Determination of Two Homogeneous Materials in a Bar with Solid-Solid Interface

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ABSTRACT

In this work, a bar fully insulated on its lateral surface. composed by two different unknown materials is considered. For the analytical solution, it is assumed a perfectly assembly solid-solid interface, so no heat loss due to friction is present. This is an ideal scenario, so this loss and possible measurement errors are included by simulating noisy data for the estimation of the thermal conductivity of the unknown materials. A stationary heat transfer process along the bar is considered where a Dirichlet condition is imposed at the left that represents a source of constant temperature. At the other end of the bar, a Robin condition that models heat dissipation by convection, is assumed. The constant thermal conductivity coefficients of both solids are determined under two different situations: a) two noisy temperature measurements are available, one at the interface and the other at the right boundary; b) a temperature measurement at the interface and a heat flow measurement at the right edge of the bar are given. The bounds for the errors in the identification of the unknown coefficients are obtained based on the data measurements, the room temperature and temperature values at the boundary and interface. Numerical examples are given to illustrate the ideas used for the parameter identification and elasticity analysis is carried out to measure the dependence of the data on the estimated parameters.

1. INTRODUCTION

Heat transfer problems in multilayer, or solid-solid interface materials, have been extensively studied in recent years due to the multiple and different applications that have been found in science and engineering [1]. These problems have direct applications in different industries, including metallurgy [2], technology and electronics [3], automotive [4], aerospace and aviation [5].

The estimation of thermal conductivity was widely studied under various approaches. Numerical techniques for inverse problems were considered in some works, for instance in [6-10]. Specifically, some authors simultaneously estimate spatially varying thermal conductivity and heat capacity, see [6]. In Ref. [7], a finite differences scheme is used while in [8] the estimation was carried out under particular conditions, using the conjugate gradient method. On the other hand, to estimate the dependency of the thermal conductivity on the temperature, Yang [9, 10] proposed a linear inverse model and also used iterative methods. Other interesting estimation strategies can be seen in the studies [11-14].

This work aims to the simultaneous estimation of the thermal conductivities of two homogeneous materials,

perfectly assembled, that form a bar of known length, laterally isolated.

The scheme used in this work represents an ideal scenario. This is an inverse problem. To solve it, we first solve the forward problem, which consist in calculating the temperature along the bar when the materials are known. In order to solve the inverse problem, we compare the measurement of the temperature with the solution to the forward problem (which depends on the material properties). Then, we obtain the values for the materials' conductivity.

In order to incorporate heat loss at interface and possible measurement errors, noisy data are simulated for the estimation of thermal conductivity of the unknown materials. For the identification problem, a constant temperature source at the left edge and dissipation by convection at the right edge, are considered. The estimations are performed based on two over-conditions under different situations: a) two noisy temperature are available: one at the interface and the other at the right end; b) a temperature measurement at the interface and a heat flow measurement at the right edge of the bar are given. The bounds for the errors in the identification of the unknown coefficients are obtained based on the data measurements, the room temperature and temperature values

at the boundary and interface. The proposed identification procedure is illustrated by few numerical examples. Moreover, elasticity analysis is carried out to measure the dependence of the data on the estimated parameters.

2. DIRECT PROBLEM

The stationary problem of heat transfer of a bar of length $L[m]$ and diameter $d[m]$, laterally isolated, with a solid-solid interface, is considered. The heat conduction is assumed to be one-dimensional. In addition, it is considered that the bar is composed of two consecutives, perfectly assembled, isotropic, homogeneous materials. The first section of the bar has a length $l[m]$ so that the second section has length $L-l$.

At the left end, a constant thermal source is assumed while the right end remains free allowing the process of convection. Figure 1 represents the stationary heat conduction problem.

$$x = 0 \qquad x = l \qquad x = L$$
$$u(0) = F \qquad u''(x) = 0 \qquad k_B u'(L) = -h(u(L) - Ta)$$
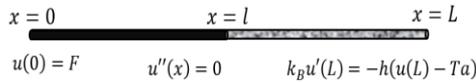
Figure 1. Scheme for the mathematical model

This problem may be modeled by the following system of ordinary differential equations with boundary and continuity conditions:

$$
\begin{cases}
u''(x) = 0, & 0 < x < l, \\
u''(x) = 0, & l < x < L, \\
u(0) = F, & \\
u(l^-) = u(l^+), & \\
\kappa_A u'(l^-) = \kappa_B u'(l^+), & \\
\kappa_B u'(L) = -h(u(L) - T_a),
\end{cases}
\tag{1}
$$

where, u [°C] is the temperature of the bar, F [°C] is the value of the point source at $x=0$, κ_A and κ_B [W/m°C] are the thermal conductivity constants of the homogeneous materials of the bar (note that A is the material on the left and B is the material on the right), h [W/m²°C] is the convection heat transfer coefficient, T_a [°C] represents the room temperature, and

$$
u(l^-) = \lim_{x \to l^-} u(x),\ u(l^+) = \lim_{x \to l^+} u(x),
$$
$$
u'(l^-) = \lim_{x \to l^-} u'(x),\ u'(l^+) = \lim_{x \to l^+} u'(x).
\tag{2}
$$

The solution to the problem (1)-(2) is given by:

$$
u(x)
$$
$$
=
\begin{cases}
F + \dfrac{\kappa_B h(T_a - F)}{\zeta} x, & 0 \leq x \leq l \\[3mm]
F + \dfrac{h(T_a - F)(l(\kappa_B - \kappa_A) + \kappa_A x)}{\zeta}, & l < x \leq L
\end{cases}
\tag{3}
$$

where,

$$
\zeta = \kappa_A \kappa_B + \kappa_A hL + (\kappa_B - \kappa_A)hl.
\tag{4}
$$

Details on the solution (3)-(4), its properties and examples, can be found in [15-17].

Without loss of generality, from now on it is assumed that $F > T_a$.

3. DETERMINATION OF THERMAL CONDUCTIVITIES

In this section, the inverse problem consisting in the estimation of the thermal conductivities of the material, is explained for two different situations. Analytical expressions for the thermal conductivities are obtained where the estimations depend on the available data for each case.

This is an inverse problem. To solve it, we first solve the forward problem, which consist in calculating the temperature along the bar when the materials are known. In order to solve the inverse problem, we compare the measurement of the temperature with the solution to the forward problem (which depends on the material properties). Then, we obtain the values for the materials' conductivity.

3.1 Case 1: Estimation based on two temperature measurements

The first estimation procedure assumes that two noisy temperature measurements are given: one at the interface, denoted by T_1, and the other one at the right edge of the bar, denoted as T_2.

The analytical solution given in (3)-(4), to the forward problem (1)-(2), is evaluated at $x=l$ and at $x=L$, this allows to obtain:

$$
\begin{cases}
u(l) = F + \dfrac{\kappa_B h(T_a - F)}{\kappa_A \kappa_B + \kappa_A hL + (\kappa_B - \kappa_A)hl} l, \\[4mm]
u(L) = F + \dfrac{h(T_a - F)(l(\kappa_B - \kappa_A) + \kappa_A L)}{\kappa_A \kappa_B + \kappa_A hL + (\kappa_B - \kappa_A)hl}.
\end{cases}
\tag{5}
$$

After some algebraic computation it results:

$$
\begin{cases}
F - u(l) = \dfrac{\kappa_B h(F - T_a)}{\kappa_A \kappa_B + \kappa_A hL + (\kappa_B - \kappa_A)hl} l, \\[4mm]
u(l) - u(L) = \dfrac{\kappa_A h(F - T_a)(L - l)}{\kappa_A \kappa_B + \kappa_A hL + (\kappa_B - \kappa_A)hl}.
\end{cases}
\tag{6}
$$

From the ratio between the expressions given in (6), the following relation between the thermal conductivities is obtained:

$$
\kappa_B = \frac{F - u(l)}{u(l) - u(L)} \frac{(L - l)}{l} \kappa_A
\tag{7}
$$

Replacing Eq. (7) in the any of the equations given in (6), the thermal conductivities are obtained as nonlinear functions on the temperature at $x=l$ and $x=L$ as follows:

$$
\kappa_A = hl \frac{u(L) - T_a}{F - u(l)}
\tag{8}
$$

$$
\kappa_B = h(L - l) \frac{u(L) - T_a}{u(l) - u(L)}
\tag{9}
$$

Note a unique expression for each coefficient κ_A and κ_B are obtained from the nonlinear equations given in (6).

The measured temperatures at $x=l$ and $x=L$, T_1 and T_2, respectively, yield the following estimates $\widehat{\kappa_A}, \widehat{\kappa_B}$, for the materials conductivities.

$$\widehat{\kappa_A} = hl \frac{T_2 - T_a}{F - T_1} \qquad (10)$$

$$\widehat{\kappa_B} = h(L - l) \frac{T_2 - T_a}{T_1 - T_2} \qquad (11)$$

The necessary and sufficient conditions for the existence of the solution to the proposed inverse problem are given by:

$$T_a < T_2 < T_1 < F \qquad (12)$$

3.2 Case 2: Estimation based on a temperature measurement and a heat flow measurement

The second estimation technique is based on a noisy temperature measurement data at the interface, T_1 and a heat flux measurement, q_1 at the right edge of the bar.

From the solution given in (3)-(4) it follows that the heat flux q at $x=L$ is:

$$q = -\kappa_B \, u'(L) = \frac{-\kappa_B \kappa_A h(T_a - F)}{\kappa_A \kappa_B + \kappa_A hL + (\kappa_B - \kappa_A)hl} \qquad (13)$$

The ratio between (13) and the first equation in (6) lead to:

$$\frac{q}{F - u(l)} = \frac{\kappa_A}{l} \qquad (14)$$

After some algebraic computations, using Eq. (14) in Eq. (13), the thermal conductivities are obtained as functions on the temperature at $x=l$ and the heat flux at $x=L$ as follows:

$$\kappa_A = \frac{ql}{F - u(l)} \qquad (15)$$

$$\kappa_B = \frac{h(L - l)}{\frac{h}{q}(u(l) - T_a) - 1} \qquad (16)$$

Estimated values $\widehat{\kappa_A}, \widehat{\kappa_B}$ for κ_A, κ_B are obtained by replacing $u(l)$ and q by the measured data T_1 and q_1, respectively, leading to:

$$\widehat{\kappa_A} = \frac{q_1 l}{F - T_1} \qquad (17)$$

$$\widehat{\kappa_B} = \frac{h(L - l)}{\frac{h}{q_1}(T_1 - T_a) - 1} \qquad (18)$$

The necessary and sufficient conditions for the existence of the solution to the inverse problem are given by:

$$T_a < T_1 < F \qquad (19)$$

and

$$q_1 < h(T_1 - T_a) \qquad (20)$$

4. ANALYSIS OF ESTIMATION ERROR

In this section we derive a bound for the error in estimating the thermal conductivities, depending on available data.

4.1 Error analysis for the Case 1

In this case, it is assumed that the errors of the measured temperatures T_1, and T_2, satisfy the following inequalities:

$$\begin{cases} |u(l) - T_1| < \varepsilon \, (F - T_a), \\ \\ |u(L) - T_2| < \varepsilon \, (F - T_a) \end{cases} \qquad (21)$$

where, the dimensionless constant $\varepsilon \in (0,1)$ depends on the error produced by the measurement tools, among others.

From the expressions for the thermal conductivities given in (8)-(11), it follows:

$$|\kappa_A - \widehat{\kappa_A}| = hl \left| \frac{u(L) - T_a}{F - u(l)} - \frac{T_2 - T_a}{F - T_1} \right| \qquad (22)$$

$$|\kappa_B - \widehat{\kappa_B}| = h(L - l) \left| \frac{u(L) - T_a}{u(l) - u(L)} - \frac{T_2 - T_a}{T_1 - T_2} \right| \qquad (23)$$

Let us consider (22). Observe that

$$\left| \frac{u(L) - T_a}{F - u(l)} - \frac{T_2 - T_a}{F - T_1} \right|$$
$$= \left| \frac{u(L)F - u(L)T_1 + T_a T_1 - T_2 F + T_2 u(l) - u(l)T_a}{\big(F - u(l)\big)(F - T_1)} \right|.$$

After adding and subtracting $T_1 T_2$ in the numerator, rearranging terms and performing some algebraic computations it results:

$$\left| \frac{u(L) - T_a}{F - u(l)} - \frac{T_2 - T_a}{F - T_1} \right|$$
$$= \frac{\left| (u(L) - T_2) + (u(l) - T_1)\frac{T_2 - T_a}{F - T_1} \right|}{|F - u(l)|} \qquad (24)$$

By using the triangular inequality and assuming that (12) and (21) hold, it follows that:

$$\left| \frac{u(L) - T_a}{F - u(l)} - \frac{T_2 - T_a}{F - T_1} \right|$$
$$\leq \frac{|u(L) - T_2| + |u(l) - T_1|\frac{T_2 - T_a}{F - T_1}}{F - u(l)} \qquad (25)$$
$$< \frac{1 + \frac{T_2 - T_a}{F - T_1}}{F - u(l)} \varepsilon \, (F - T_a)$$

Observe that $F > u(l) > T_a$. Hence,

$$0 < F - u(l) < F - T_a \qquad (26)$$

and there exists a dimensionless constant $M_1 \in (0, 1)$ such that:

$$M_1 < \frac{F - u(l)}{F - T_a} \qquad (27)$$

hence, inequality (25) can be written as:

$$\left| \frac{u(L) - T_a}{F - u(l)} - \frac{T_2 - T_a}{F - T_1} \right| < \frac{1}{M_1}\left(1 + \frac{T_2 - T_a}{F - T_1} \right)\varepsilon \qquad (28)$$

Replacing (28) in (22) it turns out that:

$$|\kappa_A - \widehat{\kappa_A}| < hl\, A_1 \varepsilon \qquad (29)$$

where, the dimensionless constant A_1 depends on the data measurements, T_1, T_2 and it is given by:

$$A_1 = \frac{1}{M_1}\left(1 + \frac{T_2 - T_a}{F - T_1} \right) \qquad (30)$$

Similarly, assuming that (12) and (21) hold, it is possible to write:

$$|\kappa_B - \widehat{\kappa_B}| = h(L - l)$$
$$\left| \frac{(T_2 - T_a)(T_1 - u(l)) + (T_1 - T_a)(u(L) - T_2)}{(u(l) - u(L))(T_1 - T_2)} \right| \qquad (31)$$
$$< h(L - l)\left(\frac{(T_2 - T_a) + (T_1 - T_a)}{u(l) - u(L)} \right)\left(\frac{F - T_a}{T_1 - T_2} \right)\varepsilon$$

Since $F > u(l) > u(L) > T_a$, the inequalities:

$$0 < u(l) - u(L) < F - T_a \qquad (32)$$

Hold and there exists a dimensionless constant $M_2 \in (0,1)$ such that:

$$M_2 < \frac{u(l) - u(L)}{F - T_a} \qquad (33)$$

Therefore,

$$|\kappa_B - \widehat{\kappa_B}| < \frac{h(L - l)}{M_2}\left(1 + 2\frac{T_2 - T_a}{T_1 - T_2} \right)\varepsilon \qquad (34)$$

or, equivalently,

$$|\kappa_B - \widehat{\kappa_B}| < \; h(L - l)B_1 \varepsilon \qquad (35)$$

where, the dimensionless constant B_1 depends on the data measurements, T_1, T_2 and it is given by:

$$B_1 = \frac{1}{M_2}\left(1 + 2\frac{T_2 - T_a}{T_1 - T_2} \right) \qquad (36)$$

hence, the inequalities (29)-(30) and (35)-(36) imply that the estimation errors for κ_A, κ_B vanish as the measurement error (or ε) approaches 0. The dimensionless constant A_1 and B_1 determine the "velocity" of the converges of the estimation errors to 0. In other words, the estimation errors vanish as the errors in data approaches 0, independently of the dimensionless constant, but if these constants are big, the data error should be small to have a good estimate for the conductivities.

4.2 Error analysis for the Case 2

In this case, it is assumed that the errors of the measured temperature T_1, and the measured heat flux q_1, satisfy the following inequalities:

$$\begin{cases} |u(l) - T_1| < \varepsilon\,(F - T_a), \\[2mm] |q - q_1| < \varepsilon\,h\,(F - T_a), \end{cases} \qquad (37)$$

where, as before, the dimensionless constant $\varepsilon \in (0,1)$ depends on the error produced by the measurement tools, among others.

From the expressions for the thermal conductivities given in Eqns. (15)-(18), we can write:

$$|\kappa_A - \widehat{\kappa_A}| = l\left| \frac{q_1}{F - T_1} - \frac{q}{F - u(l)} \right| \qquad (38)$$

and

$$|\kappa_B - \widehat{\kappa_B}| = \left| \frac{h(L - l)}{\dfrac{h(T_1 - T_a)}{q_1} - 1} - \frac{h(L - l)}{\dfrac{h(u(l) - T_a)}{q} - 1} \right| \qquad (39)$$

Let us consider Eq. (38), that is the error κ_A. Note that:

$$\left| \frac{q_1}{F - T_1} - \frac{q}{F - u(l)} \right| = \left| \frac{F(q_1 - q) - q_1 u(l) + T_1 q}{(F - T_1)(F - u(l))} \right|$$

or, equivalently, adding and subtracting $T_1 q_1$ in the numerator, rearranging terms and using condition (19) and triangular inequality it results:

$$\left| \frac{q_1}{F - T_1} - \frac{q}{F - u(l)} \right| = \left| \frac{(F - T_1)(q_1 - q) + q_1(T_1 - u(l))}{(F - T_1)(F - u(l))} \right|$$
$$\leq \frac{(F - T_1)|q_1 - q| + q_1|T_1 - u(l)|}{(F - T_1)(F - u(l))}.$$

Hence, by conditions (37), it follows that

$$\left| \frac{q_1}{F - T_1} - \frac{q}{F - u(l)} \right|$$
$$< \frac{(F - T_1)\varepsilon h\,(F - T_a) + \varepsilon\, q_1(F - T_a)}{(F - T_1)(F - u(l))} \qquad (40)$$
$$= \frac{(F - T_1)h + q_1}{F - u(l)}\,\varepsilon\left(\frac{F - T_a}{F - T_1} \right)$$

Then, replacing inequality (40) in (38), it results:

$$|\kappa_A - \widehat{\kappa_A}| < l\left(\frac{(F - T_1)h + q_1}{F - u(l)} \right)\varepsilon\left(\frac{F - T_a}{F - T_1} \right) \qquad (41)$$

From (19) and (27) it follows that:

$$|\kappa_A - \widehat{\kappa_A}| < l\left(\frac{(F - T_1)h + q_1}{M_1} \right)\left(\frac{1}{F - T_1} \right)\varepsilon \qquad (42)$$

or, equivalently,

$$|\kappa_A - \widehat{\kappa_A}| < hlA_2 \varepsilon \qquad (43)$$

where, the dimensionless constant A_2 depends on the data measurements, T_1, q_1 and it is given by:

$$A_2 = \frac{1}{M_1}\left(1 + \frac{q_1}{(F - T_1)h} \right) \qquad (44)$$

Let us consider the expression (39). Algebraic computations allow to rewrite it as:

$$|\kappa_B - \widehat{\kappa_B}| = h^2(L-l)\left|\frac{q_1(u(l)-T_a)-q(T_1-T_a)}{[h(T_1-T_a)-q_1][h(u(l)-T_a)-q]}\right| \quad (45)$$

Notice that, by the boundary condition at $x=L$ given in Eq. (1), the heat flux satisfies $q=h(u(L)-T_a)$ and the estimation error for κ_B given in (45) can also be written as:

$$|\kappa_B - \widehat{\kappa_B}| = h^2(L-l)\left|\frac{q_1(u(l)-T_a)-q(T_1-T_a)}{[h(T_1-T_a)-q_1][h(u(l)-u(L))]}\right| \quad (46)$$

Let us focus on the last factor in Eq. (46). The numerator and denominator of this factor are treated separately. Firstly, a bound depending on the measurement errors given in (37) are obtained for the numerator, while a constant bound is derived for the denominator.

Adding and subtracting q_1T_1 to the numerator, rearranging terms, taking common factor and using the triangle inequality results,

$$\begin{aligned}|q_1(u(l)-T_a)-q(T_1-T_a)| \\ = |q_1(u(l)-T_1)-(q-q_1)(T_1-T_a)| \\ \leq q_1|u(l)-T_1|+|q-q_1|(T_1-T_a)\end{aligned}$$

and conditions (37) yield

$$\begin{aligned}|q_1(u(l)-T_a)-q(T_1-T_a)| \\ < \left(q_1+h(T_1-T_a)\right)(F-T_a)\varepsilon\end{aligned} \quad (47)$$

A bound for the denominator in (46) is obtained by (33) since:

$$\begin{aligned}|[h(T_1-T_a)-q_1][h(u(l)-u(L))]| \\ > |h(T_1-T_a)-q_1|h\,M_2(F-T_a)\end{aligned} \quad (48)$$

Therefore, (46)-(48) along with condition (20) yield:

$$|\kappa_B - \widehat{\kappa_B}| < h(L-l)\frac{1}{M_2}\frac{h(T_1-T_a)+q_1}{h(T_1-T_a)-q_1}\varepsilon \quad (49)$$

or, equivalently,

$$|\kappa_B - \widehat{\kappa_B}| < h(L-l)B_2\varepsilon \quad (50)$$

where, the dimensionless constant B_2 depends on the data measurements, T_1, q_1 and it is given by:

$$B_2 = \frac{1}{M_2}\left(1+2\frac{q_1}{h(T_1-T_a)-q_1}\right) \quad (51)$$

Hence, the inequalities (43)-(44) and (50)-(51) imply that the estimation errors for κ_A, κ_B vanish as the measurement error (or ε) approaches 0. The dimensionless constant A_2 and B_2 determine the "velocity" of the converges of the estimation errors to 0. In other words, the estimation errors vanish as the errors in data approaches 0, independently of the dimensionless constant, but if these constants are big the data error should be small to have a good estimate for the conductivities.

5. LOCAL DEPENDENCE OF THE THERMAL CONDUCTIVITY WITH RESPECT TO THE TEMPERATURE

Eqns. (10)-(11) and (17)-(18) indicate that the estimated values of thermal conductivities depend on the parameters of the problem and on data. There are some tools that help to study the influence of the data on the estimated parameters. Among them, the sensitivity [18, 19] and elasticity analysis [15, 17, 19] are frequently used in the literature. The latter is widely used in economics and provides the percentage of error in the estimation for an error of 1% in the measured data.

Letting p the parameter to be estimated and d the data, the elasticity function is defined by:

$$E(d) = \frac{d}{p}\frac{\partial p}{\partial d} \quad (52)$$

The inverse problem presented here, consists in the identification of two parameters, and each of them depends on two data, hence four elasticity functions are considered for each case of estimation presented, analyzed and formalized in Section 3.

5.1 Elasticity analysis for the Case 1

For this case, the identification is based on two temperature measurements, one at the interface, T_1 and the other at the right edge, T_2. The elasticity functions of each parameter to be estimated with respect to each temperature measurement are obtained from the Eqns. (10)-(11) using the formula given in Eq. (52). This process leads to the following four elasticity functions.

$$E_1(T_1) = \frac{T_1}{\widehat{\kappa_A}}\frac{\partial\widehat{\kappa_A}}{\partial T_1} \Rightarrow E_1(T_1) = \frac{T_1}{F-T_1} \quad (53)$$

$$E_2(T_2) = \frac{T_2}{\widehat{\kappa_A}}\frac{\partial\widehat{\kappa_A}}{\partial T_2} \Rightarrow E_2(T_2) = \frac{T_2}{T_2-T_a} \quad (54)$$

$$E_3(T_1) = \frac{T_1}{\widehat{\kappa_B}}\frac{\partial\widehat{\kappa_B}}{\partial T_1} \Rightarrow E_3(T_1) = \frac{T_1}{T_2-T_1} \quad (55)$$

$$E_4(T_2) = \frac{T_2}{\widehat{\kappa_B}}\frac{\partial\widehat{\kappa_B}}{\partial T_2} \Rightarrow E_4(T_2) = \frac{T_2(T_1-T_a)}{(T_1-T_2)(T_2-T_a)} \quad (56)$$

Notice that E_1 and E_2 depend only on one of the temperature measurements, while E_3 and E_4 depend on both measurements. To analyzed E_3 a fixed value for T_2 is considered. Analogously, a fixed value of T_1 is considered in order to analyzed the elasticity function E_4.

5.2 Elasticity analysis for the Case 2

In this case, the identification is based on a temperature measurement at the interface, T_1 and a thermal flow measurement at the right edge, q_1. The elasticity functions are now obtained from Eqns. (17)-(18):

$$E_1(T_1) = \frac{T_1}{\widehat{\kappa_A}}\frac{\partial\widehat{\kappa_A}}{\partial T_1} \Rightarrow E_1(T_1) = \frac{T_1}{F-T_1} \quad (57)$$

$$E_2(q_1) = \frac{q_1}{\widehat{\kappa_A}} \frac{\partial \widehat{\kappa_A}}{\partial q_1} \Longrightarrow E_2(q_1) = 1 \qquad (58)$$

$$E_3(T_1) = \frac{T_1}{\widehat{\kappa_B}} \frac{\partial \widehat{\kappa_B}}{\partial T_1} \Longrightarrow E_3(T_1) = -\frac{T_1}{T_1 - T_a - \frac{q_1}{h}} \qquad (59)$$

$$E_4(q_1) = \frac{q_1}{\widehat{\kappa_B}} \frac{\partial \widehat{\kappa_B}}{\partial q_1} \Longrightarrow E_4(q_1) = \frac{T_1 - T_a}{T_1 - T_a - \frac{q_1}{h}} \qquad (60)$$

It can be observed from (57)-(58) that the estimation of κ_A does not depend on the value of q_1. In particular, the percentage of error in the estimation of κ_A is constant, equal to 1, regardless of the value of q_1. On the other hand, the percentage error in the estimation of κ_B depends on both measurements, as in the Case 1. Then, to analyzed E_3, a fixed value for q_1 is considered, while a fixed value of T_1 is considered in order to analyzed the elasticity function E_4.

6. NUMERICAL EXAMPLES

In this section, the thermal conductivity is estimated numerically for each case described in Section 3, where data are calculated from the solution to the forward problem and random noise is added to simulate experimental measurements. The inverse problem is solved considering the configuration described in Table 1 where the bar is assumed to be composed by Pb on the left ($0 < x < l$) and Fe on the right ($l < x < L$). The values for $\kappa_A. \kappa_B$ are obtained from [16].

Table 1. Problem setup data

L	l	F	T_a	h [20]	κ_A	κ_B
10m	4m	100°C	25°C	10W/m²°C	35W/m°C	73W/m°C

The thermal conductivity coefficients are estimated using Eqns. (10)-(11) or Eqns. (17)-(18) depending on the available data. For each case, a table is included to show the simulated data and the relative errors. Then, the absolute values of elasticity functions, given in Eqns. (53)-(56) or Eqns. (57)-(60), respectively, are plotted and the results are analyzed.

6.1 Numerical estimation for the case 1

The temperatures at the interface ($x=l$) and at the right edge ($x=L$) are given in Eq. (5). For this example, the values are $u(l)$=71.08°C and $u(L)$=50.30°C.

Table 2. Relative estimation errors of κ_A and κ_B

T_1(°C)	T_2(°C)	$\dfrac{\|\kappa_A - \widehat{\kappa_A}\|}{\kappa_A}$	$\dfrac{\|\kappa_B - \widehat{\kappa_B}\|}{\kappa_B}$
70.5	49.8	0.0392	0.0152
70.6	49.9	0.0320	0.0113
70.7	50	0.0248	0.0073
70.8	50.1	0.0176	0.0033
70.9	50.2	0.0103	0.0005
71	50.3	0.0029	0.0044
71.1	50.4	0.0044	0.0085
71.2	50.5	0.0119	0.0125
71.3	50.6	0.0194	0.0164

Table 2 shows the relative errors for the estimated parameters considering different values of simulated data T_1

and T_2, in a neighborhood of the exact values $u(l)$ and $u(L)$, respectively. The results show that, the maximum errors for the values considered here, are about 4% in the estimation of κ_A and 2% in the estimation of κ_B.

The absolute values of elasticity functions given by the expressions (53)-(56) are plotted in Figures 2-5.

The elasticity analysis shows that an error of 1% in the measurement of the temperature T_1 implies an error of 2.5% in the estimation of κ_A and of 3.5% in the estimation of κ_B. Furthermore, it is observed that an error of 1% in the measurement of the temperature T_2 translates into an error in the estimation of κ_A of 2% and of κ_B of 4.5%.
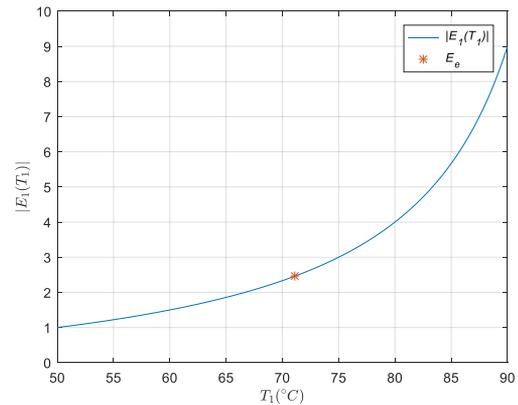
Figure 2. Plot of $|E_1(T_1)|$: Absolute value of the elasticity of $\widehat{\kappa_A}$ with respect to T_1
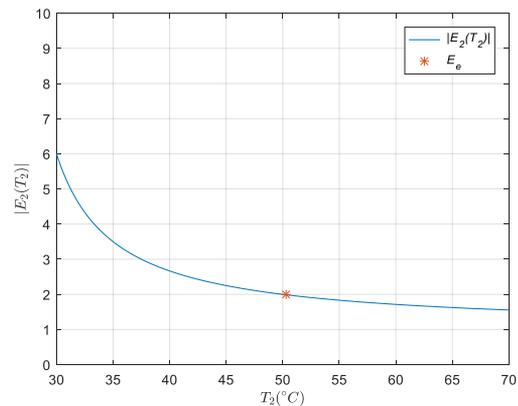
Figure 3. Plot of $|E_2(T_2)|$: Absolutve value of the elasticity of $\widehat{\kappa_A}$ with respect to T_2
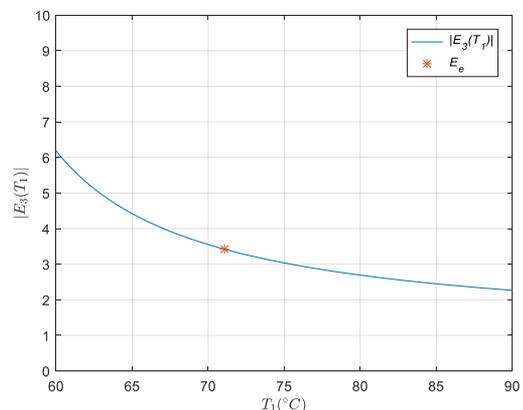
Figure 4. Plot of $|E_3(T_1)|$: Absolute value of the elasticity of $\widehat{\kappa_B}$ with respect to T_1 (T_2=50.30°C)
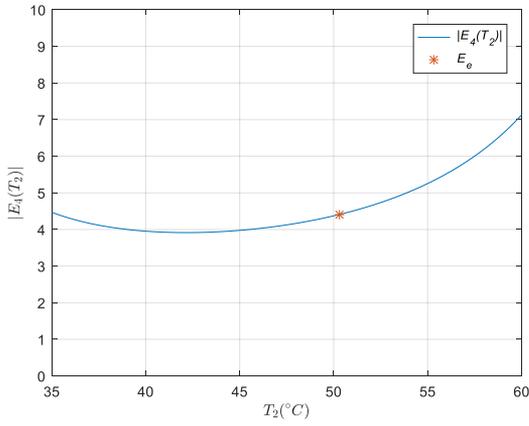
Figure 5. Plot of $|E_4(T_2)|$: Absolute value of the elasticity of $\widehat{\kappa_B}$ with respect to T_2 (T_1=71.08°C)

6.2 Numerical estimation for the case 2

The temperatures at the interface and the heat flux at the free edge are given in Eq. (5) and Eq. (13). Considering the values in Table 1, $u(l)$= 71.08°C and q=292.97 W/m^2.

Table 3 shows the relative errors obtained for the estimated parameters using different values of temperature T_1 and q_1 around the exact values of $u(l)$ and q, respectively. The results show that, the maximum errors for the values considered here, are about 2% in the estimation of κ_A and 4% in the estimation of κ_B.

Table 3. Relative estimation errors of κ_A and κ_B

T_1(°C)	q_1(W/m^2)	$\dfrac{|\kappa_A - \widehat{\kappa_A}|}{\kappa_A}$	$\dfrac{|\kappa_B - \widehat{\kappa_B}|}{\kappa_B}$
70.5	252.92	0.0201	0.0287
70.6	252.93	0.0167	0.0237
70.7	252.94	0.0134	0.0188
70.8	252.95	0.0099	0.0139
70.9	252.96	0.0065	0.0090
71	252.97	0.0030	0.0043
71.1	252.98	0.0004	0.0004
71.2	252.99	0.0039	0.0051
71.3	253.00	0.0074	0.0098

The absolute values of the elasticity functions given by the expression (57)-(60) are plotted below (Figures 6-9).
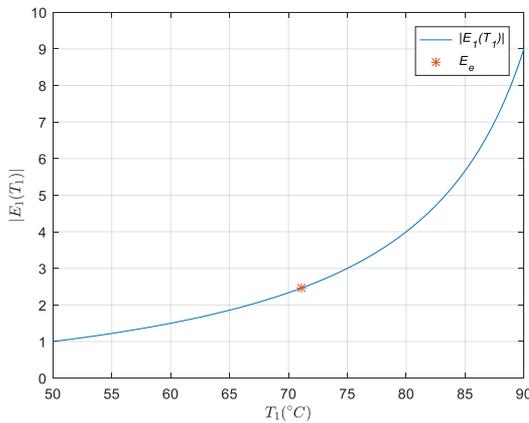
Figure 6. Plot of $|E_1(T_1)|$: Absolute value of the elasticity of $\widehat{\kappa_A}$ with respect to T_1

Figure 7. Plot of $|E_2(q_1)|$: Absolute value of the elasticity of $\widehat{\kappa_A}$ with respect to q_1
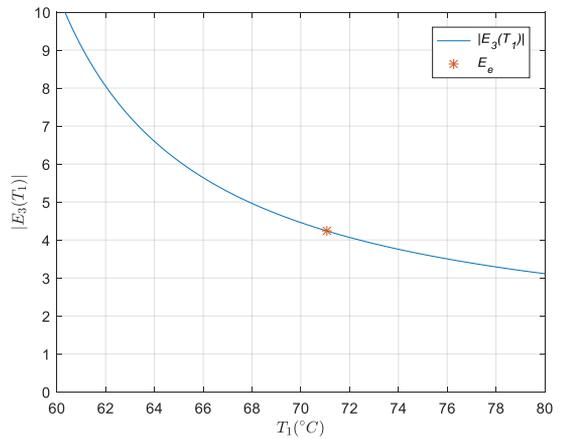
Figure 8. Plot of $|E_3(T_1)|$: Elasticity in absolute value of $\widehat{\kappa_B}$ with respect to T_1 (q_1=292.97 W/m^2)
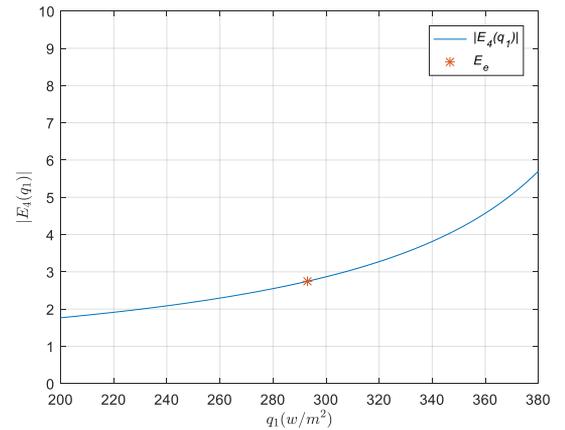
Figure 9. Plot of $|E_4(q_1)|$: Elasticity in absolute value of $\widehat{\kappa_B}$ with respect to q_1 (T_1=71.08°C)

The elasticity analysis, for this case, indicates that an error of 1% in the measurement of the temperature T_1 implies an error of 2.5% in the estimation of κ_A and of 4.2% in the estimation of κ_B. Furthermore, it is observed that an error of 1% in the measurement of the heat flux q_1 produces an error of 1% in the estimation of κ_A and about 2.8% for the estimation of κ_B.

7. CONCLUSIONS

It is considered a stationary heat transfer process in a bar made of two perfectly assembled sections of homogeneous and isotropic materials. A Dirichlet condition is assumed at the left boundary and a Robin type condition is assumed at the right one. The thermal conductivities of the materials are estimated based on two measurements. Firstly, two noisy temperature measurements, one at the interface and the other one at the right edge, are considered. Secondly, a noisy temperature measurement at the interface and a heat flow measurement at the right edge, are assumed.

The necessary and sufficient conditions are provided for the estimation of the parameters for each case. The bounds for the identification errors of the unknown coefficients are derived based on the data measurements, the room temperature and temperature values at the boundary and interface (the estimates are made from the data and the parameters of the direct problem). The error analysis makes clear that the identification is continuous with respect to data, in order words, they will vanish as the measurement errors approach zero; that is, the determination problem is a well-posed inverse problem.

An elasticity analysis is performed to study the local influence of the data on the estimated thermal conductivities and we show that the estimation errors are reasonable based on the simulated errors in the data.

The numerical examples presented here show a good performance in both cases and indicate that they could be useful to identify the materials in a bar involved in a similar heat transfer process.

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NOMENCLATURE

A_1	dimensionless auxiliary variable
A_2	dimensionless auxiliary variable
B_1	dimensionless auxiliary variable
B_2	dimensionless auxiliary variable
d	bar diameter, m
E	Elasticity
F	heat source, °C
h	convective coefficient, $W.m^{-2}.°C^{-1}$
L	interface position, m
L	bar length, m
q	heat flow at the right edge, $W. m^{-2}$
q_1	measured heat flow at the right edge, $W.m^{-2}$
M_1	dimensionless auxiliary variable
M_2	dimensionless auxiliary variable
T_1	measured temperature at the interface, °C
T_2	measured temperature at the right edge, °C
T_a	room temperature, °C
u	temperature function on the bar, °C
x	spatial variable, m

Greek symbols

ε	dimensionless measurement error
κ	thermal conductivity, $W.m^{-1}.°C^{-1}$
$\hat{\kappa}$	estimated thermal conductivity, $W.m^{-1}.°C^{-1}$
ς	auxiliary variable, $W^2.m^{-2}.°C^{-2}$

Subscripts

A	related to material A
B	related to material B
e	related to the exact value