Mathematical Modelling of Engineering Problems
Vol. 8, No. 5, October, 2021, pp. 819-825
Journal homepage: http://iieta.org/journals/mmep

Numerical Solution of Two-Parameter Singularly Perturbed Convection-Diffusion Boundary Value Problems via Fourth Order Compact Finite Difference Method

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https://doi.org/10.18280/mmep.080519

Received: 11 February 2020
Accepted: 9 August 2021

Keywords:
compact finite difference method, convection-diffusion, singular perturbation, two-parameter, uniform mesh

ABSTRACT

In this study, we have developed a fourth order compact finite difference method for finding the numerical solution of two-parameter singularly perturbed convection-diffusion boundary value problems. We have used fourth order compact finite difference method on uniform mesh which provides a tridiagonal linear system of equations. The convergence analysis of the proposed method is established through a matrix analysis approach and it is proved that present method gives fourth order convergence results. Present method is implemented on two numerical examples for checking the efficiency and precision of the method. Numerical outcomes are exhibited which supports the theoretical outcomes. Numerical outcomes are compared with other existing methods and found that present method gives more accurate approximate solution as compare to the other existing methods.

1. INTRODUCTION

We consider the following two-parameter singularly perturbed convection--diffusion boundary value problem:

$$-\varepsilon u''(x)+\mu p(x)u'(x)+q(x)u(x)=f(x), x\in(0,1) \tag{1}$$

$$u(0)=\alpha, u(1)=\beta. \tag{2}$$

with two small positive parameters, $0<\varepsilon\ll 1$ and $0<\mu\ll 1$. The function $p(x), q(x)$ and $f(x)$ are sufficiently smooth real valued function and satisfied $p(x)\geq p^*>0, q(x)\geq q^*>0$ and $f(x)\geq f^*>0$ for $x\in(0,1)$. Under these assumptions problem (1) is characterized into two cases:

1) When we put $\mu=0$, in Eq. (1), then the Eq. (1) is known as reaction-diffusion singular perturbation problem.

2) When we put $\mu=1$, in Eq. (1), then the Eq. (1) is known as convection-diffusion singular perturbation problem.

These kinds of problems arise in numerous field like applied mathematics, chemical reactor theory and control theory [1-6]. O'Malley [7] has discussed the nature of two-parameter problems by asymptotic expansion where the ratio of μ and ε have significant role in solution. The two-parameter singular perturbation boundary value problems have two cases $\frac{\mu^2}{\varepsilon\to 0}$ as $\varepsilon\to 0$ and $\frac{\varepsilon}{\mu^2\to 0}$ as $\mu\to 0$ an established sufficient condition for convergence.

We frequently discuss the boundary value problems in which one or two small positive parameter multiplies with the derivatives. A lot of research work has been done by researchers for single parameter convection-diffusion and reaction-diffusion problems [8-11]. However, only few researchers have studied two-parameter singular perturbation boundary value problems [12-21]. Shishkin and Titov [22] have discussed an exponentially fitted finite difference method based on a uniform mesh to obtain the approximate solution of two-parameter boundary value problems. Zahra and El Mhlawy [21] have solved two parameter singularly perturbed semi-linear boundary value problem via exponential spline with Shishkin mesh. Khandelwal and Khan [23] have discussed the numerical solution of problem (1) and (2) by using non-polynomial cubic spline method. For more detail about singular perturbation problems readers are referred to books [24-26] and references therein.

The remaining part of the paper is arranged as: In section 2 we have given a brief description of the proposed method for the numerical solution of problem (1) and (2). Convergence analysis of the method is presented in section 3. Section 4 presents the numerical results and comparisons are made with other existing methods. Finally, the conclusion is given at the end of the paper in section 5.

2. DESCRIPTION OF THE METHOD

We divide the interval [0,1] into N equal subinterval and choice piecewise uniform mesh points represented by $\pi=\{0=x_0,x_1,x_2,\dots,x_{N-1},x_N=1\}$ i.e., $x_i=x_0+ih$, $i=0,1,2,\dots,N$, where $h=\frac{1}{N}$. For straightforwardness, let us denote $p(x_i)=p_i, q(x_i)=q_i, f(x_i)=f_i$, $u(x_i)=u_i, u'(x_i)=u_i', u''(x_i)=u''$ and $u^{(n)}(x_i)=u_i^{(n)}$. Assume that $u(x)$ has continuous fourth order derivatives on [0,1]. Using the Taylor's series expansion, we obtain:

$$u_{i+1}=u_i+hu_i'+\frac{h^2}{2!}u_i''+\frac{h^3}{3!}u_i^{(3)}+\frac{h^4}{4!}u_i^{(4)}+\frac{h^5}{5!}u_i^{(5)}+\frac{h^6}{6!}u_i^{(6)}+\frac{h^7}{7!}u_i^{(7)}+O\left(h^8\right) \tag{3}$$

$$u_{i-1}=u_i-hu_i'+\frac{h^2}{2!}u_i''-\frac{h^3}{3!}u_i^{(3)}+\frac{h^4}{4!}u_i^{(4)}-\frac{h^5}{5!}u_i^{(5)}+\frac{h^6}{6!}u_i^{(6)}-\frac{h^7}{7!}u_i^{(7)}+O\left(h^8\right) \tag{4}$$

Subtracting Eq. (4) from Eq. (3), we obtain second order finite difference approximation for u_i':

$$\delta_x u_i=\frac{u_{i+1}-u_{i-1}}{2h}+T_1 \tag{5}$$

where, $T_1=-\frac{h^2}{6}u_i^{(3)}-\frac{h^4}{124}u_i^{(5)}+O\left(h^6\right)$.

Adding Eq. (3) from Eq. (4), we obtain second order finite difference approximation for u_i'':

$$\delta_x^2 u_i=\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}+T_2 \tag{6}$$

where, $T_2=-\frac{h^2}{12}u_i^{(4)}-\frac{h^4}{360}u_i^{(6)}+O\left(h^7\right)$.

The central difference discretization of Eq. (1) can be written as:

$$-\varepsilon\delta_x^2u_i+\mu p_i\delta_x u_i+q_iu_i+R=f_i,\quad i=1,2,\ldots,N-1, \tag{7}$$

where the truncation error R is given by:

$$R=\frac{h^2}{12}\left\{-2\mu p_i u_i^{(3)}+\varepsilon u_i^{(4)}\right\}+\frac{h^4}{360}\left\{-3\mu p_i u_i^{(5)}+\varepsilon u_i^{(6)}\right\}+O\left(h^6\right). \tag{8}$$

Discretized Eq. (1) at $x=x_i, i=0,1,2,\ldots,N$, we obtain,

$$-\varepsilon u_i''+\mu p_i u_i'+q_i u_i=f_i. \tag{9}$$

To obtain the fourth order accuracy, the terms containing h^2 in Eq. (8) must be further approximated. For approximating the $u_i^{(3)}$ and $u_i^{(4)}$. We first differentiating Eq. (1) with respect to x and then at $x=x_i$, we get,

$$u_i^{(3)}=\frac{\mu p_i}{\varepsilon}u_i''+\frac{\left(\mu p_i'+q_i\right)}{\varepsilon}u_i'+\frac{q_i'}{\varepsilon}u_i-\frac{f_i'}{\varepsilon},\quad i=0,1,2,\ldots,N. \tag{10}$$

Now, differentiating Eq. (1) twice with respect to x and then at $x=x_i$, we have:

$$u_i^{(4)}=\left\{\frac{\mu^2p_i^2}{\varepsilon^2}+\frac{2\mu p_i'}{\varepsilon}+\frac{q_i}{\varepsilon}\right\}u_i''+\left\{\frac{\mu^2p_ip_i'+\mu p_iq_i}{\varepsilon^2}+\frac{\mu p_i''}{\varepsilon}+\frac{2q_i'}{\varepsilon}\right\}u_i'+\left\{\frac{\mu p_iq_i'}{\varepsilon^2}+\frac{q_i''}{\varepsilon}\right\}u_i-\frac{\mu p_i f_i'}{\varepsilon^2}-\frac{f_i''}{\varepsilon},\ i=0,1,2,\ldots,N. \tag{11}$$

Using Eqns. (10) and (11) in Eq. (8), we obtain:

$$R=\left\{-\frac{\mu^2p_i^2h^2}{12\varepsilon}+\frac{\mu p_i'h^2}{6}+\frac{q_ih^2}{12}\right\}u_i''+\left\{-\frac{\mu^2p_ip_i'h^2}{12\varepsilon}-\frac{\mu p_iq_ih^2}{12\varepsilon}+\frac{\mu p_i''h^2}{12}+\frac{q_i'h^2}{6}\right\}u_i'+\left\{-\frac{\mu p_iq_i'h^2}{12\varepsilon}+\frac{q_i''h^2}{12}\right\}u_i+\frac{\mu p_if_i'h^2}{12\varepsilon}-\frac{f_i''h^2}{12}+\frac{h^4}{360}\left\{-3\mu p_iu_i^{(5)}+\varepsilon u_i^{(6)}\right\}+O\left(h^6\right). \tag{12}$$

Replacing u_i' and u_i'' by their central difference approximations in above equation, we get:

$$R=\left\{-\frac{\mu^2p_i^2h^2}{12\varepsilon}+\frac{\mu p_i'h^2}{6}+\frac{q_ih^2}{12}\right\}\delta_x^2u_i+\left\{-\frac{\mu^2p_ip_i'h^2}{12\varepsilon}-\frac{\mu p_iq_ih^2}{12\varepsilon}+\frac{\mu p_i''h^2}{12}+\frac{q_i'h^2}{6}\right\}\delta_xu_i+\left\{-\frac{\mu p_iq_i'h^2}{12\varepsilon}+\frac{q_i''h^2}{12}\right\}u_i+\frac{\mu p_if_i'h^2}{12\varepsilon}-\frac{f_i''h^2}{12}+\frac{h^4}{360}\left\{-3\mu p_iu_i^{(5)}+\varepsilon u_i^{(6)}\right\}+O\left(h^6\right), i=1,2,\ldots,N-1. \tag{13}$$

From Eqns. (7) and (13), we have,

$$\left\{-\varepsilon-\frac{\mu^2p_i^2h^2}{12\varepsilon}+\frac{\mu p_i'h^2}{6}+\frac{q_ih^2}{12}\right\}\delta_x^2u_i+\left\{\mu p_i-\frac{\mu^2p_ip_i'h^2}{12\varepsilon}-\frac{\mu p_iq_ih^2}{12\varepsilon}+\frac{\mu p_i''h^2}{12}+\frac{q_i'h^2}{6}\right\}\delta_xu_i+\left\{q_i-\frac{\mu p_iq_i'h^2}{12\varepsilon}+\frac{q_i''h^2}{12}\right\}u_i=f_i-\frac{\mu p_if_i'h^2}{12\varepsilon}+\frac{f_i''h^2}{12}+\frac{h^4}{360}\left\{3\mu p_iu_i^{(5)}-\varepsilon u_i^{(6)}\right\}+O\left(h^6\right),\quad i=1,2,\ldots,N-1. \tag{14}$$

Substituting the values from Eqns. (5) and (6) in Eq. (14). After simplifying, we obtain:

$$\begin{aligned}&\left\{-\varepsilon-\frac{\mu^2p_i^2h^2}{12\varepsilon}+\frac{\mu p_i'h^2}{6}+\frac{q_ih^2}{12}-\frac{\mu p_ih}{2}+\frac{\mu^2p_ip_i'h^3}{24\varepsilon}+\frac{\mu p_iq_ih^3}{24\varepsilon}-\frac{\mu p_i'''h^3}{24}-\frac{q_i'h^3}{12}\right\}u_{i-1}\\&+\left\{2\varepsilon+\frac{\mu^2p_i^2h^2}{6\varepsilon}-\frac{\mu p_i'h^2}{3}+\frac{5q_ih^2}{6}-\frac{\mu p_iq_ih^4}{12\varepsilon}+\frac{q_i'h^4}{12}\right\}u_i\\&+\left\{-\varepsilon-\frac{\mu^2p_i^2h^2}{12\varepsilon}+\frac{\mu p_i'h^2}{6}+\frac{q_ih^2}{12}+\frac{\mu p_ih}{2}-\frac{\mu^2p_ip_i'h^3}{24\varepsilon}-\frac{\mu p_iq_ih^3}{24\varepsilon}+\frac{\mu p_i'''h^3}{24}+\frac{q_i'h^3}{12}\right\}u_{i+1}\\&=f_ih^2-\frac{\mu p_if_i'h^4}{12\varepsilon}+\frac{f_i''h^4}{12}+T,\ i=1,2,\ldots,N-1.\end{aligned}\tag{15}$$

where, $T=\frac{\mu p_iu_i^{(5)}h^6}{120}-\frac{\varepsilon u_i^{(6)}h^6}{360}$ is the local truncation error.

From Eq. (15), we obtain the three term recurrence relation of the form:

$$-E_iu_{i-1}+F_iu_i-G_iu_{i+1}=H_i,\qquad i=1,2,\ldots,N-1,\tag{16}$$

where,

$$E_i=\varepsilon+\frac{\mu^2p_i^2h^2}{12\varepsilon}-\frac{\mu p_i'h^2}{6}-\frac{q_ih^2}{12}+\frac{\mu p_ih}{2}-\frac{\mu^2p_ip_i'h^3}{24\varepsilon}-\frac{\mu p_iq_ih^3}{24\varepsilon}+\frac{\mu p_i'''h^3}{24}+\frac{q_i'h^3}{12},$$

$$F_i=2\varepsilon+\frac{\mu^2p_i^2h^2}{6\varepsilon}-\frac{\mu p_i'h^2}{3}+\frac{5q_ih^2}{6}-\frac{\mu p_iq_ih^4}{12\varepsilon}+\frac{q_i'h^4}{12},$$

$$G_i=\varepsilon+\frac{\mu^2p_i^2h^2}{12\varepsilon}-\frac{\mu p_i'h^2}{6}-\frac{q_ih^2}{12}-\frac{\mu p_ih}{2}+\frac{\mu^2p_ip_i'h^3}{24\varepsilon}+\frac{\mu p_iq_ih^3}{24\varepsilon}-\frac{\mu p_i'''h^3}{24}-\frac{q_i'h^3}{12},$$

$$H_i=f_ih^2-\frac{\mu p_if_i'h^4}{12\varepsilon}+\frac{f_i''h^4}{12}.$$

It is easily seen that the Eq. (16) gives $(N-1)$ algebraic equations with $(N-1)$ unknowns. These systems of equations are written in matrix form $AX=B$, (where A is a tri-diagonal coefficient matrix, B is constant column matrix and X is an unknown's column matrix). This matrix form has been solved by Thomas algorithm in MATLAB software for getting the values of unknowns. Obtained unknowns values are called the numerical solution of problem (1) and (2).

3. CONVERGENCE ANALYSIS

Writing Eq. (15) in matrix vector form, we obtain:

$$AU=C,\tag{17}$$

where, $A=(a_{ij}),\ 1\le i,j\le N-1$ is a tri-diagonal matrix of order $(N-1)$, with

$$a_{i\,i+1}=\varepsilon+\frac{\mu^2p_i^2h^2}{12\varepsilon}-\frac{\mu p_i'h^2}{6}-\frac{q_ih^2}{12}-\frac{\mu p_ih}{2}+\frac{\mu^2p_ip_i'h^3}{24\varepsilon}+\frac{\mu p_iq_ih^3}{24\varepsilon}-\frac{\mu p_i'''h^3}{24}-\frac{q_i'h^3}{12},$$

$$a_{i\,i}=2\varepsilon+\frac{\mu^2p_i^2h^2}{6\varepsilon}-\frac{\mu p_i'h^2}{3}+\frac{5q_ih^2}{6}-\frac{\mu p_iq_ih^4}{12\varepsilon}+\frac{q_i'h^4}{12},$$

$$a_{i\,i-1}=\varepsilon+\frac{\mu^2p_i^2h^2}{12\varepsilon}-\frac{\mu p_i'h^2}{6}-\frac{q_ih^2}{12}+\frac{\mu p_ih}{2}-\frac{\mu^2p_ip_i'h^3}{24\varepsilon}-\frac{\mu p_iq_ih^3}{24\varepsilon}+\frac{\mu p_i'''h^3}{24}+\frac{q_i'h^3}{12},$$

and $C=(c_i)$ is a column vector with

$$c_i=f_ih^2-\frac{\mu p_if_i'h^4}{12\varepsilon}+\frac{f_i''h^4}{12},\ \ i=1,2,\ldots,N-1.$$

and with the local truncation error

$$T_i(h)=\frac{\mu p_iu_i^{(5)}h^6}{120}-\frac{\varepsilon u_i^{(6)}h^6}{360}\tag{18}$$

We also have

$$A\bar{U}-T(h)=C\tag{19}$$

where, $\bar{U}=(\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_{N-1})^T$ denotes the exact solution and $T(h)=(T_1(h),T_2(h),\ldots,T_{N-1}(h))^T$ denotes local truncation error. From Eq. (17) and (19), we get

$$AE=T(h)\tag{20}$$

where, $E=\bar{U}-U=(e_1,e_2,\ldots,e_{N-1})^T$.

Let S_i be the sum of elements of the i^{th} row of the matrix A, then

$$S_1=\sum_{j=1}^{N-1}a_{1j}=\varepsilon+\frac{\mu^2p_1^2h^2}{12\varepsilon}-\frac{\mu p_1'h^2}{6}+\frac{11q_1h^2}{12}+\frac{\mu p_1h}{2}-\frac{\mu^2p_1p_1'h^3}{24\varepsilon}-\frac{\mu p_1q_1h^3}{24\varepsilon}+\frac{\mu p_1'''h^3}{24}+\frac{q_1'h^3}{12}-\frac{\mu p_1q_1h^4}{12\varepsilon}+\frac{q_1'h^4}{12},\ i=1.$$

$$S_i=\sum_{j=1}^{N-1}a_{ij}=q_ih^2-\frac{\mu p_iq_ih^4}{12\varepsilon}+\frac{q_i'h^4}{12},\quad i=2,3,\ldots,N-2.$$

$$\begin{aligned}S_{N-1}&=\sum_{j=1}^{N-1}a_{N-1,j}\\&=\varepsilon+\frac{\mu^2p_{N-1}^2h^2}{12\varepsilon}-\frac{\mu p_{N-1}'h^2}{6}+\frac{11q_{N-1}h^2}{12}-\frac{\mu p_{N-1}h}{2}\\&+\frac{\mu^2p_{N-1}p_{N-1}'h^3}{24\varepsilon}+\frac{\mu p_{N-1}q_{N-1}h^3}{24\varepsilon}-\frac{\mu p_{N-1}'''h^3}{24}\end{aligned}$$

$$-\frac{q'_{N-1}h^3}{12}-\frac{\mu p_{N-1}q_{N-1}h^4}{12\varepsilon}+\frac{q'_{N-1}h^4}{12}, \quad i=N-1.$$

If we choose h sufficiently small, matrix A becomes irreducible and monotone [27]. It follows that A^{-1} exists and its elements are nonnegative. Hence, from Eq. (20), we have

$$E=A^{-1}T(h) \tag{21}$$

Let $a_{k,i}^{-1}$ is the $(k,i)^{th}$ element of the matrix A^{-1}. We define

$$\left\|a_{k,i}^{-1}\right\|=\max_{1\le k\le N-1}\sum_{i=1}^{N-1}\left|a_{k,i}^{-1}\right| \tag{22}$$

and

$$\|T\|=\max_{1\le k\le N-1}\left|T_k(h)\right| \tag{23}$$

In addition, from the theory of matrices, we have

$$\sum_{i=1}^{N-1}a_{k,i}^{-1}S_i=1, \quad k=1,2,\ldots,N-1. \tag{24}$$

Therefore

$$a_{k,i}^{-1}\le\frac{1}{\min\limits_{1\le i\le N-1}S_i}=\frac{1}{h^2Q_{i_0}}, \tag{25}$$

where $Q_{i_0}=\frac{1}{h^2}\min\limits_{1\le i\le N-1}S_i>0$, for some i_0 between 1 to N-1.

From Eqns. (18), (21), (22) and (25), we have

$$e_i=\sum_{i=1}^{N-1}a_{k,i}^{-1}T_i(h), \quad k=1,2,\ldots,N-1, \tag{26}$$

and therefore

$$\left|e_i\right|\le\frac{Kh^4}{\left|Q_{i_0}\right|}, \quad i=1,2,\ldots,N-1, \tag{27}$$

where, K is constant independent of h. It follows that $\|E\|=O(h^4)$. This implies that the present method gives a fourth order convergence.

4. NUMERICAL EXAMPLES

To exhibit the relevance of the proposed method, we have considered two numerical examples.

Example 1: Consider the following two-parameter singularly perturbed boundary value problem from [13, 15].

$-\varepsilon u''+\mu u'+u=1,\ x=(0,1)$

subject to boundary conditions:

$u(0)=0, u(1)=0.$

The exact solution of the example 1 is

$$u(x)=\frac{e^{-\frac{\mu}{2\varepsilon}}}{-1+e^{\frac{\sqrt{4\varepsilon+\mu^2}}{\varepsilon}}}\{e^{-\frac{\mu}{2\varepsilon}}+e^{\frac{\mu+2\sqrt{4\varepsilon+\mu^2}}{2\varepsilon}} -e^{\frac{(1+x)\mu+(2-x)\sqrt{4\varepsilon+\mu^2}}{2\varepsilon}}+e^{\frac{x\mu+\sqrt{4\varepsilon+\mu^2}-x\sqrt{4\varepsilon+\mu^2}}{2\varepsilon}} +e^{\frac{\mu+x\mu+x\sqrt{4\varepsilon+\mu^2}}{2\varepsilon}}-e^{\frac{x\mu+\sqrt{4\varepsilon+\mu^2}+x\sqrt{4\varepsilon+\mu^2}}{2\varepsilon}}\}.$$

The point wise absolude errors of example 1 are presented in Tables 1 and 2 for different values of ε,μ and N. Comparisons with other existing techniques are also shown in the same tables. The Tables 1 and 2 show that the present method gives better approximate solution than the other existing methods at the same number of mesh points.

Table 1. Comparison of point wise error with other existing methods of example 1

$x\downarrow$	$\varepsilon=0.1,\mu=1,N=32$			$\varepsilon=0.1,\mu=1,N=128$		
	Kadalbajoo and Yadaw [15]	Gracia et al. [13]	Present method	Kadalbajoo and Yadaw [15]	Gracia et al. [13]	Present method
1/16	2.74E-02	3.27E-06	7.35E-10	6.8E-03	1.15E-06	2.93E-12
2/16	2.59E-02	5.44E-06	3.84E-10	6.4E-03	2.18E-06	1.35E-12
4/16	2.30E-02	3.98E-06	1.41E-08	5.7E-03	3.72E-06	5.46E-11
6/16	2.04E-02	1.61E-05	6.79E-08	5.0E-03	3.92E-06	2.63E-10
12/16	2.50E-02	8.52E-04	1.94E-06	4.0E-04	4.45E-05	7.53E-09
14/16	3.30E-02	1.70E-03	3.82E-06	9.4E-03	9.61E-05	1.48E-08

Table 2. Comparison of point wise error with other existing methods of example 1

$x\downarrow$	$\varepsilon=0.01,\mu=1,N=32$			$\varepsilon=0.01,\mu=1,N=128$		
	Kadalbajoo and Yadaw [15]	Gracia et al. [13]	Present method	Kadalbajoo and Yadaw [15]	Gracia et al. [13]	Present method
1/16	2.95E-02	4.55E-06	3.70E-08	7.3E-03	2.84E-07	1.46E-10
2/16	2.78E-02	8.55E-06	6.96E-08	6.9E-03	5.35E-07	2.74E-10
4/16	2.45E-02	1.51E-05	1.23E-07	6.1E-03	9.45E-07	4.84E-10
6/16	2.17E-02	2.00E-05	1.63E-07	5.4E-03	1.25E-06	6.42E-10
12/16	1.50E-02	3.07E-05	2.24E-07	3.7E-03	1.73E-06	8.85E-10
14/16	1.29E-02	1.41E-03	1.65E-05	3.3E-03	7.31E-07	1.37E-08

The proposed method provides fourth-order convergence results whenever previous existing methods provide the second-order convergence result. Figures 1 and 2 show the graphical representation of exact and approximate solution.

Example 2: Consider the following two-parameter singularly perturbed boundary value problem from [15, 19, 21, 23].

$$-\varepsilon u'' + \mu u' + u = \cos(\pi x),\ x = (0, 1)$$

subject to boundary conditions:

$$u(0) = 0, u(1) = 0.$$

The exact solution of the example 2 is

$$u(x) = a\cos(\pi x) + b\sin(\pi x) + C_1 e^{\lambda_1 x} + C_2 e^{-\lambda_2(1-x)},$$

where,

$$a = \frac{\varepsilon\pi^2 + 1}{\mu^2\pi^2 + (\varepsilon\pi^2 + 1)^2},\ b = \frac{\mu\pi}{\mu^2\pi^2 + (\varepsilon\pi^2 + 1)^2},$$

$$C_1 = -a\frac{1 + e^{-\lambda_2}}{1 - e^{\lambda_1 - \lambda_2}},\ C_2 = a\frac{1 + e^{\lambda_1}}{1 - e^{\lambda_1 - \lambda_2}},\ \lambda_{1,2} = \frac{\mu \mp \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}.$$

The maximum absolute errors of example 2 are summarized in Tables 3 and 4 at $\varepsilon = 10^{-2}, \varepsilon = 10^{-4}$ and $N = 128$ and different values of μ, respectively. Comparisons with other existing techniques are also mentioned in Tables 3, 4. These tables depicts that the proposed method gives a more accurate approximate solution than the existing methods. This method provides fourth-order convergence results whenever previous existing methods provide the first or second-order convergence result. Figure 3 and 4 are shows the graphical representation of exact and approximate solution.

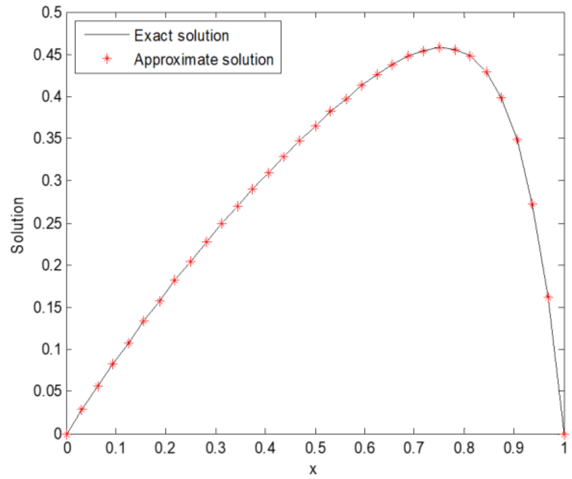

Figure 1. Graphical representation of exact and approximate solution of example 1 for $\varepsilon = 0.1, \mu = 1$ and $N = 32$

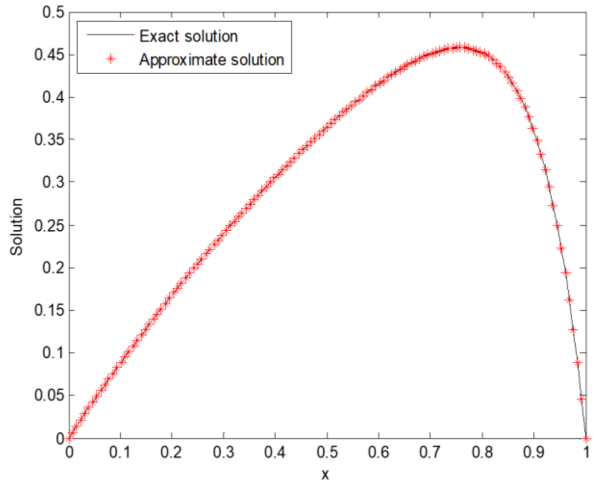

Figure 2. Graphical representation of exact and approximate solution of example 1 for $\varepsilon = 0.1, \mu = 1$ and $N = 128$

Table 3. Comparison of maximum absolute error with other existing methods of example 2 for $\varepsilon = 10^{-2}$ and $N = 128$

$\mu \downarrow$	Kadalbajoo and Yadaw [15]	Zahra and El Mhlawy [21]	Pandit and Kumar [19]	Khandelwal and Khan [23]	Present method
10^{-3}	8.3832E-05	4.1924E-05	4.2303E-05	6.0243E-06	7.2704E-07
10^{-4}	8.2686E-05	4.1296E-05	4.1318E-05)	6.1827E-07	7.3425E-08
10^{-5}	8.2572E-05	4.1232E-05	4.1220E-05	1.1455E-07	2.6931E-08
10^{-6}	8.2561E-05	4.1226E-05	4.1210E-05	7.2269E-08	2.4863E-08
10^{-7}	8.2560E-05	4.1225E-05	4.1209E-05	6.8266E-08	2.4661E-08

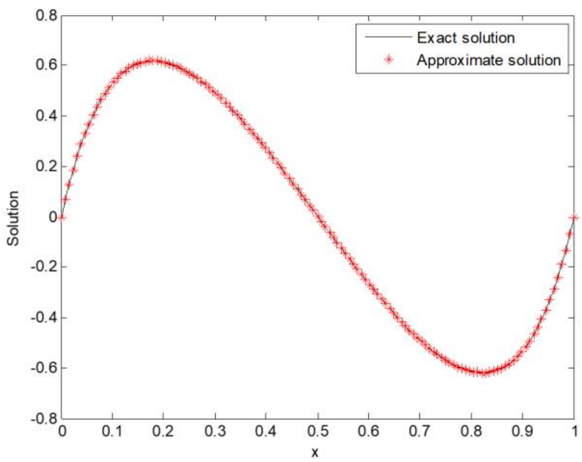

Figure 3. Graphical representation of exact and approximate solution of example 2 for $\varepsilon = 10^{-2}, \mu = 10^{-3}$ and $N = 128$

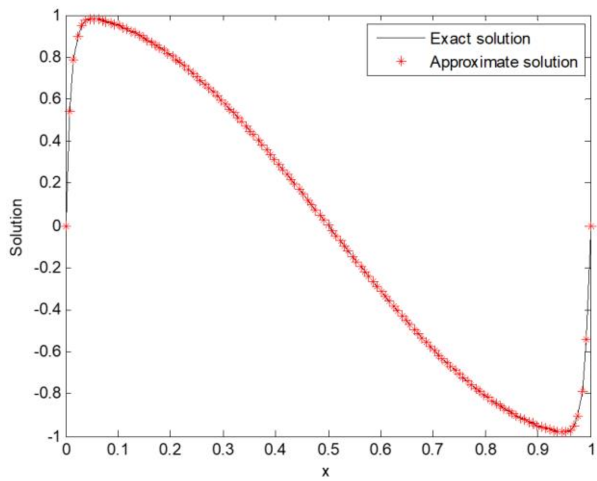

Figure 4. Graphical representation of exact and approximate solution of example 2 for $\varepsilon = 10^{-4}, \mu = 10^{-5}$ and $N = 128$

Table 4. Comparison of maximum absolute error with other existing methods of example 2 for $\varepsilon = 10^{-4}$ and $N = 128$

$\mu \downarrow$	Kadalbajoo and Yadaw [15]	Zahra and El Mhlawy [21]	Pandit and Kumar [19]	Khandelwal and Khan [23]	Present method
10^{-3}	9.4446E-03	4.7598E-03	5.1964E-03	6.2154E-03	2.7628E-04
10^{-4}	9.0436E-03	4.2856E-03	4.1710E-03	1.8330E-03	2.7220E-04
10^{-5}	9.0036E-03	4.2295E-03	4.0754E-03	1.1412E-03	2.7146E-04
10^{-6}	8.9996E-03	4.2238E-03	4.0659E-03	1.3699E-03	2.7138E-04
10^{-7}	8.9992E-03	4.2232E-03	4.0650E-03	1.3656E-03	2.7138E-04

5. CONCLUSIONS

In this communication, a fourth order compact finite difference method have studied for numerical solution of two-parameter singularly perturbed convection-diffusion boundary value problems. Present method is computationally efficient. The algorithm of this method is easy to implement on computer and it gives fourth order convergence result. Comparison of the methods are also delineated through Tables 1, 2, 3 and 4 which is indicated that this scheme gives better numerical solution as compared with previously applied techniques with the same mesh point.

ACKNOWLEDGMENT

The authors wish to express their thanks to the anonymous reviewers for very valuable comments and suggestions, which helped us to improve the quality of the article.

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