



## Analytical Study of Instability Phenomenon Using Laplace Transform

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### ABSTRACT

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*fingering method, instability, saturation of water, Laplace method, analytic*

In the present work, demonstrate the analytical method to analyze the instability phenomenon by using Laplace transform. For the analysis of instability two phase immiscible flow through the homogenous porous media is adopted. Fingering method is used to develop the Burger's equation. Which is used to obtain the saturation of water by Laplace transform method. The analytical results show that the saturation of water decreases with an increase in injected fluid and time. The obtained analytical results also satisfied and confirmed from the simulated results that shows the saturation of water.

## 1. INTRODUCTION

The fingering phenomenon has been discussed, which occurred in two immiscible phase flow through the homogeneous porous media. It has been reported that the simply investigated the unidirectional flow of immiscible fluids in the large medium. The variation in the values of pressure, saturations and speed of fluid in a single space direction corresponds to the movement direction [1].

In case of thicker porous medium, the vertical components of velocity of fluid could not be ignored. For the porous medium, the analyzed value of forces represents the interfaces, that are generally distorted (encroachment) the fronts. The front is referred to the zone of medium in which saturation of injected phase sharply rises. The encroachment on the scale of the front is known as tongue phenomenon. If it occurred on the smaller scale, then it's referred as fingering [2]. The condition of the stability of instability is directed to the encroachment. It has been reported that the instabilities depend on the mobility ratio ( $M$ ). It is also reported that the mobility ratio ( $M$ ) higher than 1 is more precisely for the appearing of instabilities [3]. The injected fluids that are more transportable than native fluid could be causes of harmful instabilities. The difference in viscosities of flowing fluids causes the occurrence of the phenomenon of fingering. Most of the reports showed that the capillary pressure has been neglected. Further Borana et al. [4] have included the capillary pressure in the analyzed of fingers. Scheidegger [1] has given a review on the instabilities of displacement fronts in porous media and represented the most practically significant method for the oil production in the oil reservoir engineering. Hence, at the time of oil recovery process the fingers should be stabilized [4].

Therefore, the authors should be attention towards the study of finger phenomenon with capillary pressure. The individual pressure of two flowing phases may be replaced by their common mean pressure and the behavior of the fingers. These can be determined by the statistical treatment. Darcy's law

(2003), the equation of continuity and certain basic assumptions yields given a nonlinear partial differential equation for the motion of saturation of injecting fluid [5].

The earlier reports have been investigated the instabilities in displacement process through the homogeneous porous media by various methods. In the earlier study, we reported the analytic solution of instability phenomena using Fourier Transform method [6]. However, no report prevails for using Laplace transform for solving the problem of instability phenomena [7-9].

Here, we demonstrate the analytical solution of instability phenomenon by using Laplace Transform method. It is also discussed the correlation and confirm the observed results of the analytical methods with simulation results.

## 2. STATEMENT OF THE PROBLEM

In the porous medium, when a fluid is displaced by another fluid of lesser viscosity, instead of regular displacement of the whole front, with relatively great speed the protuberance occurs. These protuberances known as fingers and the phenomenon is referred as fingering.

We assume that a finite cylindrical piece of homogeneous porous medium of length  $L$  that are fully saturated with oil. This cylindrical piece displaced by the injecting water that give rise to fingers (protuberance).

At the initial boundary, the entire oil is displaced through a small distance due to the injection of water. Therefore, assumed that the compete saturation exists at the initial boundary condition. The value of  $x$  being measured in the direction of displacement.

For the mathematical formulation, the Darcy's Law considered as valid for investigate the flow of system and macroscopic behavior of fingers that obtained by statistical method.

In the statistical treatment of finger, the average behavior of

the two fluids (oil and water) are considered. Scheideger [1] introduced the process of motion with the concept of fictitious relative permeability. They described the two immiscible of injected fluid flow through the porous media.

In the injected fluid level ( $x$ ) at time  $t$ , the saturation of injected fluid ( $S_i$ ) is defined as the average cross-sectional area that occupied in the porous media by fingers, and can be obtained by the analytical expression.

### 3. MATHEMATICAL FORMULATION OF THE PROBLEM

#### 3.1 Fundamental equation

From the validity of Darcy's law, let assumed that the seepage velocity of water and oil are ( $V_w$ ) and ( $V_o$ ) respectively and can be written as:

$$V_w = -\frac{k_w}{\mu_w} k \frac{\partial P_w}{\partial x} \quad (1)$$

$$V_o = -\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x} \quad (2)$$

where,  $k$  is the permeability of the homogeneous medium,  $k_w$ ,  $P_w$  and  $\mu_w$  are represents the relative permeability, pressure and viscosity of water similarly,  $k_o$ ,  $P_o$  and  $\mu_o$  are the relative permeability, pressure and viscosity of oil.

Let assume that the relative permeability of water and oil are the functions of water saturation ( $S_w$ ) and oil saturation ( $S_o$ ), respectively.

The equation of continuity can be written as:

$$\varphi \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \quad (3)$$

$$\varphi \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \quad (4)$$

where,  $\varphi$  is the porosity of the medium, From the definition of phase saturation, the porous medium considered as fully saturated, and it is evident from:

$$S_w + S_o = 1 \quad (5)$$

The capillary pressure ( $P_c$ ) is also known as pressure discontinuity of the flowing phases across their common interface and can be written as:

$$P_c = P_o - P_w \quad (6)$$

For the determination of the mathematical analysis, assumed for instabilities the standard form of an analytical expression for the relationship between the relative permeability, phase saturation and capillary pressure, is given as:

$$k_w = S_w, k_o = S_o = 1 - S_w \quad (7)$$

$$P_c = -\beta S_w \quad (8)$$

Here negative sign indicates the direction of water saturation opposite to the capillary pressure and  $\beta$  assumed a constant with smaller parameter.

The value of oil pressure ( $P_o$ ) is given by [3]:

$$P_o = \frac{1}{2}(P_o + P_w) + \frac{1}{2}(P_o - P_w) \quad (9)$$

$$P_o = \bar{P} + \frac{P_c}{2}, P_o = \frac{1}{2}(P_o + P_w)$$

where,  $\bar{P}$  is mean pressure and considered as constant.

#### 3.2 Equation of motion for saturation

Equation of motion for the saturation obtained by the substitute the values of  $V_w$  from Eq. (1) in Eq. (3) and value of  $V_o$  from Eq. (2) in (4), can be written as:

$$\varphi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left[ -\frac{k_w}{\mu_w} k \frac{\partial P_w}{\partial x} \right] \quad (10)$$

$$\varphi \frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x} \left[ -\frac{k_o}{\mu_o} k \frac{\partial P_o}{\partial x} \right] \quad (11)$$

From the Eq. (6) and (10), after eliminating the term  $\frac{\partial P_w}{\partial x}$ , we get:

$$\varphi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left[ -\frac{k_w}{\mu_w} k \left\{ \frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \right\} \right] \quad (12)$$

From (11) and (12), using Eq. (5), we obtained:

$$\frac{\partial}{\partial x} \left[ \left\{ \frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right\} k \frac{\partial P_o}{\partial x} - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x} \right] = 0 \quad (13)$$

After integrating the above equation with respect to the  $x$ , we get:

$$\left[ \left\{ \frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right\} k \frac{\partial P_o}{\partial x} - \frac{k_w}{\mu_w} k \frac{\partial P_c}{\partial x} \right] = -A \quad (14)$$

where,  $A$  is integration constant.

Simplifying the Eq. (14):

$$\frac{\partial P_o}{\partial x} = -\frac{A}{k \left\{ \frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right\}} + \frac{k_w}{\mu_w \left\{ \frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right\}} \frac{\partial P_c}{\partial x} \quad (15)$$

After simplification the value of  $\frac{\partial P_o}{\partial x}$  obtained from Eq. (15) and substitute in Eq. (12), we obtained:

$$\varphi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left[ \frac{k_w}{\mu_w} k \left\{ -\frac{A}{k \left\{ \frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right\}} + \frac{k_w}{\mu_w \left\{ \frac{k_o}{\mu_o} + \frac{k_w}{\mu_w} \right\}} \frac{\partial P_c}{\partial x} - \frac{\partial P_c}{\partial x} \right\} \right] \quad (16)$$

$$\varphi \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} \left[ \frac{k_o}{\mu_o} k \frac{\partial P_c}{\partial x} \frac{1}{\left\{ 1 + \frac{k_o \mu_w}{k_w \mu_o} \right\}} + \frac{A}{\left\{ 1 + \frac{k_o \mu_w}{k_w \mu_o} \right\}} \right] = 0$$

Since, from Eq. (9):

$$\frac{\partial P_o}{\partial x} = \frac{1}{2} \frac{\partial P_c}{\partial x}$$

The Eq. (14) becomes after substitute the above value:

$$A = \left\{ \frac{k_w}{\mu_w} - \frac{k_o}{\mu_o} \right\} \frac{k}{2} \frac{\partial P_c}{\partial x} \quad (17)$$

Now, substituting the value of A in (16), we get:

$$\begin{aligned} \varphi \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} \left[ k \frac{k_w}{2\mu_w} \frac{\partial P_c}{\partial x} \right] &= 0 \\ \varphi \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} \left[ k \frac{k_w}{2\mu_w} \frac{dP_c}{dS_w} \frac{\partial S_w}{\partial x} \right] &= 0 \end{aligned} \quad (18)$$

Using Eq. (7), (8) and (18), we get:

$$\varphi \frac{\partial S_w}{\partial t} - \frac{\beta k k_w}{2\mu_w} \frac{\partial^2 S_w}{\partial x^2} = 0 \quad (19)$$

Let consider:

$$k = C_0 \tau \frac{\varphi^3}{M_s (1 - \varphi)^2}, \tau = \left( \frac{L}{L_e} \right)^2 \quad (20)$$

where,  $\varphi$  is porosity;  $M_s$  is specific surface area;  $C_0$  Kozeny constant;  $L_e$  Effective length of the path of the fluid.

From Eqns. (19) and (20), we get:

$$\frac{\partial S_w}{\partial t} - \frac{\beta C_0 \tau k_w \varphi^2}{2\mu_w M_s (1 - \varphi)^2} \frac{\partial^2 S_w}{\partial x^2} = 0 \quad (21)$$

The obtained equation represents the non-linear partial differential equation of motion for the saturation of injected fluid through the homogeneous porous medium.

$$P \frac{\partial S_i}{\partial t} + \left( \frac{\beta K_c}{\delta_n} \right) \frac{\partial}{\partial x} \left[ \left( P(\alpha S_i - 1) \frac{\partial S_i}{\partial x} \right) \right] = 0 \quad [10]$$

Let consider the capillary pressure co-efficient is to small enough use for smaller perturbation parameter.

The Eq. (21) becomes a singular perturbation problem equation by multiplying  $\beta$  with highest derivative term. Such the problem with appropriate conditions has been solved by either numerically or analytically.

The set of conditions can be written as:

$$S_w(0, t) = 0, S_w(1, t) = 1 \quad (22)$$

$$S_w(x, 0) = 0 \quad (23)$$

Let assume that at  $x=L$ , there is no flow because of consider as impermeable limit, i.e.

Let

$$X = \frac{x}{L}, T = \frac{C_0 \tau k_w \varphi^2}{2\mu_w M_s (1 - \varphi)^2 L^2} t$$

From (21), we get:

$$\frac{\partial S_w}{\partial T} - \beta \frac{\partial^2 S_w}{\partial X^2} = 0 \quad (24)$$

$$S_w(0, t) = 0, S_w(1, t) = 1 \quad (25)$$

$$S_w(X, 0) = 0 \quad (26)$$

The above equations now can be solve using Laplace transform.

### 3.3 Mathematical solution by using Laplace transform

Now from Eq. (24), we have:

$$\frac{\partial S_w}{\partial T} = \beta \frac{\partial^2 S_w}{\partial X^2}$$

With the boundary conditions are  $S_w(0, t)=0$ ,  $S_w(1, t)=1$  and initial condition is  $S_w(X, 0)=0$ .

The solution of above partial differential equation using Laplace transform method [4], From the Laplace transform:

$$\begin{cases} \frac{\partial S_w}{\partial T} = s \overline{S_w} - S_w(X, 0) \\ \frac{\partial^2 S_w}{\partial X^2} = \frac{\partial^2 \overline{S_w}}{\partial X^2} \end{cases} \quad (27)$$

Substitute Eq. (27) in to Eq. (24), so we get:

$$\beta \frac{\partial^2 \overline{S_w}}{\partial X^2} = s \overline{S_w} - S_w(X, 0) \quad (28)$$

Given initial condition is  $S_w(X, 0)=0$ .

So, (28) becomes:

$$\begin{aligned} \beta \frac{\partial^2 \overline{S_w}}{\partial X^2} - s \overline{S_w} &= 0 \\ \beta \frac{d^2 \overline{S_w}}{dX^2} - s \overline{S_w} &= 0 \\ \frac{d^2 \overline{S_w}}{dX^2} - \frac{s}{\beta} \overline{S_w} &= 0 \end{aligned} \quad (29)$$

Solution of (3) will be:

$$\overline{S_w}(X, s) = c_1 e^{-\sqrt{\frac{s}{\beta}} X} + c_2 e^{\sqrt{\frac{s}{\beta}} X} \quad (30)$$

where,  $c_1$  and  $c_2$  are arbitrary constants.

By initial conditions,  $\overline{S_w}(0, s) = L\{S_w(0, T)\} = L\{0\} = 0$ .

Substitute this value in (30),  $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$ .

So, (30) becomes:

$$\overline{S_w}(X, s) = c_2 \left( -e^{-\sqrt{\frac{s}{\beta}} X} + e^{\sqrt{\frac{s}{\beta}} X} \right) \quad (31)$$

Another initial condition,  $\overline{S_w}(1, s) = L\{S_w(1, T)\} = L\{1\} = \frac{1}{s}$ .

Substitute this condition in (31):

$$\begin{aligned} \frac{1}{s} &= c_2 \left( -e^{-\sqrt{\frac{s}{\beta}} X} + e^{\sqrt{\frac{s}{\beta}} X} \right) \\ c_2 &= \frac{1}{2s \sinh \left( \sqrt{\frac{s}{\beta}} \right)} \end{aligned} \quad (32)$$

By using (31) and (32) we can find the value of  $c_1$ :

$$c_1 = -\frac{1}{2s \sinh\left(\sqrt{\frac{s}{\beta}}\right)} \quad (33)$$

Substitute (32) and (33) in (30) we get:

$$\bar{S}_w(X, s) = -\frac{1}{2s \sinh\left(\sqrt{\frac{s}{\beta}}\right)} e^{-\sqrt{\frac{s}{\beta}}X} + \frac{1}{2s \sinh\left(\sqrt{\frac{s}{\beta}}\right)} e^{\sqrt{\frac{s}{\beta}}X}$$

$$\bar{S}_w(X, s) = \frac{\sinh\left(\sqrt{\frac{s}{\beta}}X\right)}{s \sinh\left(\sqrt{\frac{s}{\beta}}\right)}$$

By applying Laplace transform, we get the solution:

$$S_w(X, T) = \sum_{n=0}^{\infty} \left[ \operatorname{erfc}\left(\frac{1-x+2n}{2\sqrt{T\beta}}\right) - \operatorname{erfc}\left(\frac{1+x+2n}{2\sqrt{T\beta}}\right) \right]$$

### 3.4 Simulation

Simulation Programme:

```

X=0;
X1=0;
temp=0;
temp1=0;
row=0;
col=0;
count=0;
result=zeros(11,4);

for x=0:0.1:1
    row=row+1;
    col=0;
    for t=0.1:0.1:0.4
        col=col+1;
        for n=0:1000

            X=(1-x+(2*n))/(2*sqrt(t*0.1));
            X1=(1+x+(2*n))/(2*sqrt(t*0.1));
            temp=temp+erfc(X);
            temp1=temp1+erfc(X1);
            X=0;
            X1=0;
        end

        result(row,col)=temp-temp1;
        temp=0;
        temp1=0;
    end
end

mat=zeros(11,4);
for i=1:11
    for j=1:4
        mat(i,j)=result(12-i,5-j);
    end
end
end

```

The obtained results from the analytic and simulation of

the variation of the saturation in water with respect to the injected fluid and time are shown in Figure 1 and Figure 2, respectively. The results demonstrate that the saturation in water is decreased as increased the injected fluid at fixed time. Whereas,  $S_w$  slightly decreases with an increase in time for the fixed amount of injected fluid.

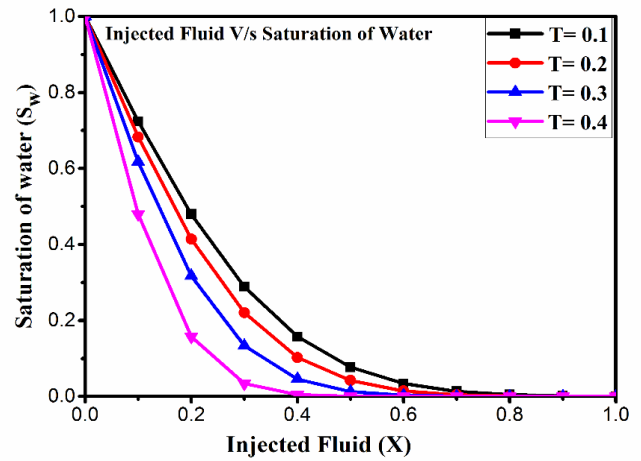


Figure 1. Plot between the injected fluid (%) Vs saturation of water (%)

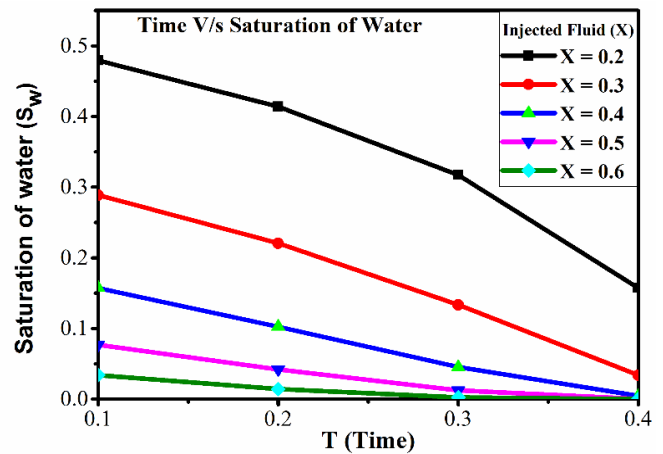


Figure 2. Variation in saturation in water (%) with respect to the changes in time (sec.)

### 4. CONCLUSIONS

In conclusions, the analytical and programmable solution of instability phenomena of two phase flow of fluid using by Laplace transform demonstrate that as increase the injected fluid (X) in the porous media results in exponentially decrease the saturation of water ( $S_w$ ) at constant time. Similarly, at constant injected fluid increase in time (T) linearly decrease in  $S_w$ . The results indicate and useful application in the achieved appropriate high recovery rate of oil from the given porous media.

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