- \tag{14}$$
$$-\alpha_1\alpha_2\alpha_3 y=0,$$

and the total decision is presented as an amount.

$$\bar{y}(x)=\sum_{i=0}^{3}C_i y_i(x)=C_1F\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right)+$$
$$+C_2x^{1-\beta_1}F\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1\\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right)+ \tag{15}$$
$$+C_3x^{1-\beta_2}F\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2\\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right).$$

2.1 Features of the Frobenius -Latysheva method

By the Frobenius -Latysheva method [10], the classification of regular and irregular special points of ordinary differential equations is defined by the concept of rank:

$$p=1+k, k=\max_{(1\le s\le n)}\frac{\beta_s-\beta_0}{s} \tag{16}$$

which introduced by A. Poincare (1886) and antirank:

$$m=-1-\lambda, \lambda=\min_{(1\le s\le n)}\frac{\pi_s-\pi_0}{s} \tag{17}$$

added by L. Tome. These notions of K.Y. Latysheva were used as a basis for the classification of regular and irregular special points of ordinary differential Eq. [10].

According to this classification, if both rank $p\le 0$ and antirange ≤ 0 , then features $x=0$ and $x=\infty$ special regular.

Theorem 2.3. Let be a generalized inhomogeneous Clausen equation given:

$$L[y]=x^2(1-x)y'''+[1+\beta_1+\beta_2-(3+\alpha_1+\alpha_2+\alpha_3)x]xy''+$$
$$+[\beta_1\beta_2-(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)x]y'- \tag{18}$$
$$-\alpha_1\alpha_2\alpha_3 y=x^\rho,$$

where, ρ- permanent. Then the general solution of the generalized heterogeneous Clausen Eq. (18) is represented as the sum of the general solution $\bar{y}(x)$ of the corresponding homogeneous Clausen Eq. (14) and the private solution $Y(x)$ of the heterogeneous Eq. (18), that is, it has the form:

$$y(x)=\bar{y}(x)+Y(x)=A_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right)+$$
$$+Bx^{1-\beta_1}{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1\\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right)+$$
$$+Cx^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2\\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right)+ \tag{19}$$
$$+\frac{x^\rho}{\rho(\rho-1+\beta_1)(\rho-1+\beta_2)}{}_3F_2\left(\begin{matrix}\rho-\alpha_1, & \rho-\alpha_2, & \rho-\alpha_3\\ \rho-1+\beta_1, & \rho-1+\beta_2 & \end{matrix}\middle| x\right).$$

Proof. We know the general solution $\bar{y}(x)$ of the respective homogeneous equation. In the $A_0\neq 0$ defining equation relative to a special point x=0 there are three distinct real roots: $\lambda_1=0$, $\lambda_1=1-\beta_1$, $\lambda_2=1-\beta_2$. These roots are linear-independent partial solutions in the form of generalized power series of the homogeneous Clausen Eq. (14):

$$y_1(x)={}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right),$$
$$y_2(x)=x^{1-\beta_1}\cdot{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1\\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right), \tag{20}$$
$$y_3(x)=x^{1-\beta_2}\cdot{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2\\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right).$$

(20) constitutes the fundamental system of solutions to the Clausen equation.

It remains to construct a partial solution of the inhomogeneous Eq. (18), at $\lambda=\rho$. To this end, we construct a characteristic Frobenius function of the Eq. (18), placing the Eq. (18) in the left side of the equation $y=x^\lambda$:

$$L[x^\lambda]\equiv x^{\lambda-1}\cdot\{f_0(\lambda)+f_1(\lambda)\cdot x\}=x^\rho \tag{21}$$

where,

$$f_0(\lambda)=\lambda(\lambda-1)(\lambda-2)+(1+\beta_1+\beta_2)\cdot\lambda(\lambda-1)+\beta_1\beta_2\lambda,$$
$$f_1(\lambda)=\lambda(\lambda-1)(\lambda-2)+(3+\alpha_1+\alpha_2+\alpha_3)\cdot\lambda(\lambda-1)+ \tag{22}$$
$$+(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\rho-\alpha_1\alpha_2\alpha_3.$$

From the system the unknown constant coefficients A_j (j=0, 1, 2, ...) of the series (2.3), at $\lambda=\rho$. The coefficient A_0 shall be determined from the first equation of the system, taking into account α_0 at x^ρ that the coefficient at: $\alpha_0=1$

$$A_0=\frac{1}{\rho(\rho-1+\beta_1)\cdot(\rho-1+\beta_2)} \tag{23}$$

Since all other coefficients α_l (l=1, 2, ...) to $\alpha_0$are zero, the partial solution of the heterogeneous Eq. (18) is represented as:

$$Y(x)=\frac{x^{\rho}}{\rho(\rho-1+\beta_1)(\rho-1+\beta_2)}\{1+\frac{(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)}{(\rho+1)(\rho+\beta_1)(\rho+\beta_2)}x+$$
$$+\frac{(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)(\rho+1-\alpha_1)(\rho+1-\alpha_2)(\rho+1-\alpha_3)}{(\rho+1)(\rho+2)(\rho+\beta_1)(\rho+1+\beta_1)(\rho+\beta_2)(\rho+1+\beta_2)(\rho+\beta_3)(\rho+1+\beta_3)}\cdot$$
$$\cdot x^2+...\}=x^{\rho}\sum_{n=0}^{\infty}\frac{(\rho-\alpha_1)_n(\rho-\alpha_2)_n(\rho-\alpha_3)_n}{(\rho)_n(\rho-1+\beta_1)_n(\rho-1+\beta_2)_n}=\frac{x^{\rho}}{\rho(\rho-1+\beta_1)(\rho-1+\beta_2)}\cdot \quad (24)$$
$$\cdot{}_3F_2\left(\begin{matrix}\rho-\alpha_1, & \rho-\alpha_2, & \rho-\alpha_3\\ \rho-1+\beta_1, & \rho-1+\beta_2 & \end{matrix}\middle| x\right).$$

Therefore, the general solution of the inhomogeneous equation in the neighborhood of a special point x=0 as the sum of the general solution $\overline{y}(x)$ of the corresponding uniform Eq. (14) and the partial solution of the inhomogeneous Eq. (18) is represented (19).

Babister also called the Frobenius method for constructing a particular solution to an inhomogeneous equation. The method was used exclusively to construct the second-order decisions. The generalization to Clausen equations is the first application. The newly constructed partial solution of the inhomogeneous Clausen equation is further denoted by:

$$K_{\rho}\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right)=\frac{x^{\rho}}{\rho(\rho-1+\beta_1)(\rho-1+\beta_2)}\cdot$$
$$\cdot{}_3F_2\left(\begin{matrix}\rho-\alpha_1, & \rho-\alpha_2, & \rho-\alpha_3\\ \rho-1+\beta_1, & \rho-1+\beta_2 & \end{matrix}\middle| x\right). \quad (25)$$

2.2 Constructions of a solution to the Clausen equation in the vicinity of a regular special point at infinity

All of the special points of the Clausen equation x=0, x=1 and $x=\infty$ the regular ones. The case x=1 can be traced back to the previous case by converting x-1=t. Now we move to build the solution in the vicinity of a special point $x=\infty$ and study their properties.

Theorem 2.4. A generalized uniform Clausen Eq. (14) in the vicinity of a regular special point $x=\infty$ has three linear-independent private solutions $y_j(x)$(j=1, 2, 3) and the general solution is represented as a sum

$$\overline{y}(x)=\sum_{j=1}^{3}C_j y_j(x)=$$
$$=C_1x^{\alpha_1}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_1)_n[-(\alpha_1+\beta_1)]_n[-(\alpha_1+\beta_2)]_n}{n![-(\alpha_1-\alpha_2)]_n[-(\alpha_1-\alpha_3)]_n x^n}+$$
$$+C_2x^{\alpha_2}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_2)_n[-(\alpha_2+\beta_1)]_n[-(\alpha_2+\beta_2)]_n}{n![-(\alpha_2-\alpha_1)]_n[-(\alpha_2-\alpha_3)]_n x^n}+ \quad (26)$$
$$+C_3x^{\alpha_3}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_3)_n[-(\alpha_3+\beta_1)]_n[-(\alpha_3+\beta_2)]_n}{n![-(\alpha_3-\alpha_1)]_n[-(\alpha_3-\alpha_2)]_n x^n}.$$

Proof. To construct a fundamental system of solutions $y_j(x)$(j=1, 2, 3), we use the Frobenius-Latysheva method. For this purpose, we rewrite the characteristic function from (21) in the following form:

$$L[x^{\lambda}]\equiv x^{\lambda}\{\varphi_0(\lambda)+\frac{\varphi_1(\lambda)}{x}\}, \quad (27)$$

where, $\phi_0(\lambda)=f_1(\lambda)$, $\phi_1(\lambda)=f_0(\lambda)$.

Now in (27):

$$\varphi_0(\lambda)=f_1(\lambda)=\lambda(\lambda-1)(\lambda-2)+(3+\alpha_1+\alpha_2+\alpha_3)\lambda(\lambda-1)+$$
$$+(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)\rho-\alpha_1\alpha_2\alpha_3= \quad (28)$$
$$=(\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)=0$$

is the defining equation, the Clausen Eq. (14) with respect to a special point $x=\infty$.

The solution is found as a generalized power series in descending degrees of an independent variable x:

$$y(x)=x^{\lambda}\sum_{n=0}^{\infty}C_n x^{-n}, C_0\neq 0. \quad (29)$$

Unknown constants C_n (n=0, 1, ...) are defined from the next recurrence sequence system:

$$\begin{aligned}&C_0\phi_0(\lambda)=0,(C_0\neq 0)\\&C_1\phi_0(\lambda-1)+C_0\phi_1(\lambda)=0,\\&C_2\phi_0(\lambda-2)+C_1\phi_1(\lambda-1)+C_0\phi_2(\lambda)=0,\\&C_3\phi_0(\lambda-3)+C_2\phi_1(\lambda-2)+C_1\phi_2(\lambda-1)+C_0\phi_3(\lambda)=0,\end{aligned} \quad (30)$$

From the first equation of the system (30):

$$C_0\varphi_0(\lambda)=C_0(\lambda-\alpha_1)(\lambda-\alpha_2)(\lambda-\alpha_3)=0 \quad (31)$$

when, $C_0\neq 0$, we find three roots: $\lambda_1=\alpha_1$, $\lambda_2=\alpha_2$, $\lambda_3=\alpha_3$ defining Eq. (28) with respect to a special point $x=\infty$. By inserting the found roots into the system (30), one successively defines the unknown constant $C_n^j(j=1,2,3)$ generalized power series (29):

$$y_1(x)=x^{\alpha_1}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_1)_n[-(\alpha_1+\beta_1)]_n[-(\alpha_1+\beta_2)]_n}{n![-(\alpha_1-\alpha_2)]_n\cdot[-(\alpha_1-\alpha_3)]_n\cdot x^n},$$
$$y_2(x)=x^{\alpha_2}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_2)_n[-(\alpha_2+\beta_1)]_n[-(\alpha_2+\beta_2)]_n}{n![-(\alpha_2-\alpha_1)]_n[-(\alpha_2-\alpha_3)]_n x^n}, \quad (32)$$
$$y_3(x)=x^{\alpha_3}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_3)_n[-(\alpha_3+\beta_1)]_n[-(\alpha_3+\beta_2)]_n}{n![-(\alpha_3-\alpha_1)]_n[-(\alpha_3-\alpha_2)]_n x^n},$$

where to use the notation $\alpha\cdot(\alpha-1)\cdot(\alpha-2)\ldots(\alpha-n+1)=(-1)^n\cdot(-\alpha)_n$.

The sum of three decisions $y_j(x)$(j=1, 2, 3)represents the general solution (26) of the homogeneous Clausen Eq. (4). What needed to be proved. These solutions (32) constitute a fundamental system of solutions Clausen's Eq. (14) in the vicinity of a regular special point $x=\infty$.

Theorem 2.5. The partial solution of the heterogeneous Clausen Eq. (18) in the vicinity of a special point $x=\infty$ is represented as:

$$Y(x)=x^{\rho-1}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\rho)_n[-(\rho+\beta_1)]_n[-(\rho+\beta_2)]_n}{n![-(\rho-\alpha_1)]_n[-(\rho-\alpha_2)]_n[-(\rho-\alpha_3)]_n x^n}, \qquad (33)$$

where the species designation is used $\alpha(\alpha-1)\ldots(\alpha-n+1)=(-1)^n(-\alpha)_n$.

Proof. To prove the theorem we will use the information of the previous theorem 2.3. The solution is sought as a generalized power series (29). Only, in this case, unknown constants C_n (n=0, 1, ...) are defined from another system of recurrence sequences:

$$\begin{aligned}
&C_0\phi_0(\lambda)=\alpha_0, (\alpha_0\neq 0)\\
&C_1\phi_0(\lambda-1)+C_0\phi_1(\lambda)=\alpha_1,\\
&C_2\phi_0(\lambda-2)+C_1\phi_1(\lambda-1)+C_0\phi_2(\lambda)=\alpha_2,\\
&C_3\phi_0(\lambda-3)+C_2\phi_1(\lambda-2)+C_1\phi_2(\lambda-1)+C_0\phi_3(\lambda)=\alpha_3,
\end{aligned} \qquad (34)$$

here, we define

$$C_0=\frac{1}{(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)} \qquad (35)$$

as a coefficient $x^{\rho-1}$ is using the defining Eq. (28) from is (34). Since all subsequent coefficients α_l (l=1, 2, ...) are zero, the coefficients of the partial solution (33) C_l (l=1, 2, ...) of the heterogeneous Eq. (18) are defined as homogeneous, only expressed in (35):

$$\begin{aligned}
&Y(x)=\frac{x^{\rho-1}}{(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)}\{1+\frac{\rho(\rho+\beta_1-1)(\rho+\beta_2-1)}{(\rho-1-\alpha_1)(\rho-1-\alpha_2)(\rho-1-\alpha_3)x}+\\
&+\frac{\rho(\rho-1)(\rho+\beta_1-1)(\rho+\beta_1-2)(\rho+\beta_2-1)(\rho+\beta_2-2)}{(\rho-1-\alpha_1)(\rho-2-\alpha_1)(\rho-1-\alpha_2)(\rho-2-\alpha_2)(\rho-1-\alpha_3)(\rho-2-\alpha_3)x^2}+\ldots\}=\\
&=x^{\rho-1}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\rho)_n[-(\rho+\beta_1-1)]_n[-(\rho+\beta_2-1)]_n}{n![-(\rho-\alpha_1)]_n[-(\rho-\alpha_2)]_n[-(\rho-\alpha_3)]_n x^n}=\\
&\frac{x^{\rho}}{(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)}{}_3F_2\left(\begin{matrix}-\rho, & 1-\rho-\beta_1, & 1-\rho-\beta_2\\ \rho-\alpha_1, & \rho-\alpha_2, & \rho-\alpha_3\end{matrix}\middle| x\right).
\end{aligned} \qquad (36)$$

where, $\rho\cdot(\rho-1)\cdot(\rho-2)\cdot\ldots\cdot(\rho-n+1)=(-1)^n\cdot(-\rho)_n$. Partial decision received (2.21) nonhomogeneous Clausen equation is further denoted by:

$$\begin{aligned}
&K_\rho\left(\begin{matrix}\beta_1, & \beta_2 & \\ \alpha_1, & \alpha_2, & \alpha_3\end{matrix}\middle| x\right)=\frac{x^\rho}{(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3)}\cdot\\
&\cdot{}_3F_2\left(\begin{matrix}-\rho, & 1-\rho-\beta_1, & 1-\rho-\beta_2\\ \rho-\alpha_1, & \rho-\alpha_2, & \rho-\alpha_3\end{matrix}\middle| x\right).
\end{aligned} \qquad (37)$$

On the basis of proven theorems three and four, it is possible to formulate a general theorem on the representation of a general solution of the heterogeneous Clausen Eq. (18).

Theorem 2.6. The general solution of the heterogeneous Clausen Eq. (18) in the vicinity of the regular special point x=∞ is represented as the sum of the general solution $\overline{y}(x)$ (26) of the corresponding homogeneous Eq. (2) and the partial solution $Y(x)$ (36) of the inhomogeneous Eq. (18), that is, has the form:

$$\begin{aligned}
&y(x)=\overline{y}(x)+Y(x)=C_1x^{\alpha_1}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_1)_n[-(\alpha_1+\beta_1)]_n[-(\alpha_1+\beta_2)]_n}{n![-(\alpha_1-\alpha_2)]_n[-(\alpha_1-\alpha_3)]_n x^n}+\\
&+C_2x^{\alpha_2}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_2)_n[-(\alpha_2+\beta_1)]_n[-(\alpha_2+\beta_2)]_n}{n![-(\alpha_2-\alpha_1)]_n[-(\alpha_2-\alpha_3)]_n x^n}+\\
&+C_3x^{\alpha_3}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\alpha_3)_n[-(\alpha_3+\beta_1)]_n[-(\alpha_3+\beta_2)]_n}{n![-(\alpha_3-\alpha_1)]_n[-(\alpha_3-\alpha_2)]_n x^n}+\\
&+x^{\rho-1}\sum_{n=0}^{\infty}\frac{(-1)^{3n}(-\rho)_n[-(\rho+\beta_1-1)]_n[-(\rho+\beta_2-1)]_n}{n![-(\rho-\alpha_1)]_n[-(\rho-\alpha_2)]_n[-(\rho-\alpha_3)]_n\cdot x^n},
\end{aligned} \qquad (38)$$

where the notation is used.

$$\begin{aligned}
&\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)=(-1)^n(-\alpha)_n,\\
&\rho(\rho-1)(\rho-2)\ldots(\rho-n+1)=(-1)^n(-\rho)_n.
\end{aligned} \qquad (39)$$

Let us consider the differentiability property of Clausen functions.

Theorem 2.7. The generalized hypergeometric Clausen function (3) has m-a derivative order

$$\begin{aligned}
&\frac{d^mF}{dx^m}=\frac{\alpha_1(\alpha_1+1)\ldots(\alpha_1+m-1)\alpha_2(\alpha_2+1)\ldots(\alpha_2+m-1)\alpha_3\ldots(\alpha_3+m-1)}{\beta_1(\beta_1+1)\ldots(\beta_1+m-1)\beta_2(\beta_2+1)\ldots(\beta_2+m-1)}\cdot\\
&\cdot{}_3F_2\left(\begin{matrix}\alpha_1+m, & \alpha_2+m, & \alpha_3+m\\ \beta_1+m, & \beta_2+m & \end{matrix}\middle| x\right)=\frac{(\alpha_1)_m(\alpha_2)_m(\alpha_3)_m}{(\beta_1)_m(\beta_2)_m}\\
&\cdot{}_3F_2\left(\begin{matrix}\alpha_1+m, & \alpha_2+m, & \alpha_3+m\\ \beta_1+m, & \beta_2+m & \end{matrix}\middle| x\right).
\end{aligned} \qquad (40)$$

From here, for different values, you can get derivatives of the 1st, the 2nd, etc. of other orders. So you get m=1 the first-order derivative:

$$\frac{dF}{dx}=\frac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1+1, & \alpha_2+1, & \alpha_3+1\\ \beta_1+1, & \beta_2+1 & \end{matrix}\middle| x\right). \qquad (41)$$

In the same way, derivatives of the second and the third solutions can be found:

$$y_2(x)=x^{1-\beta_1}{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1\\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right), \qquad (42)$$

and

$$y_3(x)=x^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2\\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right). \qquad (43)$$

To determine their derivatives, we shall use the general formula [11]:

$$\begin{aligned}
&\frac{d^m}{dx^m}\left[x^\sigma\cdot{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| x\right)\right]=(\sigma-m+1)_m\cdot x^{\sigma-m}\cdot\\
&{}_{p+1}F_{q+1}\left(\begin{matrix}\sigma+1, & \alpha_p\\ \sigma+1-m, & \rho_q\end{matrix}\middle| x\right).
\end{aligned} \qquad (44)$$

For p=3, q=2, m- private derivative solutions (42) and (43) are presented as:

$$\frac{d^m y_2}{dx^m}=\frac{d^m}{dx^m}\left[x^{1-\beta_1}\cdot{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1\\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right)\right]=$$
$$(2-\beta_1-m)\cdot x^{1-\beta_1-m}\cdot \tag{45}$$
$$\cdot{}_4F_3\left(\begin{matrix}2-\beta_1, & \alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1\\ 2-\beta_1-m, & \beta_2+1-\beta_1 & & \end{matrix}\middle| x\right),$$

$$\frac{d^m y_3}{dx^m}=\frac{d^m}{dx^m}\left[x^{1-\beta_2}\cdot{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2\\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right)\right]=$$
$$=(2-\beta_2-m)\cdot x^{1-\beta_2-m}\cdot \tag{46}$$
$$\cdot{}_4F_3\left(\begin{matrix}2-\beta_2, & \alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2\\ \beta_1+1-\beta_2, & 2-\beta_2-m & & \end{matrix}\middle| x\right).$$

3. SPECIFIC FEATURES OF THE CLAUSEN TYPE SOLUTION

Problem Statement. The construction of solutions (0, 0) near a regular homogeneous system consisting of two differential equations in the third-order partial derivatives at the type is investigated:

$$\left.\begin{aligned}&\sum_{j+k=0}^{j+k=\omega+1}\left(r_{j,k}-\alpha_{j,k}\cdot x^h\right)\cdot x^j\cdot y^k\cdot p_{j,k}=0,\\ &\sum_{j+k=0}^{j+k=\omega+1}\left(t_{j,k}-\beta_{j,k}\cdot y^h\right)\cdot x^j\cdot y^k\cdot p_{j,k}=0,\end{aligned}\right\} \tag{47}$$

where, $p_{0,0}$ $(x, y)=Z(x, y)(j=0, k=0)$the common is not known for the two equations of the system (1); through the different order $p_{j,k}$ of the partial derivatives of the unknown function $Z(x, y)$. The order depends on the value ω. If $\omega=1$ we get the second order systems.

$$\begin{aligned}&x^2\left(r_{2,0}-\alpha_{2,0}\cdot x^h\right)\cdot p_{2,0}+xy\left(r_{1,1}-\alpha_{1,1}\cdot x^h\right)\cdot p_{1,1}+x\left(r_{1,0}-\alpha_{1,0}\cdot x^h\right)\cdot p_{1,0}+\\ &+y\left(r_{0,1}-\alpha_{0,1}\cdot x^h\right)\cdot p_{0,1}+\left(r_{0,0}-\alpha_{0,0}\cdot x^h\right)\cdot p_{0,0}=0,\\ &y^2\left(t_{0,2}-\beta_{0,2}\cdot y^h\right)\cdot p_{0,2}+xy\left(t_{1,1}-\beta_{1,1}\cdot y^h\right)\cdot p_{1,1}+x\left(t_{1,0}-\beta_{1,0}\cdot y^h\right)\cdot p_{1,0}+\\ &+y\left(t_{0,1}-\beta_{0,1}\cdot y^h\right)\cdot p_{0,1}+\left(t_{0,0}-\beta_{0,0}\cdot y^h\right)\cdot p_{0,0}=0,\end{aligned} \tag{48}$$

where, $p_{0,0}$ $(x, y)=Z(x, y)$ common unknown, $r_{j,k}$, $\alpha_{j,k}$, $t_{j,k}$, $\beta_{j,k}$ $(j, k=0, 2)$unknown constants; coefficients (3) - polynomials of two variables.

The case $h=1$ is the most explored. Ya.Horn proved that all 34 known hypergeometric functions, in particular four hypergeometric functions of two variables. P.Appell F_1-F_4 are solutions to individual cases of such systems. If you get systems $h=2$ whose solutions are orthogonal polynomials of two variables. Thus the solutions of the system [12] in type (48) are more than forty special functions of two variables. The establishment of this relation is important in the approximate calculation on the values of the hypergeometric function at the two variables. Some work in this direction began to be carried out in the works of the American scientist O. I. Marichev.

When $\omega=2$ we get third-order systems. The most interesting ones are systems like Clausen. The research on these systems is not well developed. In this work we will study two systems like Clausen. From them, the construction of a simple Clausen-type system solution as a product of two Clausen functions allows to reveal a number of different properties of Clausen functions of two variables.

3.1 A simple Clausen-type system with Clausen function solutions

This paper examined two types of Clausen-type system: simple and basic Clausen systems

Theorem 3.1. Clausen type system

$$\begin{aligned}&x^2(1-x)p_{30}+\left[1+\beta_1+\beta_2-(3+\alpha_1+\alpha_2+\alpha_3)x\right]xp_{20}+\\ &+\left[\beta_1\beta_2-(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)x\right]p_{10}-\alpha_1\alpha_2\alpha_3p_{00}=0,\\ &y^2(1-y)p_{03}+\left[1+\beta_1'+\beta_2'-(3+\alpha_1'+\alpha_2'+\alpha_3')y\right]yp_{02}+\\ &+\left[\beta_1'\beta_2'-(1+\alpha_1'+\alpha_2'+\alpha_3'+\alpha_1'\alpha_2'+\alpha_1'\alpha_3'+\alpha_2'\alpha_3')y\right]p_{01}-\alpha_1'\alpha_2'\alpha_3'p_{00}=0\end{aligned} \tag{49}$$

has nine linearly independent private solutions as a product of various Clausen functions, and one of them is a Clausen type function.

$$\begin{aligned}&Z_1(x,y)={}_3F_2\left(\begin{matrix}\alpha_1,\alpha_1', & \alpha_2,\alpha_2', & \alpha_3,\alpha_3'\\ \beta_1,\beta_1', & \beta_2,\beta_2' & \end{matrix}\middle| x,y\right)=\\ &=\sum_{m,n=0}^{\infty}\frac{(\alpha_1)_m(\alpha_1')_n(\alpha_2)_m(\alpha_2')_n(\alpha_3)_m(\alpha_3')_n}{(\beta_1)_m(\beta_1')_n(\beta_2)_m(\beta_2')_n}\frac{x^m}{m!}\frac{y^n}{n!}.\end{aligned} \tag{50}$$

Proof. The system (49) consists of two joint ordinary differential equations. They are united by a common unknown equation $p_{0,0}$ $(x, y)=Z(x, y)$. Applying the Frobenius-Latysheva method as in the common case where the solution of the system is sought as a generalized power series of two variables:

$$Z(x,y)=x^\rho y^\sigma\sum_{m,n=0}^{\infty}C_{m,n}x^my^n, C_{0,0}\neq 0 \tag{51}$$

ρ, σ, $C_{m,n}$ $(m,n=0, 1, 2, ...)$-unknown constants) by inserting $Z(x,y)=x^\rho y^\sigma$ into the system (47) define the system of characteristic functions of Frobenius.

The series (51) shall be determined from the system of defining equations with respect to the feature (0, 0):

$$\left.\begin{aligned}&f_{0,0}^{(1)}(\rho,\sigma)=\rho(\rho-1+\beta_1)(\rho-1+\beta_2)=0,\\ &f_{0,0}^{(2)}(\rho,\sigma)=\sigma(\sigma-1+\beta_1')(\sigma-1+\beta_2')=0,\end{aligned}\right\} \tag{52}$$

in the form of pairs:

$$\begin{aligned}&(\rho_1,\sigma_1)=(0,0),(\rho_1=0,\sigma_2=1-\beta_1),(\rho_1=0,\sigma_2=1-\beta_2),\\ &(\rho_2=1-\beta_1,\sigma_1=0),(\rho_2=1-\beta_1,\sigma_2=1-\beta_1'),(\rho_2=1-\beta_1,\sigma_3=1-\beta_2'),\\ &(\rho_3=1-\beta_2,\sigma_1=0),(\rho_3=1-\beta_2,\sigma_2=1-\beta_1'),(\rho_3=1-\beta_2,\sigma_3=1-\beta_2').\end{aligned} \tag{53}$$

These pairs determined the series (51) and the unknown coefficients $C_{m,n}(m, n=0, 1, 2, ...)$ are derived from the recurrence sequence system.

$$\sum_{m,n=0}^{\infty}C_{\mu-m,\nu-n}f_{m,n}^{(j)}(\rho+\mu-m,\sigma+\nu-n)=0,(j=1,2;\mu,\nu=0,1,2,...). \tag{54}$$

This gives nine linearly independent private solutions:

$$Z_1(x,y) = {}_3F_2(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,x)\,{}_3F_2(\alpha_1',\alpha_2',\alpha_3',\beta_1',\beta_2',y) = F\left(\begin{matrix}\alpha_1,\alpha_1', & \alpha_2,\alpha_2', & \alpha_3,\alpha_3' \\ \beta_1,\beta_1', & \beta_2,\beta_2' & \end{matrix}\middle| x,y\right) = \sum_{m,n=0}^{\infty}\frac{(\alpha_1)_m(\alpha_1')_n(\alpha_2)_m(\alpha_2')_n(\alpha_3)_m(\alpha_3')_n}{(\beta_1)_m(\beta_1')_n(\beta_2)_m(\beta_2')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{55}$$

$$Z_2(x,y) = F\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right) y^{1-\beta_1'} F\left(\begin{matrix}\alpha_1'+1-\beta_1', & \alpha_2'+1-\beta_1', & \alpha_3'+1-\beta_1' \\ 2-\beta_1', & \beta_2'+1-\beta_1' & \end{matrix}\middle| y\right), \tag{56}$$

$$Z_3(x,y) = F\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right) y^{1-\beta_2'} F\left(\begin{matrix}\alpha_1'+1-\beta_2', & \alpha_2'+1-\beta_2', & \alpha_3'+1-\beta_2' \\ \beta_2'+1-\beta_2', & 2-\beta_2' & \end{matrix}\middle| y\right), \tag{57}$$

$$Z_4(x,y) = x^{1-\beta_1}\cdot F\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right)\cdot F\left(\begin{matrix}\alpha_1', & \alpha_2', & \alpha_3' \\ \beta_1, & \beta_2 & \end{matrix}\middle| y\right), \tag{58}$$

$$Z_5(x,y) = x^{1-\beta_1}\cdot F\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_2+1-\beta_1 & \end{matrix}\middle| x\right)\cdot y^{1-\beta_1'}\cdot F\left(\begin{matrix}\alpha_1'+1-\beta_1', & \alpha_2'+1-\beta_1', & \alpha_3'+1-\beta_1' \\ 2-\beta_1', & \beta_2'+1-\beta_1' & \end{matrix}\middle| y\right), \tag{59}$$

$$Z_6(x,y) = x^{1-\beta_1}\cdot F\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_2+1-\beta_1, & \end{matrix}\middle| x\right)\cdot y^{1-\beta_2'}\cdot F\left(\begin{matrix}\alpha_1'+1-\beta_2', & \alpha_2'+1-\beta_2', & \alpha_3'+1-\beta_2' \\ \beta_1'+1-\beta_2', & 2-\beta_2' & \end{matrix}\middle| y\right), \tag{60}$$

$$Z_7(x,y) = x^{1-\beta_2}\cdot F\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2 \\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right)\cdot F\left(\begin{matrix}\alpha_1', & \alpha_2', & \alpha_3' \\ \beta_1', & \beta_2' & \end{matrix}\middle| y\right), \tag{61}$$

$$Z_8(x,y) = x^{1-\beta_2}\cdot F\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2 \\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right)\cdot y^{1-\beta_1'}\cdot F\left(\begin{matrix}\alpha_1'+1-\beta_1', & \alpha_2'+1-\beta_1', & \alpha_3'+1-\beta_1' \\ 2-\beta_1', & \beta_2'+1-\beta_1' & \end{matrix}\middle| y\right), \tag{62}$$

$$Z_9(x,y) = x^{1-\beta_2}\cdot F\left(\begin{matrix}\alpha_1+1-\beta_2, & \alpha_2+1-\beta_2, & \alpha_3+1-\beta_2 \\ \beta_1+1-\beta_2, & 2-\beta_2 & \end{matrix}\middle| x\right)\cdot y^{1-\beta_2'}\cdot F\left(\begin{matrix}\alpha_1'+1-\beta_2', & \alpha_2'+1-\beta_2', & \alpha_3'+1-\beta_2' \\ \beta_1'+1-\beta_2', & 2-\beta_2' & \end{matrix}\middle| y\right). \tag{63}$$

Theorem 3.2. The general solution of the Clausen system (49) is represented as a sum

$$\overline{Z}(x,y) = \sum_{i=1}^{9} C_i\cdot Z_i(x,y), \tag{64}$$

where, $C_i\left(i=\overline{1,9}\right)$-arbitrary permanent, Z_i $(x,\ y)$ $\left(i=\overline{1,9}\right)$-nine linearly independent private decisions (50), (56)-(63).

Simple heterogeneous system partial solution

Theorem 3.3. The simple heterogeneous system of the Clausen type

$$\begin{aligned}
&x^2(1-x)Z_{xxx} + [1+\beta_1+\beta_2-(3+\alpha_1+\alpha_2+\alpha_3)x]xZ_{xx} + \\
&+[\beta_1\beta_2-(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)x]Z_x - \\
&-\alpha_1\alpha_2\alpha_3 Z = f_1(x,y), \\
&y^2(1-y)Z_{yyy} + [1+\beta_1'+\beta_2'-(3+\alpha_1'+\alpha_2'+\alpha_3')x]xZ_{yy} + \\
&+[\beta_1'\beta_2'-(1+\alpha_1+\alpha_2+\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)x]Z_y - \\
&-\alpha_1'\alpha_2'\alpha_3' Z = f_2(x,y),
\end{aligned} \tag{65}$$

with the right part

$$\begin{aligned}
f_1(x,y) &= x^{\rho}y^{\sigma}\frac{1}{\sigma(\sigma-1+\beta_1')(\sigma-1+\beta_2')}, \\
f_2(x,y) &= x^{\rho}y^{\sigma}\frac{1}{\rho(\rho-1+\beta_1)(\rho-1+\beta_2)},
\end{aligned} \tag{66}$$

has a specific solution of type.

$$K_{\rho,\sigma}(x,y) = x^{\rho}y^{\sigma}\sum_{m,n=0}^{\infty}\frac{(\rho-\alpha_1)_m(\rho-\alpha_2)_m(\rho-\alpha_3)_m(\sigma-\alpha_1')_n(\sigma-\alpha_2')_n(\sigma-\alpha_3')_n\,x^m y^n}{(\rho)_m(\rho-1+\beta_1)_m(\rho-1+\beta_2)_m(\sigma)_n(\sigma-1+\beta_1')_n(\sigma-1+\beta_2')_n}. \tag{67}$$

Evidence. By analogy with the ordinary case (paragraph 2), we first construct a system of characteristic functions of Frobenius of the given system (62) and by substituting in place $Z(x,y)=x^{\rho}y^{\sigma}$:

$$\left.\begin{aligned}
L_1[x^{\rho}y^{\sigma}] &\equiv x^{\rho-1}y^{\sigma}\left\{f_{0,0}^{(1)}(\rho,\sigma)+f_{1,0}^{(1)}(\rho,\sigma)x\right\} \\
L_2[x^{\rho}y^{\sigma}] &\equiv x^{\rho}y^{\sigma-1}\left\{f_{0,0}^{(2)}(\rho,\sigma)+f_{0,1}^{2}(\rho,\sigma)y\right\}
\end{aligned}\right\} \tag{68}$$

where,

$$\left.\begin{aligned}
f_{0,0}^{(1)}(\rho,\sigma) &= \rho(\rho-1)(\rho-2)+(1+\beta_1+\beta_2)\rho(\rho-1)+\beta_1\beta_2\rho \\
f_{0,0}^{(2)}(\rho,\sigma) &= \sigma(\sigma-1)(\sigma-2)+(1+\beta_1'+\beta_2')\sigma(\sigma-1)+\beta_1'\beta_2'\sigma \\
f_{1,0}^{(1)}(\rho,\sigma) &= -(\rho-\alpha_1)(\rho-\alpha_3)(\rho-\alpha_3) \\
f_{0,1}^{(1)}(\rho,\sigma) &= -(\sigma-\alpha_1')(\sigma-\alpha_3')(\sigma-\alpha_3')
\end{aligned}\right\} \tag{69}$$

of these,

$$\left.\begin{aligned}
f_{0,0}^{(1)}(\rho,\sigma) &= \rho(\rho-1+\beta_1)(\rho-1+\beta_2)=0, \\
f_{0,0}^{(2)}(\rho,\sigma) &= \sigma(\sigma-1+\beta_1')(\sigma-1+\beta_2')=0,
\end{aligned}\right\} \tag{70}$$

forms a system of defining equations with respect to the feature (0,0). (70) has nine pairs of roots (53). They are indicators of decisions (56)-(63) and define the overall solution (3.17) of the homogeneous system (49).

A heterogeneous system solution is sought as a generalized power series of two variables (51). Only unlike a homogeneous system solution, in this case, unknown constants $A_{m,n}$ (m, n=0, 1, 2, ...) are defined from the next recurrence sequence system:

$$\begin{aligned}
&A_{0,0}f_{0,0}^{(j)}(\rho,\sigma)=\alpha_{0,0}^{(j)},(j=1,2) \\
&A_{1,0}f_{0,0}^{(j)}(\rho+1,\sigma)+A_{0,0}f_{1,0}^{(j)}(\rho,\sigma)=\alpha_{1,0}^{(j)}, \\
&A_{0,1}f_{0,0}^{(j)}(\rho,\sigma+1)+A_{0,0}f_{0,1}^{(j)}(\rho,\sigma)=\alpha_{0,1}^{(j)}, \\
&A_{2,0}f_{0,0}^{(j)}(\rho+2,\sigma)+A_{1,0}f_{1,0}^{(j)}(\rho+1,\sigma)+A_{1,0}f_{2,0}^{(j)}(\rho,\sigma)=\alpha_{2,0}^{(j)}, \\
&A_{1,1}f_{0,0}^{(j)}(\rho+1,\sigma+1)+A_{1,0}f_{0,1}^{(j)}(\rho+1,\sigma)+A_{0,1}f_{1,0}^{(j)}(\rho,\sigma+1)+A_{0,0}f_{1,1}^{(j)}(\rho,\sigma)=\alpha_{1,1}^{(j)}, \\
&A_{0,2}f_{0,0}^{(j)}(\rho,\sigma+2)+A_{0,1}f_{0,1}^{(j)}(\rho,\sigma+1)+A_{0,0}f_{0,2}^{(j)}(\rho,\sigma)=\alpha_{0,2}^{(j)},
\end{aligned} \tag{71}$$

$A_{0,0}$- define from the first equation of the recurrence sequence system (71) taking into account that in (64) at j=1, 2:

$$\alpha_{0,0}^{(1)}=\frac{1}{\sigma\left(\sigma-1+\beta_1^{'}\right)\left(\sigma-1+\beta_2^{'}\right)},\qquad \alpha_{0,0}^{(2)}=\frac{1}{\rho\left(\rho-1+\beta_1\right)\left(\rho-1+\beta_2\right)} \tag{72}$$

So

$$A_{0,0}=\frac{1}{\sigma\left(\sigma-1+\beta_1^{'}\right)\left(\sigma-1+\beta_2^{'}\right)}\cdot\frac{1}{\rho\left(\rho-1+\beta_1\right)\left(\rho-1+\beta_2\right)} \tag{73}$$

Then, the coefficients at the values $A_{1,0}, A_{0,1}, A_{2,0}, A_{1,1}, A_{0,2}$, ... are sequentially determined $\alpha_{1,0}, \alpha_{0,1}, \alpha_{2,0}, \alpha_{1,1}, \alpha_{0,2}$, ... to zero from the system (71). The partial solution obtained is represented as

$$\begin{aligned}K_{\rho,\sigma}(x,y)=&\frac{x^{\rho}\cdot y^{\sigma}}{\rho\cdot(\rho-1+\beta_1)\cdot(\rho-1+\beta_2)\cdot\sigma\cdot\left(\sigma-1+\beta_1^{'}\right)\cdot\left(\sigma-1+\beta_2^{'}\right)}\cdot\\ &\{1+\frac{(\rho-\alpha_1)\cdot(\rho-\alpha_2)\cdot(\rho-\alpha_3)}{(\rho+1)\cdot(\rho+\beta_1)\cdot(\rho+\beta_2)}\cdot x+\frac{\left(\sigma-\alpha_1^{'}\right)\cdot\left(\sigma-\alpha_2^{'}\right)\cdot\left(\sigma-\alpha_3^{'}\right)}{(\sigma+1)\cdot\left(\sigma+\beta_1^{'}\right)\cdot\left(\sigma+\beta_2^{'}\right)}\cdot y+\\ &++\frac{(\rho-\alpha_1)\cdot(\rho-\alpha_2)\cdot(\rho-\alpha_3)\cdot\left(\sigma-\alpha_1^{'}\right)\cdot\left(\sigma-\alpha_2^{'}\right)\cdot\left(\sigma-\alpha_3^{'}\right)}{(\rho+1)\cdot(\rho+\beta_1)\cdot(\rho+\beta_2)\cdot(\sigma+1)\cdot\left(\sigma+\beta_1^{'}\right)\cdot\left(\sigma+\beta_2^{'}\right)}\cdot xy+...\}=\\ &=x^{\rho}\cdot y^{\sigma}\cdot\sum_{m,n=0}^{\infty}\frac{(\rho-\alpha_1)_m\cdot(\rho-\alpha_2)_m\cdot(\rho-\alpha_3)_m\cdot\left(\sigma-\alpha_1^{'}\right)_n\cdot\left(\sigma-\alpha_2^{'}\right)_n\cdot\left(\sigma-\alpha_3^{'}\right)_n}{(\rho)_m\cdot(\rho-1+\beta_1)_m\cdot(\rho-1+\beta_2)_m\cdot(\sigma)_n\cdot\left(\sigma-1+\beta_1^{'}\right)_n\cdot\left(\sigma-1+\beta_2^{'}\right)_n}\cdot x^m\cdot y^n.\end{aligned} \tag{74}$$

On the basis of theorem 3.2 and 3.3, it is possible to formulate a general solution theorem for a heterogeneous simple Clausen system (65).

Theorem 3.4. The general solution of a heterogeneous simple system of Clausen type (65) with right part (66) is represented as the sum of the general solution $\overline{Z}(x,y)$ of the relevant homogeneous system (49) and the private solution of the heterogeneous system (65), that is, it has the form:

$$\begin{aligned}Z(x,y)=&\overline{Z}(x,y)+K_{\rho,\sigma}(x,y)=\sum_{i=1}^{9}C_iZ_i(x,y)+\\ &+x^{\rho}y^{\sigma}\sum_{m,n=0}^{\infty}\frac{(\rho-\alpha_1)_m(\rho-\alpha_2)_m(\rho-\alpha_3)_m\left(\sigma-\alpha_1^{'}\right)_n\left(\sigma-\alpha_2^{'}\right)_n\left(\sigma-\alpha_3^{'}\right)_n}{(\rho)_m(\rho-1+\beta_1)_m(\rho-1+\beta_2)_m(\sigma)_n\left(\sigma-1+\beta_1^{'}\right)_n\left(\sigma-1+\beta_2^{'}\right)_n}x^my^n.\end{aligned} \tag{75}$$

where, C_i (i=1, 2, 3)-arbitrary permanent, (74) nine linearly independent private decisions (50)-(59). In the future, the newly constructed partial solution of a heterogeneous simple Clausen-type system (65) is denoted by:

$$K_{\rho,\sigma}\left(\begin{matrix}\alpha_1,\alpha_1^{'}, & \alpha_2,\alpha_2^{'}, & \alpha_3,\alpha_3^{'}\\ \beta_1,\beta_1^{'}, & \beta_2,\beta_2^{'} & \end{matrix}\middle| x,y\right). \tag{76}$$

Theorem 3.5. Derivatives of generalized hypergeometric Clausen functions of two functions have:

The first-order derivatives for an independent variable x

$$\begin{aligned}&\frac{\partial}{\partial x}\left[{}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle|x\right){}_3F_2\left(\begin{matrix}\alpha_1^{'}, & \alpha_2^{'}, & \alpha_3^{'}\\ \beta_1^{'}, & \beta_2^{'} & \end{matrix}\middle|y\right)\right]=\\ &=\frac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1+1, & \alpha_2+1, & \alpha_3+1\\ \beta_1+1, & \beta_2+1 & \end{matrix}\middle|x\right){}_3F_2\left(\begin{matrix}\alpha_1^{'}, & \alpha_2^{'}, & \alpha_3^{'}\\ \beta_1^{'}, & \beta_2^{'} & \end{matrix}\middle|y\right)=\\ &=\frac{\alpha_1\alpha_2\alpha_3}{\beta_1\beta_2}\sum_{m,n=0}^{\infty}\frac{(\alpha_1+1)_n\left(\alpha_1^{'}\right)_n(\alpha_2+1)_n\left(\alpha_2^{'}\right)_n(\alpha_3+1)_n\left(\alpha_3^{'}\right)_n}{(\beta_1+1)_m\left(\beta_1^{'}\right)_n(\beta_2+1)_m\left(\beta_2^{'}\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}.\end{aligned} \tag{77}$$

Next, you can define the derivatives of two functions relative to an independent variable y. Then

$$\begin{aligned}&\frac{\partial}{\partial y}\left[{}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle|x\right){}_3F_2\left(\begin{matrix}\alpha_1^{'}, & \alpha_2^{'}, & \alpha_3^{'}\\ \beta_1^{'}, & \beta_2^{'} & \end{matrix}\middle|y\right)\right]=\\ &=\frac{\alpha_1^{'}\alpha_2^{'}\alpha_3^{'}}{\beta_1^{'}\beta_2^{'}}{}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2 & \end{matrix}\middle|x\right){}_3F_2\left(\begin{matrix}\alpha_1^{'}+1, & \alpha_2^{'}+1, & \alpha_3^{'}+1\\ \beta_1^{'}+1, & \beta_2^{'}+1 & \end{matrix}\middle|y\right)=\\ &=\frac{\alpha_1^{'}\alpha_2^{'}\alpha_3^{'}}{\beta_1^{'}\beta_2^{'}}\sum_{m,n=0}^{\infty}\frac{(\alpha_1)_m\left(\alpha_1^{'}+1\right)_n(\alpha_2)_m\left(\alpha_2^{'}+1\right)_n(\alpha_3)_m\left(\alpha_3^{'}+1\right)_n}{(\beta_1)_m\left(\beta_1^{'}+1\right)_n(\beta_2)_m\left(\beta_2^{'}+1\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}.\end{aligned} \tag{78}$$

In the same way, derivatives of two higher order Clausen functions (77) can be found near the feature (0,0). The above general formula (86) can also be applied to find a derivative solution of a simple Clausen system (56)-(63). Here is another new property of the formula (86).

Indeed, if σ+1-$n$is a negative integer or zero, the ratio (2.27) can be written in a more convenient way, in one of the following three forms [11]:

$$\frac{d^n}{dx^n}\left[x^{\sigma}{}_pF_q\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle|x\right)\right]=\frac{(\alpha_p)_{n-\sigma}n!}{(\rho_q)_{n-\sigma}(n-\sigma)!}\cdot{}_{p+1}F_{q+1}\left(\begin{matrix}\alpha_p+n-\sigma, & n+1\\ \rho_q+n-\sigma, & n+1-\sigma\end{matrix}\middle|x\right), \tag{79}$$

$$\frac{d^n}{dx^n}\left[x^{\sigma+n-1}\cdot{}_{p+1}F_q\left(\begin{matrix}\sigma, & \alpha_p\\ \rho_q & \end{matrix}\middle|x\right)\right]=(\sigma)_nx^{\sigma-1}\cdot{}_{p+1}F_q\left(\begin{matrix}\sigma+n, & \alpha_p\\ \rho_q & \end{matrix}\middle|x\right), \tag{80}$$

$$\frac{d^n}{dx^n}\left[x^{\sigma-1}\,{}_pF_{q+1}\left(\begin{matrix}\alpha_p \\ \sigma \quad \rho_q\end{matrix}\middle| x\right)\right]=(\sigma-n)_n x^{\sigma-1-n}\cdot{}_pF_{q+1}\left(\begin{matrix}\alpha_p, \\ \sigma-n, \quad \rho_q\end{matrix}\middle| x\right). \quad (81)$$

For example, if in formula (56) σ=1-$\beta_1$and $\sigma+1-n=1-\beta_1+1-n=2-\beta_1-n<0$, to m- the derivative of the product:

$$\frac{d^m}{dy^m}\left[y^{1-\beta_1}\cdot{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_1+1-\beta_2 & \end{matrix}\middle| x\right)\right] \quad (82)$$

can be defined using formula (80), at σ=1-β_1, p=3, q=2:

$$\begin{aligned}&\frac{d^m}{dy^m}\left[y^{1-\beta_1+n-1}\,{}_4F_2\left(\begin{matrix}1-\beta_1, & \alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1 & \beta_1+1-\beta_2 & & \end{matrix}\middle| y\right)\right]= \\ &=(1-\beta_1)_m x^{-\beta_1}\,{}_4F_2\left(\begin{matrix}1-\beta_1+m, & \alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_1+1-\beta_2 & & \end{matrix}\middle| y\right).\end{aligned} \quad (83)$$

The derivative m- of the private decision Z_2 (x, y) is then presented as:

$$\begin{aligned}&\frac{d^m Z_2(x,y)}{dy^m}=\frac{d^m}{dy^m}\left[{}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right)\right]y^{1-\beta_1}\cdot \\ &\cdot{}_3F_2\left(\begin{matrix}\alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_1+1-\beta_2 & \end{matrix}\middle| y\right)= \\ &=(1-\beta_1)_m x^{-\beta_1}\,{}_4F_2\left(\begin{matrix}1-\beta_1+m, & \alpha_1+1-\beta_1, & \alpha_2+1-\beta_1, & \alpha_3+1-\beta_1 \\ 2-\beta_1, & \beta_1+1-\beta_2 & & \end{matrix}\middle| y\right)\cdot \\ &\cdot{}_3F_2\left(\begin{matrix}\alpha_1, & \alpha_2, & \alpha_3 \\ \beta_1, & \beta_2 & \end{matrix}\middle| x\right).\end{aligned} \quad (84)$$

3.2 Solving and studying the features of constructing solutions of the main Clausen system

Theorem 3.6. Hypergeometric system Clausen

$$\begin{aligned}&x^2(1-x)p_{30}+xyp_{21}+[\gamma+\delta+1-(3+\beta_1+\beta_2+\beta_3)x]xp_{20}+\delta yp_{11}+ \\ &+[\gamma\delta-(1+\beta_1+\beta_2+\beta_3+\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)x]p_{10}-\beta_1\beta_2\beta_3 p_{00}=0, \\ &y^2(1-y)p_{03}+xyp_{12}+[\gamma+\delta'+1-(3+\beta_1'+\beta_2'+\beta_3')y]yp_{02}+\delta' xp_{11}+ \\ &+[\gamma\delta'-(1+\beta_1'+\beta_2'+\beta_3'+\beta_1'\beta_2'+\beta_1'\beta_3'+\beta_2'\beta_3')y]p_{01}-\beta_1'\beta_2'\beta_3' p_{00}=0\end{aligned} \quad (85)$$

near regular feature (0,0) has nine linearly independent private solutions of type

$$Z(x,y)=x^\rho y^\sigma\sum_{m,n=0}^{\infty}A_{m,n}x^m y^n, \quad (86)$$

where, ρ, σ and $A_{m,n}$ (m, n − unknown permanent), if the conditions of

$$\frac{P(m,n+1)Q(m,n)}{R(m,n+1)S(m,n)}=\frac{P(m,n)Q(m+1,n)}{R(m,n)S(m+1,n)} \quad (87)$$

coefficients $A_{m,n}$ (m, n=0, 1, 2, ...) of series (86) and integration conditions [4]:

$$\begin{aligned}&\Delta_1=1-a_{12}b_{21}=1\neq 0, \\ &\Delta_2=\Delta_1^2-(a_{21}+a_{12}b_{12})(b_{12}+b_{21}a_{21})= \\ &=1^2-\left[\frac{y}{x(1-x)}\cdot\frac{x}{y(1-y)}\right]=1-\frac{1}{(1-x)(1-y)}\neq 0.\end{aligned} \quad (88)$$

One particular solution is a generalized hypergeometric series of two variables.

$$\begin{aligned}&F_1=F_1\left(\begin{matrix}\beta_1,\beta_1', & \beta_2,\beta_2', & \beta_3,\beta_3' \\ \gamma, & \delta,\delta' & \end{matrix}\middle| x,y\right)= \\ &=\sum_{m,n=0}^{\infty}\frac{(\beta_1)_m(\beta_1')_n(\beta_2)_m(\beta_2')_n(\beta_3)_m(\beta_3')_n}{(\gamma)_{m+n}(\delta)_m(\delta')_n}\frac{x^m}{m!}\frac{y^n}{n!}.\end{aligned} \quad (89)$$

Proof. For the classification of special curves, a simple rule applies [10]:

Rule 3.1. If the coefficients $r_{3,0}\neq0$, $t_{0,3}\neq0$, at the characteristic (0.0) for the system (3.26) is special regular. Only in this case, the system of defining equations relative to the feature (0.0) has up to nine different root pairs $(\rho_t,\sigma_t)(t=\overline{1,9})$ .Then by the Frobenius- Latysheva method [10], the system (54) has up to nine linearly independent regular solutions close to the feature (0.0) in the form of generalized power series of two variables:

$$Z(x,y)=x^\rho y^\sigma\sum_{m,n=0}^{\infty}A_{m,n}x^m y^n, \quad (90)$$

where, ρ, σ and $A_{m,n}$ (m, n=0, 1, 2, ...)– unknown coefficients.

It should be noted that the nine linearly independent private solutions of the Clausen system have only if the conditions of the common area (87) and integrability (88) are met. Due to the difficulty of constructing different solutions, it is usually limited to constructing one solution in the form of Clausen functions (89). However, we built all nine linearly independent private solutions.

The Frobenius- Latysheva method [10] is used to construct a private solution (89). To this end, we make up a system of characteristic Frobenius functions of the form (68), from which the system of determining equations with respect to the feature (0.0) is defined as:

$$\left.\begin{aligned}&f_{0,0}^{(1)}(\rho,\sigma)=\rho(\rho-1+\delta)(\rho-1+\gamma)(\rho+\delta'-\gamma)=0, \\ &f_{0,0}^{(2)}(\rho,\sigma)=\sigma(\sigma-1+\delta')(\sigma-1+\gamma)(\sigma+\delta-\gamma)=0.\end{aligned}\right\} \quad (91)$$

It has nine pairs of roots:

$$\begin{aligned}&(\rho_1=0,\sigma_1=0),(\rho_2=1-\delta,\sigma_1=0),(\rho_1=0,\\&\sigma_2=1-\delta^{'}),(\rho_2=1-\delta,\sigma_2=1-\delta^{'}),(\rho_1=0,\sigma_3=1-\gamma),\\&(\rho_3=1-\gamma,\sigma_1=0),(\rho_3=1-\gamma,\sigma_3=1-\gamma),\\&(\rho_2=1-\delta,\sigma_4=\delta-\gamma),(\rho_4=\delta^{'}-\gamma,\sigma_2=1-\delta^{'}),\end{aligned} \quad (92)$$

which are indicators of the series (86). These different pairs (ρ_t, σ_t) (t=1, 2, ..., 9) correspond to nine linear-independent private solutions of the type (86). Let's show now that one of the private solutions is a row (89). The coefficients $A_{m,n}$ (m, n=0, 1, 2, ...) of the series (86) are determined by the use of recurrence sequence systems:

$$\sum_{m,n=0}^{\infty} A_{\mu-m,\nu-n} f_{m,n}^{(j)}(\rho+\mu-m,\sigma+\nu-n) = 0,\ (j=1,2;\mu,\nu=0,1,2,\ldots). \quad (93)$$

Indeed, the system of determining Eq. (91) of the Clausen source system (85) defines nine pairs of roots (92). The first partial solution corresponds to the indicator (ρ_1=0, σ_1=0), that is, by setting the values (ρ_1=0) and (σ_1=0) in the system (93) one successively defines the unknown coefficients of the series (86):

$$\begin{aligned}&A_{1,0}=\frac{\beta_1\beta_2\beta_3}{\gamma\delta},A_{0,1}=\frac{\beta^{'}_1\beta^{'}_2\beta^{'}_3}{\gamma\delta^{'}},A_{1,1}=\frac{\beta_1\beta_2\beta_3\beta^{'}_1\beta^{'}_2\beta^{'}_3}{\gamma(\gamma+1)\delta\delta^{'}},\\&A_{2,0}=\frac{\beta_1(\beta_1+1)\beta_2(\beta_2+1)\beta_3(\beta_3+1)}{\gamma(\gamma+1)\delta(\delta+1)\delta^{'}},\ldots\end{aligned} \quad (94)$$

that is, the series (90) is represented as (89). Similarly, the other eight linear-independent private solutions are constructed:

$$\begin{aligned}
&Z_2(x,y)=y^{1-\delta^{'}}\sum_{m,n=0}^{\infty}\frac{(\beta_1)_m(\beta_2)_m(\beta_3)_m(1-\delta^{'}+\beta^{'}_1)_n(1-\delta^{'}+\beta^{'}_2)_n(1-\delta^{'}+\beta^{'}_3)_n}{(\delta)_m(2-\delta^{'})_n(1-\delta^{'}+\gamma)_{m+n}}\\
&\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_3(x,y)=x^{1-\delta}\sum_{m,n=0}^{\infty}\frac{(1-\delta+\beta_1)_m(1-\delta+\beta_2)_m(1-\delta+\beta_3)_m(\beta^{'}_1)_n(\beta^{'}_2)_n(\beta^{'}_3)_n}{(2-\delta)_m(1-\delta+\gamma)_{m+n}(\delta^{'})_n}\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_4(x,y)=x^{1-\delta}y^{1-\delta^{'}}\cdot\\
&\cdot\sum_{m,n=0}^{\infty}\frac{(1-\delta+\beta_1)_m(1-\delta+\beta_2)_m(1-\delta+\beta_3)_m(1-\delta^{'}+\beta^{'}_1)_n(1-\delta^{'}+\beta^{'}_2)_n(1-\delta^{'}+\beta^{'}_3)_n}{(2-\delta)_m(2-\delta-\delta^{'}+\gamma)_{m+n}(2-\delta^{'})_n}\\
&\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_5(x,y)=y^{1-\gamma}\sum_{m,n=0}^{\infty}\frac{(\beta_1)_m(\beta_2)_m(\beta_3)_m(1-\delta^{'}+\beta^{'}_1)_n(1-\delta^{'}+\beta^{'}_2)_n(1-\delta^{'}+\beta^{'}_3)_n}{(\delta)_m(2-\gamma)_m(1-\gamma+\delta^{'})_n}\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_6(x,y)=x^{1-\gamma}\sum_{m,n=0}^{\infty}\frac{(1-\gamma+\beta_1)_m(1-\gamma+\beta_2)_m(1-\gamma+\beta_3)_m(\beta^{'}_1)_n(\beta^{'}_2)_n(\beta^{'}_3)_n}{(2-\gamma)_m\cdot(1-\gamma+\delta)_m\cdot(\delta^{'})_n}\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_7(x,y)=x^{1-\gamma}y^{1-\gamma}\cdot\\
&\cdot\sum_{m,n=0}^{\infty}\frac{(1-\gamma+\beta_1)_m(1-\gamma+\beta_2)_m(1-\gamma+\beta_3)_m}{(2-\gamma)_m(2-\gamma)_n(1+\delta-\gamma)_m(2-\delta)_n}\frac{(1-\gamma+\beta^{'}_1)_n(1-\gamma+\beta^{'}_2)_n(1-\gamma+\beta^{'}_3)_n}{(2-\delta^{'})_m(1+\delta^{'}-\gamma)_n}\\
&\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_8(x,y)=x^{1-\delta}y^{\delta-\gamma}\sum_{m,n=0}^{\infty}\frac{(1-\delta+\beta_1)_m(1-\delta+\beta_2)_m}{(2-\delta)_m(\delta-\gamma+1)_n}\cdot\\
&\cdot\frac{(1-\delta+\beta_3)_m(\delta-\gamma+\beta^{'}_1)_n(\delta-\gamma+\beta^{'}_2)_n(\delta-\gamma+\beta^{'}_3)_n}{(\delta-\gamma-\delta^{'})_n}\frac{x^m}{m!}\frac{y^n}{n!},\\
&Z_9(x,y)=x^{\delta^{'}-\gamma}y^{1-\delta^{'}}\sum_{m,n=0}^{\infty}\frac{(1-\delta^{'}+\beta^{'}_1)_n\cdot(1-\delta^{'}+\beta^{'}_2)_n}{(2-\delta^{'})_n\cdot(\delta^{'}-\gamma+1)_m}\cdot\\
&\cdot\frac{(1-\delta^{'}+\beta^{'}_3)_n(\delta^{'}-\gamma+\beta_1)_m(\delta^{'}-\gamma+\beta_2)_m(\delta^{'}-\gamma+\beta_3)_m}{(\delta^{'}-\gamma-\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}.
\end{aligned} \quad (95)$$

For them, the assertion is true.

Theorem 3.7. The general solution of the Clausen system (85) is represented as a sum:

$$\overline{Z}(x,y)=\sum_{t=1}^{9}C_iZ_i(x,y), \quad (96)$$

where, $Z_t(x,y)(t=\overline{1,9})$ linear-independent private decisions in series (86) and depend on arbitrary constants $C_t(t=\overline{1,9})$.

It should be noted that, to date, the construction of private solutions of the heterogeneous Clausen system has not been studied.

Theorem 3.8. An inhomogeneous system of the Clausen hypergeometric type

$$x^2(1-x)p_{30}+xyp_{21}+[\gamma+\delta+1-(3+\beta_1+\beta_2+\beta_3)x]xp_{20}+\delta yp_{11}+$$
$$+[\gamma\delta-(1+\beta_1+\beta_2+\beta_3+\beta_1\beta_2+\beta_2\beta_3+\beta_1\beta_3)x]p_{10}-\beta_1\beta_2\beta_3 p_{00}=f_1(x,y),$$
$$y^2(1-y)p_{03}+xyp_{12}+[\gamma+\delta'+1-(3+\beta_1'+\beta_2'+\beta_3')y]yp_{02}+\delta' xp_{11}+$$
$$+[\gamma\delta-(1+\beta_1'+\beta_2'+\beta_3'+\beta_1'\beta_2'+\beta_2'\beta_3'+\beta_1'\beta_3')y]p_{01}-\beta_1'\beta_2'\beta_3' p_{00}=f_2(x,y), \quad (97)$$

with the right side

$$f_1(x,y)=\frac{x^\rho y^\sigma}{\sigma(\sigma-1+\delta')(\sigma-1+\gamma)(\sigma-\delta+\gamma)},$$
$$f_2(x,y)=\frac{x^\rho y^\sigma}{\rho(\rho-1+\delta)(\rho-1+\gamma)(\rho-\delta'+\gamma)}, \quad (98)$$

has a specific solution of type.

$$K_{\rho,\sigma}(x,y)=x^\rho y^\sigma\cdot$$
$$\cdot\sum_{m,n=0}^{\infty}\frac{(\rho-\beta_1)_m(\rho-\beta_2)_m(\rho-\beta_3)_m(\sigma-\beta_1')_n(\sigma-\beta_2')_n(\sigma-\beta_3')_n}{(\rho)_m(\rho-1+\delta)_m(\rho-1+\gamma)_m(\rho-\delta'+\gamma)_m(\sigma)_n(\sigma-1+\beta_1')_n(\sigma-1+\beta_2')_n(\sigma-\delta+\gamma)_n}x^m y^n. \quad (99)$$

Evidence. On the basis of the characteristic Frobenius system of the homogeneous Clausen system, we construct a system of characteristic Frobenius equations of an inhomogeneous system of the hypergeometric type Clausen (97), taking into account the right part (98):

$$L_1[x^\rho y^\sigma]\equiv x^{\rho-1}y^\sigma\{f_{0,0}^{(1)}(\rho,\sigma)+f_{1,0}^{(1)}(\rho,\sigma)x\}=\frac{x^{\rho_1}y^{\sigma_1}}{\sigma(\sigma-1+\delta')(\sigma-1+\gamma)(\sigma-\delta+\gamma)},$$
$$L_2[x^\rho y^\sigma]\equiv x^\rho y^{\sigma-1}\{f_{0,0}^{(1)}(\rho,\sigma)+f_{0,1}^{(1)}(\rho,\sigma)y\}=\frac{x^{\rho_1}y^{\sigma_1}}{\rho(\rho-1+\delta)(\rho-1+\gamma)(\rho-\delta'+\gamma)}, \quad (100)$$

where,

$$f_{0,0}^{(1)}(\rho,\sigma)=\rho(\rho-1+\delta)(\rho-1+\gamma)(\rho-\delta'+\gamma),$$
$$f_{0,0}^{(1)}(\rho,\sigma)=\sigma(\sigma-1+\delta')(\sigma-1+\gamma)(\sigma-\delta+\gamma),$$
$$f_{1,0}^{(1)}(\rho,\sigma)=(\rho-\beta_1)(\rho-\beta_2)(\rho-\beta_3),$$
$$f_{0,1}^{(2)}(\rho,\sigma)=(\sigma-\beta_1')(\sigma-\beta_2')(\sigma-\beta_3'). \quad (101)$$

The system $f_{00}^{(j)}(\rho,\sigma)=0$defines a system of determining Eq. (91), allowing alsoρ-1=ρ_1 and σ-1=σ_1.

An inhomogeneous equation solution is also sought as a series (86), the coefficients $A_{m,n}$ (m,n=0,1,2,...) are consistently defined as in the case of a simple Clausen-type system from a recurrence sequence system (71).

The newly obtained partial solution (99) of the heterogeneous Clausen equation shall be denoted by:

$$K_{\rho,\sigma}\left(\begin{matrix}\beta_1,\beta_1', & \beta_2,\beta_2', & \beta_3,\beta_3'\\ \gamma, & \delta,\delta' & \end{matrix}\middle| x,y\right)=$$
$$\cdot\frac{x^\rho y^\sigma}{\rho(\rho-1+\delta)(\rho-1+\gamma)(\rho-\delta'+\gamma)\sigma(\sigma-1+\delta')(\sigma-1+\gamma)(\sigma-\delta+\gamma)}\cdot$$
$$\cdot\{1+\frac{(\rho-\beta_1)(\rho-\beta_2)(\rho-\beta_3)}{(\rho+1)(\rho+\delta)(\rho+\gamma)(\rho-\delta'+\gamma+1)(\sigma+1)}x+$$
$$\frac{(\sigma-\beta_1')(\sigma-\beta_2')(\sigma-\beta_3')}{(\sigma+1)(\sigma+\delta')(\sigma+\gamma)(\sigma-\delta+\gamma+1)}y+$$
$$+\frac{(\rho-\beta_1)(\rho-\beta_2)(\rho-\beta_3)(\sigma-\beta_1')(\sigma-\beta_2')(\sigma-\beta_3')}{(\rho+1)(\rho+\delta)(\rho+\gamma)(\rho-\delta'+\gamma+1)(\sigma+1)(\sigma+1)(\sigma+\delta')(\sigma+\gamma)(\sigma-\delta+\gamma+1)}xy+...\}=$$
$$=x^\rho y^\sigma\cdot$$
$$\cdot\sum_{m,n=0}^{\infty}\frac{(\rho-\beta_1)_m(\rho-\beta_2)_m(\rho-\beta_3)_m(\sigma-\beta_1')_n(\sigma-\beta_2')_n(\sigma-\beta_3')_n}{(\rho)_m(\rho-1+\delta)_m(\rho-1+\gamma)_m(\rho-\delta'+\gamma)_m(\sigma)_n(\sigma-1+\beta_1')_n(\sigma-1+\beta_2')_n(\sigma-\delta+\gamma)_n}\cdot$$
$$\cdot x^m y^n \quad (102)$$

Let's focus on the individual properties of Clausen functions.

Differentiatibility property

Theorem 3.9. The generalized hypergeometric Clausen function of two variables (32) have:

The first-order derivatives for independent variables x and y

$$\frac{\partial F}{\partial x}=\frac{\beta_1\beta_2\beta_3}{\gamma\delta}F\left(\begin{matrix}\beta_1+1,\beta_2+1,\beta_3+1; & \beta_1', & \beta_2', & \beta_3'\\ \delta+1, & \delta', & \gamma+1 & \end{matrix}\middle| x,y\right), \quad (103)$$

$$\frac{\partial F}{\partial y}=\frac{\beta_1'\beta_2'\beta_3'}{\gamma\delta'}F\left(\begin{matrix}\beta_1,\beta_2,\beta_3; & \beta_1'+1, & \beta_2'+1, & \beta_3'+1\\ \delta, & \delta'+1, & \gamma+1 & \end{matrix}\middle| x,y\right), \quad (104)$$

The higher-order derivatives

$$\frac{\partial^2 F}{\partial x\partial y}=\frac{\beta_1\beta_2\beta_3\beta_1'\beta_2'\beta_3'}{\gamma(\gamma+1)\delta\delta'}F$$
$$\left(\begin{matrix}\beta_1+1,\beta_2+1,\beta_3+1; & \beta_1'+1, & \beta_2'+1, & \beta_3'+1\\ \delta+1, & \delta'+1, & \gamma+2 & \end{matrix}\middle| x,y\right), \quad (105)$$

$$\frac{\partial^2 F}{\partial x^2}=\frac{\beta_1(\beta_1+1)\beta_2(\beta_2+1)\beta_3(\beta_3+1)}{\gamma(\gamma+1)\delta(\delta+1)}$$
$$F\left(\begin{matrix}\beta_1+2,\beta_2+2,\beta_3+2; & \beta_1', & \beta_2', & \beta_3'\\ \delta+2, & \delta', & \gamma+2 & \end{matrix}\middle| x,y\right), \quad (106)$$

$$\frac{\partial^2 F}{\partial y^2}=\frac{\beta_1'(\beta_1'+1)\beta_2'(\beta_2'+1)\beta_3'(\beta_3'+1)}{\gamma(\gamma+1)\delta'(\delta'+1)}$$
$$F\left(\begin{matrix}\beta_1,\beta_2,\beta_3; & \beta_1'+2, & \beta_2'+2, & \beta_3'+2\\ \delta, & \delta'+2, & \gamma+2 & \end{matrix}\middle| x,y\right) \quad (107)$$

$$\frac{\partial^m F}{\partial x^m}=\frac{\beta_1(\beta_1+1)..(\beta_1+m-1)\beta_2(\beta_2+1)..(\beta_2+m-1)\beta_3(\beta_3+1)..(\beta_3+m-1)}{\gamma(\gamma+1)..(\gamma+m-1)\delta(\delta+1)..(\delta+m-1)}\cdot$$
$$\cdot F\left(\begin{matrix}\beta_1+m,\beta_2+m,\beta_3+m; & \beta_1', & \beta_2', & \beta_3'\\ \delta+m, & \delta', & \gamma+m & \end{matrix}\middle| x,y\right), \quad (108)$$

$$\frac{\partial^n F}{\partial y^n}=\frac{\beta_1'(\beta_1'+1)..(\beta_1'+n-1)\beta_2'(\beta_2'+1)..(\beta_2'+n-1)\beta_3'(\beta_3'+1)..(\beta_3'+n-1)}{\gamma(\gamma+1)..(\gamma+n-1)\delta'(\delta'+1)..(\delta'+n-1)}\cdot$$
$$\cdot F\left(\begin{matrix}\beta_1,\beta_2,\beta_3; & \beta_1'+n, & \beta_2'+n, & \beta_3'+n\\ \delta, & \delta'+n, & \gamma+n & \end{matrix}\middle| x,y\right), \quad (109)$$

$$\frac{\partial^{m+n}F}{\partial x^m \partial y^n}=\frac{\beta_1(\beta_1+1)..(\beta_1+m-1)\cdot...\cdot\beta_3^{'}(\beta_3^{'}+1)..(\beta_3^{'}+n-1)}{\gamma(\gamma+1)..(\gamma+m+n-1)\delta(\delta+1)..(\delta+m-1)\delta^{'}...(\delta^{'}+n-1)}\cdot$$
$$\cdot F\left(\begin{matrix}\beta_1+m,\beta_2+m,\beta_3+m; & \beta_1^{'}+n, & \beta_2^{'}+n, & \beta_3^{'}+n\\ \delta+m, & \delta^{'}+n, & \gamma+m+n & \end{matrix}\middle| x,y\right). \tag{110}$$

All formulas (103)-(109) can be derived from the general formula (110).

Numerical computation of the Clausen function. A small summary of the difficulties of numerical estimation of the Gaussian function is given in the monograph L. Slater [7]. Difficulties arise from the increasing number of variables and parameters. The Clausen function ${}_3F_2(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2;x)$– depends on one variable and five parameters, and the Clausen function $F\left(\begin{matrix}\beta_1,\beta_1^{'}, & \beta_2,\beta_2^{'}, & \beta_3,\beta_3^{'}\\ \gamma, & \delta,\delta^{'} & \end{matrix}\middle| x,y\right)$– depends on nine parameters and two variables x and y. Therefore, there is not much information about the approximate calculation of special functions of two variables. Here we will limit ourselves to adding up the Zaalchutz theorem [12].

Definition 3.1. If the parameters are:

$$\alpha_1+\alpha_2+...+\alpha_{q+1}=-1+\rho_1+...+\rho_q, \tag{111}$$

The series is called the Zaalchutz species. Zaalchutz's theorem

$${}_3F_2[a,b,-n;c,d;1]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \tag{112}$$

allows you to calculate the sum of any finite series. ${}_3F_2$Zaalschutz species under conditions $c+d=a+b-n+1$.

Example 3.1. Apply the formula (112) to calculate the sum of a series

$${}_3F_2[a,b,-n;c,d;1]={}_3F_2[-6,-1,-3;-4,-5;1]=$$
$$=\frac{(2)_3\cdot(-3)_3}{(-4)_3\cdot(3)_3}=\frac{2\cdot3\cdot4\cdot(-3)\cdot(-2)\cdot(-1)}{(-4)\cdot(-3)\cdot(-2)\cdot3\cdot4\cdot5}=\frac{1}{10}. \tag{113}$$

at$x$=1 the series ${}_3F_2$ can be summed by the theorems of Dixon, Watson, Whipple, Dugall and others [8].

Example 3.2. If some of the parameters β_j и $\beta_j^{'}(j=1,2,3)$ are negative, then the Clausen function becomes polynomial. Thus: 1) $\beta_1=-1,\beta_1^{'}=-1$ or 2) $\beta_2=-1,\beta_2^{'}=-1$ or 3) $\beta_3=-1,\beta_3^{'}=-1$. The Clausen series becomes a polynomial of two variables type:

$$F\left(\begin{matrix}-1, & \beta_2,\beta_3, & -1, & \beta_2^{'},\beta_3^{'}\\ \delta, & \delta^{'}, & \gamma & \end{matrix}\middle| x,y\right)=1-\frac{\beta_2\beta_3}{\gamma\delta}x-\frac{\beta_2^{'}\beta_3^{'}}{\gamma\delta^{'}}y+$$
$$+\frac{\beta_2\beta_3\beta_2^{'}\beta_3^{'}}{\gamma(\gamma+1)\delta\delta^{'}}xy. \tag{114}$$

In addition to the above three cases, other combinations are possible.

The polynomial depends on six parameters and two variables to be taken into account in the numerical evaluation of the Clausen function in (114).

4. CONCLUSIONS

Thus, the possibilities of constructing solutions to the inhomogeneous the third-order equation and systems of differential equations in the third-order partial derivatives, in particular Clausen-type systems, have been studied in this paper.

Recently, due to the active study of multidimensional degenerate equations, the properties of generalized hypergeometric functions have often been used as solutions to the Clausen equation. For example, the solution of differential equations with one line of degeneracy uses the properties in the solution of Clausen's equation [6]. Therefore, it is important to study properties in the solution of the Clausen equation in the vicinity of the singular points x=0 and x=∞. The features of constructing a general and a private solution of the heterogeneous Clausen equation by the method of undefined coefficients are shown in part two.

The third part extends these ideas to a simple Clausen system near a regular feature (0, 0). The solutions to such systems are series in the form of a product of two hypergeometric series, each of which depends on one variable. Horn found that such rows also belong to second-order hypergeometric rows, - along with 34 hypergeometric rows from the Horn list [12]. A number of properties of a product of functions of the Clausen type constructed near the feature (0, 0).

The solution properties of the degenerate hypergeometric Clausen system, derived from the Clausen core system by means of the boundary transition, are used to construct the solution to the multidimensional degenerate equation of the third order with three independent variables of the form $Lu\equiv x^n\cdot y^m\cdot u_t-t^k\cdot y^m\cdot u_{xx}-t^k\cdot x^n\cdot u_{yyy}=0, m,n,k=const>0$ in the field $\Omega=\{(x,y,t): x>0, y>0, t>0\}$ [6].

The features of the construction of a common solution to the main heterogeneous Clausen system are discussed in the third part of this paper. It should be noted that the work has studied cases where the right-hand side of the Clausen equation $f(x)=x^\rho$- is a power function, the right-hand side of systems like Clausen's product $f_j(x,y)=x^\rho\cdot y^\sigma$, ($j$=1, 2) of power functions $f_j(x)=x^\rho$ and $f_j(y)=y^\rho$, (j=1, 2). In the first part, it was noted that the right-hand parts of the Clausen equation are generalized power rows of increasing (9) and descending degrees of the species (29). The right-hand parts of Clausen-type systems can be represented as (86) a generalized power series of two degree-increasing variables. Depending on this submission, we have received four new functions. Two of them (24) and (36) are partial solutions of the heterogeneous Clausen Eq. (18) near regular points x=0 and x=∞. Two functions (35) and (86) in the form of private solutions of simple and basic heterogeneous Clausen systems (33) and (97) respectively.

In conclusion, we would like to point out that the basis of the study of heterogeneous equations and systems like Clausen is laid down near regular features, at the above right-hand parts. However, this is only the beginning of the great research that lies ahead. For example, if the equation and the Clausen system have irregular features, then the right side is represented as a product of power and exponential functions, then research becomes much more complicated. In addition, related functions with Clausen functions of one and two variables should be established.

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