



Wegstein's Method for Calculating the Global Extremum

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ABSTRACT

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This study discusses an economical and efficient method for calculating the global optimum of a function of many variables. The proposed algorithm can be attributed to methods based on auxiliary functions. The auxiliary function itself is obtained by converting the objective function using the Lebesgue integral and is a function of one variable. In a previously published paper by one of the authors of this article, this auxiliary function was used to calculate the global minimum of smooth multiextremal functions on convex closed sets. In the same article, an algorithm was proposed for dividing a segment into half to find a global minimum. And in this paper we consider the problem of finding the global minimum of continuous functions defined on bounded closed subsets of an n -dimensional Euclidean space. In addition, curious properties of the auxiliary function are established that are valid for any continuous objective function. For example, its non-negativity, positive homogeneity of some order, uniform continuity, differentiability and strict convexity are proved, and higher-order derivatives are calculated. The optimality criterion is established. The essence of this optimality criterion is that the value of a variable at which the auxiliary function and its derivatives are equal to zero up to a certain order turns out to be equal to the global minimum of the objective function. It follows from this optimality criterion that to calculate the global minimum of the objective function, it is sufficient to find the zero of the auxiliary function or its derivative up to the m -th order. Therefore, Wegstein's algorithm was used as a way to find the root of an equation with one unknown. In addition, the advantage of the Wegstein's method is that it always converges. And in this situation, it turned out to be more efficient, despite its slow convergence, since it requires almost half the number of calculations of the values of the auxiliary function and that halves the need for numerical calculations of multiple integrals with a large number of variables.

1. INTRODUCTION

The essence of the problem of finding the global extremum is to find a solution with the minimum (maximum) value of the objective function. Currently, a lot is known and written about methods and problems of global optimization [1-5]. Nevertheless, due to their practical importance, global optimization problems continuously arise and are being investigated in many fields of science [6-13]. Thus, there is a constant need for the development of new methods and approaches to solve them, since there is no universal algorithm for solving them.

To date, all known methods of global optimization can be divided into two categories [1, 4, 5], deterministic methods [1, 2] and stochastic methods [14, 15]. And optimization tasks are usually divided into two types: static and dynamic. The static type includes tasks in which it is necessary to determine the values of the arguments that make up the extreme value of the objective function. A dynamic type is considered a task when a function called a control function is to be defined, in which

the target functional takes a maximum or minimum value.

When solving static problems with a nonlinear objective function, great difficulties arise. The main difficulties are associated with multiextremality, large dimension and non-convexity of the objective function. To date, a considerable number of works devoted to overcoming the above difficulties have been published [12, 16-18].

In this paper, a new method for finding the global minimum based on the idea of auxiliary functions is proposed. At the same time, the object of research is not the objective function itself, but an auxiliary function of one variable constructed by converting the original goal function using the Lebesgue integral. It should be noted that the auxiliary function studied in this paper was considered in Ref. [19]. In it, with its help, an algorithm was proposed for dividing a segment in half to calculate the global extremum for smooth and multiextremal functions of several variables defined on convex compact sets. In this article, in contrast to the work [19], the main properties of this auxiliary function for any continuous goal function are studied in detail. An optimality criterion is established, and

Wegstein's method for finding the global minimum of continuous on bounded and closed sets of functions of many variables is proposed, which is more effective for this situation.

2. PROBLEM STATEMENT

Consider a measurable space with measure $(\mathbb{R}^n, \mathfrak{B}, \mu)$, where \mathbb{R}^n is a Euclidean n -dimensional space, \mathfrak{B} is a Borel σ -algebra in the space \mathbb{R}^n , μ is a Lebesgue measure over \mathfrak{B} . Let E be a closed set of space \mathbb{R}^n and $F: E \rightarrow \mathbb{R}$ is a continuous objective function with real values. From the continuity of F follows the measurability of the set $E(F, \alpha) = \{x \in E | F(x) \leq \alpha\}$ for each $\alpha \in \mathbb{R}$. In addition, we will assume that $\exists \alpha \in \mathbb{R}$ is a set $E(F, \alpha)$ compact, nonempty and $F \in L_m(E, \mu)$, for some positive integer m .

Consider the problem:

$$\bar{\alpha} = \underset{x \in E}{\text{globmin}} F(x). \quad (1)$$

Let's introduce an auxiliary function:

$$g_m(F, \alpha) = \int_E [|F(x) - \alpha| - F(x) + \alpha]^m d\mu, \quad (2)$$

which plays an important role in the fourth section of this paper. Here m is some given positive integer number. The purpose of this work is to study the basic properties of the function (2) and on their basis to construct an effective numerical method for solving the problem (1).

3. PROPERTIES OF THE AUXILIARY FUNCTION

Lemma 1. *The auxiliary function (2) has the following properties:*

- 1) $g_m(F, \alpha) \geq 0$;
- 2) $g_m(C, \alpha) = 0$ for any constant with $C \geq \alpha$;
- 3) $g_m(kF, k\alpha) = k^m g_m(F, \alpha)$, for a real number $k \geq 0$, $\alpha \geq \bar{\alpha}$;
- 4) $g_m(k + F, k + \alpha) = g_m(F, \alpha)$, for any real k and all $\alpha \geq \bar{\alpha}$;
- 5) $g_m(\alpha + F, \alpha) = g_m(F, 0)$, for all $\alpha \geq \bar{\alpha}$;
- 6) $g_m(\alpha F, \alpha) = \alpha^m g_m(F, 1)$, for each $\alpha \geq 0$;
- 7) $g_m(F, \alpha)$ is uniformly continuous on the interval $\alpha \geq \alpha_0$, where $\forall \alpha_0 > \bar{\alpha}$;
- 8) $g_m(F, \alpha)$ is midpoint convex on the interval $\alpha \geq \alpha_0$, where $\forall \alpha_0 > \bar{\alpha}$.

Proof.

$$\begin{aligned} 1) \quad g_m(F, \alpha) &= \int_{E \setminus E(F, \alpha)} [|F(x) - \alpha| - F(x) + \alpha]^m d\mu + \\ &+ \int_{E(F, \alpha)} [|F(x) - \alpha| - F(x) + \alpha]^m d\mu = \\ &= \int_{E \setminus E(F, \alpha)} [(F(x) - \alpha) - F(x) + \alpha]^m d\mu \\ &+ \int_{E(F, \alpha)} [-(F(x) - \alpha) - F(x) + \alpha]^m d\mu = \\ &= \int_{E(F, \alpha)} [2(\alpha - F(x))]^m d\mu \geq 0, \end{aligned}$$

Since the expression under the last integral is non-negative on the set $E(F, \alpha)$.

- 2) $g_m(C, \alpha) = \int_E [|C - \alpha| - C + \alpha]^m d\mu = \int_E [(C - \alpha) - C + \alpha]^m d\mu = 0$.
- 3) $g_m(kF, k\alpha) = \int_E [|kF(x) - k\alpha| - kF(x) + k\alpha]^m d\mu = k^m \int_E [|F(x) - \alpha| - F(x) + \alpha]^m d\mu = k^m g_m(F, \alpha)$.
- 4) $g_m(k + F, k + \alpha) = \int_E [|k + F(x) - k - \alpha| - k - F(x) + k + \alpha]^m d\mu = \int_E [|F(x) - \alpha| - F(x) + \alpha]^m d\mu = g_m(F, \alpha)$.
- 5) $g_m(\alpha + F, \alpha) = \int_E [| \alpha + F(x) - \alpha | - \alpha - F(x) + \alpha]^m d\mu = \int_E [|F(x) - 0| - F(x) + 0]^m d\mu = g_m(F, 0)$.
- 6) $g_m(\alpha F, \alpha) = \int_E [| \alpha F(x) - \alpha | - \alpha F(x) + \alpha]^m d\mu = \alpha^m \int_E [|F(x) - 1| - F(x) + 1]^m d\mu = \alpha^m g_m(F, 1)$.
- 7) Let's choose arbitrary α_1, α_2 such that $\alpha_1 > \alpha_2 \geq \alpha_0$. Let's estimate the difference modulus:

$$\begin{aligned} |g_m(F, \alpha_1) - g_m(F, \alpha_2)| &= \\ &= \left| \int_E [|F(x) - \alpha_1| - F(x) + \alpha_1]^m d\mu - \int_E [|F(x) - \alpha_2| - F(x) + \alpha_2]^m d\mu \right| = \\ &= \left| \int_{E(F, \alpha_1)} [2(\alpha_1 - F(x))]^m d\mu - \int_{E(F, \alpha_2)} [2(\alpha_2 - F(x))]^m d\mu \right| \\ &= \left| \int_{E(F, \alpha_2)} [2(\alpha_1 - F(x))]^m d\mu + \int_{E(F, \alpha_1) \setminus E(F, \alpha_2)} [2(\alpha_1 - F(x))]^m d\mu - \int_{E(F, \alpha_2)} [2(\alpha_2 - F(x))]^m d\mu \right| \leq \\ &\leq \left| \int_{E(F, \alpha_2)} \left[2(\alpha_1 - \alpha_2) \left((2(\alpha_1 - F(x)))^{m-1} + (2(\alpha_1 - F(x)))^{m-2} (2(\alpha_2 - F(x))) + \dots + (2(\alpha_1 - F(x))) (2(\alpha_2 - F(x)))^{m-2} + (2(\alpha_2 - F(x)))^{m-1} \right) \right] d\mu \right| + \\ &+ \left| \int_{E(F, \alpha_1) \setminus E(F, \alpha_2)} [2(\alpha_1 - F(x))]^m d\mu \right| = J_1 + J_2. \end{aligned}$$

Note that the following inequalities are met. On the set $E(F, \alpha_2)$: $(2(\alpha_1 - F(x)))^{m-1} \leq (2(\alpha_1 - \alpha_0))^{m-1} = K$,

$$(2(\alpha_2 - F(x)))^{m-1} \leq (2(\alpha_1 - \alpha_0))^{m-1} = K.$$

And on the set $E(F, \alpha_1) \setminus E(F, \alpha_2)$:

$$(2(\alpha_1 - F(x)))^m = (2(\alpha_1 - F(x)))^{m-1} 2(\alpha_1 - F(x)) \leq 2K|\alpha_1 - \alpha_2|.$$

Using them it is easy to get estimates:

$$J_1 \leq 2mK|\alpha_1 - \alpha_2|\mu(E(F, \alpha_2)) \leq 2mK|\alpha_1 - \alpha_2|\mu(E) = B_1|\alpha_1 - \alpha_2|, \text{ где } B_1 = 2mK\mu(E);$$

$$J_2 \leq 2K|\alpha_1 - \alpha_2|(\mu(E(F, \alpha_1)) - \mu(E(F, \alpha_2))) = 2K|\alpha_1 - \alpha_2|\mu(E(F, \alpha_1)) \leq 2K\mu(E) \leq B_2|\alpha_1 - \alpha_2|, \text{ где } B_2 = 2K\mu(E).$$

Thus, for any $\varepsilon > 0$, it is sufficient to take the number $\delta = \varepsilon/(B_1 + B_2)$, that for arbitrary α_1, α_2 for which $|\alpha_1 - \alpha_2| < \delta$ there is an inequality:

$$|g_m(F, \alpha_1) - g_m(F, \alpha_2)| \leq J_1 + J_2 < (B_1 + B_2)|\alpha_1 - \alpha_2| < \frac{(B_1 + B_2)\varepsilon}{(B_1 + B_2)} = \varepsilon.$$

This proves that (2) is uniformly continuous.

8) Let's choose arbitrary α_1, α_2 such that $\alpha_1 > \alpha_2 \geq \alpha_0$. Using the Cauchy inequality $(u_1 + u_2)^m \leq 2^{m-1}(u_1^m + u_2^m)$, for any $u_1, u_2 \geq 0$ and $m \in \mathbf{N}$. Let's show the fairness of inequality:

$$g_m\left(F, \frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{2}(g_m(F, \alpha_1) + g_m(F, \alpha_2)).$$

$$g_m\left(F, \frac{\alpha_1 + \alpha_2}{2}\right) = \int_E \left[\left| F(x) - \frac{\alpha_1 + \alpha_2}{2} \right| - F(x) + \frac{\alpha_1 + \alpha_2}{2} \right]^m d\mu =$$

$$= \frac{1}{2^m} \int_E [|2F(x) - (\alpha_1 + \alpha_2)| - 2F(x) + (\alpha_1 + \alpha_2)]^m d\mu \leq$$

$$\frac{1}{2^m} \int_E [|F(x) - \alpha_1 + F(x) - \alpha_2| - (F(x) - \alpha_1) - (F(x) - \alpha_2)]^m d\mu \leq$$

$$\leq \frac{1}{2^m} \int_E [(|F(x) - \alpha_1| - F(x) + \alpha_1) + (|F(x) - \alpha_2| - F(x) + \alpha_2)]^m d\mu = I.$$

According to the statement 1) of Lemma 1, two expressions in parentheses under the last integral have non-negative values, so applying the above Cauchy inequality we get the following estimate.

$$I \leq \frac{1}{2^m} \int_E 2^{m-1} [(|F(x) - \alpha_1| - F(x) + \alpha_1)^m + (|F(x) - \alpha_2| - F(x) + \alpha_2)^m] d\mu = \frac{1}{2} (g_m(F, \alpha_1) + g_m(F, \alpha_2)).$$

The lemma is proved.

Lemma 2. The auxiliary function (2) is differentiable at each point $\alpha > \bar{\alpha}$, where $\bar{\alpha} = \min_E F(x)$.

Proof. Let $\alpha + h > \alpha > \bar{\alpha}$, consider the increment of the auxiliary function.

$$g_m(F, \alpha + h) - g_m(F, \alpha) = \int_E [|F(x) - (\alpha + h)| - F(x) + (\alpha + h)]^m d\mu - \int_E [|F(x) - \alpha| - F(x) + \alpha]^m d\mu =$$

$$= \int_{E(F, \alpha+h)} [2(\alpha + h - F(x))]^m d\mu - \int_{E(F, \alpha)} [2(\alpha - F(x))]^m d\mu =$$

$$= \int_{E(F, \alpha)} [2(\alpha + h - F(x))]^m d\mu + \int_{E(F, \alpha+h) \setminus E(F, \alpha)} [2(\alpha + h - F(x))]^m d\mu - \int_{E(F, \alpha)} [2(\alpha - F(x))]^m d\mu =$$

$$= \int_{E(F, \alpha)} [2h((\alpha + h - F(x))^{m-1} + (\alpha + h - F(x))^{m-2}(\alpha - F(x)) + \dots + (\alpha - F(x))^{m-2}(\alpha + h - F(x)) + (\alpha - F(x))^{m-1})] d\mu +$$

$$+ \int_{E(F, \alpha+h) \setminus E(F, \alpha)} [2(\alpha + h - F(x))]^m d\mu = I_1 h + I_2.$$

It is easy to check that

$$\lim_{h \rightarrow 0} I_1 = 2 \int_{E(F, \alpha)} m [2(\alpha - F(x))]^{m-1} d\mu = 2m g_{m-1}((F, \alpha)).$$

It is not difficult to see that $I_2 = \int_{E(F, \alpha+h) \setminus E(F, \alpha)} [2(\alpha + h - F(x))]^m d\mu$ is infinitesimal for $h \rightarrow 0$, since the expression under the integral in square brackets does not exceed $2h$, and due to the continuity of the measure μ takes place:

$$0 \leq I_2 \leq (2h)^m \mu(E(F, \alpha + h) - \mu(E(F, \alpha))) \rightarrow 0$$

Thus,

$$\frac{dg_m}{d\alpha} = 2m g_{m-1}((F, \alpha)).$$

Lemma 2 is proved.

Corollary 1. When $m=1$

$$\frac{dg_1}{d\alpha} = 2\mu(E(F, \alpha)).$$

Corollary 2. For any given natural number m , the equality holds,

$$\frac{d^m g_m}{d\alpha^m} = (2)^m m! \mu(E(F, \alpha)).$$

Lemma 3. Let $\{\alpha_i\}$ be a decreasing sequence whose limit for $i \rightarrow \infty$ is α_0 and $g_m(F, \alpha_0) > 0$. Then $\alpha_0 > \bar{\alpha} = \min_E F(x)$.

Proof. Suppose the opposite, let $\alpha_0 \leq \bar{\alpha}$. Since $g_m(F, \alpha_0) > 0$, the set $E(F, \alpha_0) = \{x \in E | F(x) < \alpha_0\}$ is not empty. Therefore, the inequality $F(x) < \bar{\alpha}$ holds on this set $E(F, \alpha_0)$. And this contradicts the fact that $\bar{\alpha}$ is the value of the global minimum. Hence, $\alpha_0 > \bar{\alpha}$.

Lemma 3 is proved.

4. THE MAIN RESULT

Theorem 1. Let E be a closed set of space \mathbb{R}^n and $F: E \rightarrow \mathbb{R}$ is a continuous objective function with real values. The global minimum $\bar{\alpha}$ of the problem (1) is reached at the point \bar{x} (where $\bar{\alpha} = F(\bar{x})$ and $\bar{\alpha} = \sup\{\alpha \in \mathbb{R} \mid g_m(F, \alpha) = 0\}$ if and only if

$$g_m(F, \bar{\alpha}) = 0 \quad (3)$$

Proof of necessity. Suppose \bar{x} is the point of the global minimum, where $\bar{\alpha} = F(\bar{x})$ and $\bar{\alpha} = \sup\{\alpha \in \mathbb{R} \mid g_m(F, \alpha) = 0\}$ takes place, so $F(x) \geq \bar{\alpha}$, for any $x \in E$. Hence,

$$\begin{aligned} g_m(F, \bar{\alpha}) &= \int_E [|F(x) - \bar{\alpha}| - F(x) + \bar{\alpha}]^m d\mu \\ &= \int_E [(F(x) - \bar{\alpha}) - F(x) + \bar{\alpha}]^m d\mu = 0. \end{aligned}$$

The necessity is proved.

Let's prove sufficiency in the opposite way. Assume that (3) is fulfilled and $\bar{\alpha}$ is not the point of the global minimum, where $\bar{\alpha} = F(\bar{x})$ and $\bar{\alpha} = \sup\{\alpha \in \mathbb{R} \mid g_m(F, \alpha) = 0\}$ takes place. Let the value of the global minimum be $\tilde{\alpha}$. Note $2\beta = \bar{\alpha} - \tilde{\alpha} > 0$ and note that in this case $E(F, \tilde{\alpha})$ is strictly contained in the set $E(F, \bar{\alpha})$ and hence $\mu(E(F, \bar{\alpha})) > \mu(E(F, \tilde{\alpha}))$.

$$\begin{aligned} g_m(F, \bar{\alpha}) &= \int_E [|F(x) - \bar{\alpha}| - F(x) + \bar{\alpha}]^m d\mu \\ &= \int_{E \setminus E(F, \bar{\alpha})} [|F(x) - \bar{\alpha}| - F(x) + \bar{\alpha}]^m d\mu + \\ &+ \int_{E(F, \bar{\alpha})} [|F(x) - \bar{\alpha}| - F(x) + \bar{\alpha}]^m d\mu = \\ &= \int_{E \setminus E(F, \bar{\alpha})} [(F(x) - \bar{\alpha}) - F(x) + \bar{\alpha}]^m d\mu \\ &+ \int_{E(F, \bar{\alpha})} [-(F(x) - \bar{\alpha}) - F(x) + \bar{\alpha}]^m d\mu = \\ &= \int_{E(F, \bar{\alpha})} [2(\bar{\alpha} - F(x))]^m d\mu. \end{aligned}$$

Thus,

$$g_m(F, \bar{\alpha}) = \int_{E(F, \bar{\alpha})} [2(\bar{\alpha} - F(x))]^m d\mu. \quad (4)$$

From the equality $2\beta = \bar{\alpha} - \tilde{\alpha}$, we express $\bar{\alpha} = \tilde{\alpha} + 2\beta$ and note that $\bar{\alpha} > \tilde{\alpha} + \beta > \tilde{\alpha}$. Since $E(F, \bar{\alpha}) = \{x \in E \mid F(x) \leq \bar{\alpha}\}$, $E(F, \tilde{\alpha} + \beta) = \{x \in E \mid F(x) \leq \tilde{\alpha} + \beta\}$, $E(F, \tilde{\alpha}) = \{x \in E \mid F(x) \leq \tilde{\alpha}\}$, it is obvious that $E(F, \tilde{\alpha}) \subset E(F, \tilde{\alpha} + \beta) \subset E(F, \bar{\alpha})$.

When $\tilde{\alpha} < F(x) \leq \tilde{\alpha} + \beta < \bar{\alpha}$ it is easy to see that $\bar{\alpha} - F(x) > \bar{\alpha} - (\tilde{\alpha} + \beta) = \beta$. Now let's evaluate (4).

$$\begin{aligned} g_m(F, \bar{\alpha}) &= \int_{E(F, \bar{\alpha}) \setminus E(F, \tilde{\alpha} + \beta)} [2(\bar{\alpha} - F(x))]^m d\mu \\ &+ \int_{E(F, \tilde{\alpha} + \beta)} [2(\bar{\alpha} - F(x))]^m d\mu \geq \end{aligned}$$

$$\begin{aligned} &\geq \int_{E(F, \tilde{\alpha} + \beta)} [2(\bar{\alpha} - F(x))]^m d\mu > [2\beta]^m \int_{E(F, \tilde{\alpha} + \beta)} d\mu \\ &> [2\beta]^m \mu(E(F, \tilde{\alpha} + \beta)) > 0. \end{aligned}$$

And this contradicts equality (3). The theorem is proved.

The proved theorem is valid for every fixed positive integer m , since no restrictions were imposed on m during the proof.

Corollary. Let the conditions of Theorem 1 and $m = 1$ be fulfilled. Then

$$\bar{\alpha} = \frac{1}{\mu(E(F, \bar{\alpha}))} \int_{E(F, \bar{\alpha})} F(x) d\mu. \quad (5)$$

For $m = 1$, equality (4) has the form

$$\int_{E(F, \bar{\alpha})} (2(\bar{\alpha} - F(x))) d\mu = 0.$$

From here, expressing $\bar{\alpha}$, we get (5).

5. WEGSTEIN'S METHOD

Despite the fact that the Wegstein's method [20] converges slower than some other methods, it requires almost twice as many calculations of the values of the auxiliary function (2). And calculating the values of the auxiliary function leads to the numerical calculation of multiple integrals with a large number of variables, which is associated with great difficulties. Therefore, Wegstein's method turns out to be more effective for this situation. Besides, it always converges. Next, we will give a brief description of this method. According to (3) we will solve the equation $g_m(F, \alpha) = 0$ with respect to α . Below we will need:

$$\begin{aligned} \Psi(\alpha) &= \alpha - A g_m(F, \alpha), \\ A &= \frac{1}{2^{m+p} m B}, B = \int_E |F(x)|^{m-1} d\mu. \end{aligned}$$

In accordance with the study [20], we can choose the coefficient A at our discretion. Therefore, here we choose the smallest positive integer p in such a way that the following inequalities are satisfied:

$$0 < 1 - 2m A g_{m-1}(F, \tau) < 1. \quad (6)$$

The iterative procedure, according to Wegstein [20], consists of the following two formulas:

$$\alpha_{n+1} = \Psi(\tilde{\alpha}_n); \quad (7)$$

$$\tilde{\alpha}_{n+1} = \alpha_{n+1} - \frac{[\alpha_{n+1} - \alpha_n][\alpha_{n+1} - \tilde{\alpha}_n]}{\alpha_{n+1} - \alpha_n - \tilde{\alpha}_n + \tilde{\alpha}_{n-1}}; \quad (8)$$

After choosing a suitable initial approximation $x_0 \in E$, $\varepsilon > 0$ and accepting $\alpha_0 = F(x_0)$, $\alpha_1 = \Psi(\alpha_0)$, $\tilde{\alpha}_0 := \alpha_0$, $\tilde{\alpha}_1 := \alpha_1$, we apply formulas (7) and (8). Then we check the condition $\frac{|\tilde{\alpha}_n - \tilde{\alpha}_{n+1}|}{|\tilde{\alpha}_n|} \geq \varepsilon$. If it is executed, then we repeat the process; otherwise, the iteration ends and we take the root equal to $\tilde{\alpha}_{n+1}$.

To determine the coordinates of the global minimum points, we apply the method of statistical tests [21]. Suppose that l Wegstein iterations have been performed, that is, $\tilde{\alpha}_{n+1} = \tilde{\alpha}_l$.

Following [21], we randomly select s different n -dimensional points $x_j = (x_{j1}, \dots, x_{jn})$ from the set E . Calculate the values of the objective function at each of these points, denote $F_j = F(x_j)$, $j = 1, \dots, s$. Next, define the sets $W_k = \{x_j \in E: F(x_j) \leq \tilde{\alpha}_k, k = 0, 1, \dots, l\}$, denote ξ_k the number of elements in W_k . For each W_k , we define the values of n functions:

$$\bar{x}_i(\tilde{\alpha}_k) = \frac{\sum_{j=1}^{\xi_k} x_{ji} [2(\tilde{\alpha}_k - F_j)]^m}{\sum_{j=1}^{\xi_k} [2(\tilde{\alpha}_k - F_j)]^m}, i = 1, \dots, n. \quad (9)$$

For each of the functions (9) we construct an approximating parabola:

$$x_i(\tilde{\alpha}) = a_i \tilde{\alpha}^2 + b_i \tilde{\alpha} + c_i, i = 1, \dots, n. \quad (10)$$

The coefficients of the quadratic trinomial (10) can be calculated by solving the following systems of linear equations.

$$\begin{cases} M_4 a_i + M_3 b_i + M_2 c_i = M_{2,1}, \\ M_3 a_i + M_2 b_i + M_1 c_i = M_{1,1}, i = 1, \dots, n. \\ M_2 a_i + M_1 b_i + M_0 c_i = M_{0,1}; \end{cases}$$

Here the coefficients and free terms of the system are determined by the following formulas [21].

$$\begin{aligned} M_0 &= 1, M_1 = \frac{1}{l} \sum_{k=1}^l \tilde{\alpha}_k, M_2 = \frac{1}{l} \sum_{k=1}^l \tilde{\alpha}_k^2, M_3 = \frac{1}{l} \sum_{k=1}^l \tilde{\alpha}_k^3, \\ M_4 &= \frac{1}{l} \sum_{k=1}^l \tilde{\alpha}_k^4. \\ M_{0,1} &= \frac{1}{l} \sum_{k=1}^l \bar{x}_i(\tilde{\alpha}_k), M_{1,1} = \frac{1}{l} \sum_{k=1}^l \tilde{\alpha}_k \cdot \bar{x}_i(\tilde{\alpha}_k), M_{2,1} = \\ &= \frac{1}{l} \sum_{k=1}^l \tilde{\alpha}_k^2 \cdot \bar{x}_i(\tilde{\alpha}_k). \end{aligned}$$

Substituting the value $\tilde{\alpha}_l$ instead of $\tilde{\alpha}$ in (10), we obtain approximate coordinates of the global minimum $x^* = (x_1(\tilde{\alpha}_l), \dots, x_n(\tilde{\alpha}_l))$. It should be noted that if the objective function is multiextremal, then the set W_k at the k -th iteration can be decomposed into the union of several sets $W_k = \bigcup_{\eta=1}^{r_k} W_k^{(\eta)}$. After that, the above procedure for determining the coordinates of the global minimum is performed for each partial set $W_k^{(\eta)}$.

Next, we formulate and prove a theorem on the convergence rate of this algorithm.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Then the iterative sequence $\{\tilde{\alpha}_n\}$ constructed according to the Wegstein's algorithm (8) converges to the global minimum of the objective function $F(x)$ with linear convergence rate, that is, there are $\beta \in (0,1)$ and for $N \in \mathbb{N}$ and such that $\forall n > N$ is fulfilled $|\tilde{\alpha}_{n+1} - \bar{\alpha}| < \beta |\tilde{\alpha}_n - \bar{\alpha}|$.*

Proof of Theorem 2. The convergence of the iterative sequence $\{\tilde{\alpha}_n\}$ to the global minimum $\bar{\alpha}$ follows from Lemma 3 and Theorem 1. Next, we show that for all $n \geq 3$ there is $\alpha_{n+1} \leq \tilde{\alpha}_n$. It follows from Wegstein's algorithm that $\alpha_{n+1} = \Psi(\tilde{\alpha}_n)$. It is equivalent to the equality $\alpha_{n+1} - \tilde{\alpha}_n = -A \int_E [|F(x) - \tilde{\alpha}_n| - F(x) + \tilde{\alpha}_n]^m d\mu$. The right part of the last equality is non-positive by virtue of statement 1 of Lemma 1, therefore $\alpha_{n+1} - \tilde{\alpha}_n \leq 0$. Thus, for all $n \geq 3$ it is fulfilled:

$$\alpha_{n+1} \leq \tilde{\alpha}_n, \quad (11)$$

Since Wegstein's algorithm is applied starting from $n = 3$. Now we note that based on the basic idea of this method [20], it follows that for any $n \geq 3$: $\tilde{\alpha}_n$ always lies in the interval between α_{n+1} and α_n . Therefore, taking into account (11), the inequalities $\alpha_{n+1} \leq \tilde{\alpha}_n \leq \alpha_n$ are satisfied for all $n \geq 3$. Therefore,

$$\alpha_{n+2} \leq \tilde{\alpha}_{n+1} \leq \alpha_{n+1}. \quad (12)$$

Further, by (12) we get the following estimate of the absolute value of the difference:

$$|\tilde{\alpha}_{n+1} - \bar{\alpha}| \leq |\tilde{\alpha}_{n+1} - \bar{\alpha}| = |\Psi(\tilde{\alpha}_n) - \Psi(\bar{\alpha})| = |(\tilde{\alpha}_n - \bar{\alpha})\Psi'(\tau)| \quad (13)$$

where, τ lies between $\tilde{\alpha}_n$ and $\bar{\alpha}$. Note that from the convergence of the sequence $\{\tilde{\alpha}_n\}$ it follows that there will be $N \in \mathbb{N}$ such that $\forall n > N$ there will be an estimate $|\tau - F(x)| \leq |F(x)|$. Now we estimate the derivative $\Psi'(\tau)$ on the set E , taking into account inequalities (6).

$$\begin{aligned} 0 \leq |\Psi'(\tau)| &= \left| (\alpha - Ag_m(F, \alpha))'_{\alpha=\tau} \right| \\ &= \left| 1 - 2m Ag_{m-1}((F, \tau)) \right| = \\ &= \left| 1 - \frac{2m \int_E [|F(x) - \tau| - F(x) + \tau]^{m-1} d\mu}{2^{m+p} m \int_E |F(x)|^{m-1} d\mu} \right| \\ &= \left| 1 - \frac{2m \int_{E(F, \tau)} [2(\tau - F(x))]^{m-1} d\mu}{2^{m+p} m \int_E |F(x)|^{m-1} d\mu} \right| \leq \\ &\leq \left| 1 - \frac{2^m m \int_{E(F, \tau)} |F(x)|^{m-1} d\mu}{2^{m+p} m \int_E |F(x)|^{m-1} d\mu} \right| = \beta < 1. \end{aligned}$$

Thus, based on (13), we obtain an estimate:

$$|\tilde{\alpha}_{n+1} - \bar{\alpha}| \leq |\Psi'(\tau)| |\tilde{\alpha}_n - \bar{\alpha}| < \beta |\tilde{\alpha}_n - \bar{\alpha}|.$$

Theorem 2 is proved.

From the proof of this theorem, we can notice the following fact. Given that for any τ inequality $\int_{E(F, \tau)} |F(x)|^{m-1} d\mu \leq \int_E |F(x)|^{m-1} d\mu$, we get the following estimate of the rate of convergence:

$$\beta = \left| 1 - \frac{2^m m \int_{E(F, \tau)} |F(x)|^{m-1} d\mu}{2^{m+p} m \int_E |F(x)|^{m-1} d\mu} \right| \leq \left| 1 - \frac{1}{2^p} \right|.$$

6. CONCLUSION

The basic properties of the auxiliary function are investigated. It turned out that the auxiliary function for any objective function is non-negative, positively homogeneous, uniformly continuous and differentiable. Higher-order derivatives are calculated. The criterion of optimality is established. A numerical method for finding the global minimum of the objective function is constructed, which is based on the Wegstein method. The proposed algorithm converges in any case. An estimate of the convergence rate of the iterative procedure is obtained.

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