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A Numerical Method for Solving the Mobile/Immobile Diffusion Equation with Non-Local Conditions

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ABSTRACT

The purpose of this work is to use a new numerical technique for solving the two-sided multi-dimensional variable order fractional mobile/immobile diffusion equation with non-local conditions (TSMDVOF-MIDENLCs) model using the variable time fractional derivative of Caputo, as well as an initial boundary value problem of modified treatment. We used the fractional variational iteration method (FVIM) to mix initial and boundary conditions, resulting in for each iteration, a new initial solution. Convergence, sufficient conditions (SC) for system convergence, and error estimation are discussed. Some examples are given to illustrate the applicability of the novel suggested method, demonstrating that the numerical solution matches the exact solution and that the error is zero. Furthermore, this algorithm is easy and inexpensive to implement, and it demonstrates efficiency and accuracy.

1. INTRODUCTION

The Fractional calculus is the generalization of the ordinary calculus which examines the integration and derivatives of the real or complex. Numerical methods have been used to solve functional equations that contain fractional derivatives, such as [1-7].

Thus, the theory and applications of partial calculus were rapidly developed. Fractional partial differential equations (FDEs) have been the focus of attention in recent decades as a possible representation for explaining anomalous diffusion and relaxation phenomena seen in a wide range of science and engineering fields [8-11], with applications in porous media fluid transport, plasma diffusion, liquid surface diffusion, surface production, and two-dimensional rotational flow. Also put a many of numerical methods for solving PDEs [12-18].

Zhang et al. [19] suggested an implicit numerical method with for solution variable fractional mobile-immobile advection-dispersion model subject to the Dirichlet condition, stability, and convergence.

Abdelkawy et al. [20] use numerical approach to solve the mobile / immobile advection-dispersion fractional time variable. Cheng et al. [21] determine the Caputo derivative order and the coefficient of diffusion. In addition, numerical treatment based upon finite difference methods for FDEs was presented [22-24]. While Finite element methods were introduced to obtain the numerical solutions of FDEs [25-27].

In other papers different variable fractional operator definitions for solving variable FDEs were discussed [28-34].

Additionally, the most developed methods today are finite difference methods for the numerical approximation of variable-order FDEs [35-38].

In this work, we aim to solution TSMDVOF-MIDENLCs model. The rest of this paper is arranged as follows. In section 2, mathematical aspects. In section 3, mobile/immobile diffusion equation with non-local conditions (MIDENLCs) model. In section 4, proposed method and its convergence. In section 5, test problems. Finally, we present conclusion about solution TSMDVOF-MIDENLCs in section 6.

2. MATHEMATICAL ASPECTS

2.1 Definition 1 Coimbra [31]

Order Caputo fractional derivative operator defined as form follows:

$$D^{\propto(\sigma,\zeta)}f(\sigma)=\frac{1}{\Gamma(m-\alpha(\sigma,\zeta))}\int_0^{\sigma}\frac{f^{(m)}(\zeta)}{(\sigma-\zeta)^{\alpha(\sigma,\zeta)-m+1}}d\zeta.$$

where, $m-1<\alpha(\sigma)<m,\ m\in N,\ \sigma>0.$

Caputo's derivative of the variable order, we have:

$$D_{L_+}^{\alpha(\sigma)}(\sigma-L)^n=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(\sigma))}(\sigma-L)^{n-\alpha(\sigma)},$$

and

$$D_{R-}^{\alpha(\sigma)}(R-\sigma)^n=\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(w))}(R-\sigma)^{n-\alpha(\sigma)}.$$

2.2 Definition 2 Coimbra [31]

If ∃ a real no. $C_{\vartheta},\vartheta\in R$ is called in space where, $k(x),x>0$ a real function. $b(>\vartheta),\ s.t\ k(x)=x^bk_1(x),k_1\in C\,[0,\infty]$. If $y\leq\vartheta$ then $C_{\vartheta}\subset C_y$.

2.3 Definition 3 Coimbra [31]

If $k^{(m)}\in C_{\vartheta}$ then $C_{\vartheta}^m,m\in N\cup\{0\}$ it is called in space where, $k(x), x > 0$ a real function.

3. MOBILE/IMMOBILE DIFFUSION EQUATION WITH NON-LOCAL CONDITIONS (MIDENLCS) MODEL

In this section, we present the TSMDVOF-MIDENLCs model:

$$\beta_1 D_\zeta \Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) + \beta_2 D_{+\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)}\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) + \beta_3 D_{-\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)}\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) + \beta_4\sum_{i=1}^{n} D_{\sigma_i}\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) - \beta_5\sum_{i=1}^{n} D_{\sigma_i}^2\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = Q(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta), i = 1,2,\dots,n. \quad (1)$$

the initial condition (IC):

$$\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,0) = f(\sigma_1,\sigma_2,\dots,\sigma_n), \quad 0 \le \sigma_i \le 1, \quad (2)$$

and the non-local boundary conditions (N-LBCs):

$$\begin{aligned}
&\Omega(0,\sigma_2,\dots,\sigma_n,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_1 + L_1(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\Omega(\sigma_1,0,\dots,\sigma_n,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\,\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_2 + L_2(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\vdots\\
&\Omega(\sigma_1,\sigma_2,\dots,0,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_n + L_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\Omega(1,\sigma_2,\dots,\sigma_n,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_1 + M_1(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\Omega(\sigma_1,1,\dots,\sigma_n,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_2 + M_2(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\vdots\\
&\Omega(\sigma_1,\sigma_2,\dots,1,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_n + M_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta).
\end{aligned} \quad (3)$$

where, $\beta_1,\beta_2,\beta_3 \ge 0, \beta_4,\beta_5 > 0,\ 0 < \gamma \le \gamma(\sigma,\zeta) \le \bar{\gamma} \le 1$, and $Q, f, L_1, L_2,\dots,L_n$, and $M_1, M_2,\dots,M_n$ are known functions, and $D^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)}$ in our problem we define in terms of Caputo variable order fractional derivatives.

In this work, we used a new technique to calculate the zeroth approximation Ω_0^* by mixed initial conditions with boundary conditions at every iteration for get a new initial solution Ω_n^* by as follows:

(1) the initial solution can be written as:

Let

$$\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,0) = f_0(\sigma_1,\sigma_2,\dots,\sigma_n), \quad D_\zeta\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,0) = f_1(\sigma_1,\sigma_2,\dots,\sigma_n). \quad (4)$$

Then,

$$\Omega_0(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = f_0(\sigma_1,\sigma_2,\dots,\sigma_n) + \zeta f_1(\sigma_1,\sigma_2,\dots,\sigma_n), \quad (5)$$

(2) We create a new successive initial solution Ω_n^* by applying a new technique at each iteration:

As,

$$\begin{aligned}
&\Omega(0,\sigma_2,\dots,\sigma_n,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_1 + L_1(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\vdots\\
&\Omega(\sigma_1,\sigma_2,\dots,0,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_n + L_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\Omega(1,\sigma_2,\dots,\sigma_n,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\rho)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_1 \\
&\qquad M_1(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\\
&\vdots\\
&\Omega(\sigma_1,\sigma_2,\dots,1,\zeta) = \int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_n + M_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta).
\end{aligned} \quad (6)$$

Then,

$$\begin{aligned}
\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = {}& \Omega_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) + (1-\sigma_1^2)\left[\int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_1 + L_1(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) - \Omega_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\right]\\
&+ \sigma_1^2\left[\int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_1 + M_1(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) - \Omega_n(1,\sigma_2,\dots,\sigma_n,\zeta)\right] + \dots +\\
&(1-\sigma_n^2)\left[\int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_n + L_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) - \Omega_n(\sigma_1,\sigma_2,\dots,\sigma_{n-1},0,\zeta)\right]\\
&+ \sigma_n^2\left[\int_0^1 \varphi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)d\sigma_n + M_n(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) - \Omega_n(\sigma_1,\sigma_2,\dots,\sigma_{n-1},1,\zeta)\right].
\end{aligned} \quad (7)$$

4. SUGGESTED METHOD AND ITS CONVERGENCE

In this work, to demonstrate the procedure of solution by FVIM, we consider the TSMDVOF-MIDENLCB model:

$$\beta_1 D_\zeta \Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) +\beta_2 D_{+\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)}\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) +\beta_3 D_{-\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)}\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) +\beta_4\sum_{i=1}^{n} D_{\sigma_i}\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) -\beta_5\sum_{i=1}^{n} D_{\sigma_i}^2\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = Q(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta),\quad i=1,2,\dots,n.$$

Correction functional by initial boundary value problems and FVIM new procedure is formed as:

$$\Omega_{n+1}(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) + \int_0^\zeta \lambda(\beta_1 D_\zeta \Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + D_{+\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\tau)(}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_3 D_{-\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\tau)(}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_4\sum_{i=1}^{n} D_{\sigma_i}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) - \beta_5\sum_{i=1}^{n} D_{\sigma_i}^2\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) - Q(\sigma_1,\sigma_2,\dots,\sigma_n,\tau))d\tau,\quad \text{i} = 1,2,\dots,\text{n} \tag{8}$$

where, $\lambda = \frac{(-1)^m(\tau-\zeta)^{m-1}}{(m-1)!},\quad for \quad m \geq 1.$

Obtains the Lagrange multipliers as the following:

$$\lambda = -1, for\ m = 1,$$

$$\lambda = \tau - t,\ for\ m = 2,$$

from Eq. (4), we get:

$$\Omega_{n+1}(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) - \int_0^\zeta (\beta_1 D_\zeta \Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + D_{+\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\tau)(}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_3 D_{-\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\tau)(}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_4\sum_{i=1}^{n} D_{\sigma_i}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) - \beta_5\sum_{i=1}^{n} D_{\sigma_i}^2\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) - Q(\sigma_1,\sigma_2,\dots,\sigma_n,\tau))d\tau,\quad i = 1,2,\dots,n. \tag{9}$$

We can obtain approximations of successive $\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)$, $n \geq 0$ from Eq. (9). The function ξ_n^* is constrained variation which means $\delta\ \Omega_n^* = 0$. In this way, we get sequences $\Omega_{n+1}(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta), n \geq 0$. Then, the exact solution is obtaining as:

$$(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \lim_{n\to\infty}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta).$$

Now, in order to discuss the convergence and error estimate for FVIM applied to Eq. (1), we will be presented SC for convergence of method and error estimation [39].

We define the operator H as:

$$H = -\int_0^\zeta (\beta_1 D_\zeta \Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_2 D_{+\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\tau)(}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_3 D_{-\zeta}^{\gamma(\sigma_1,\sigma_2,\dots,\sigma_n,\tau)(}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) + \beta_4\sum_{i=1}^{n} D_{\sigma_i}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) - \beta_5\sum_{i=1}^{n} D_{\sigma_i}^2\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\tau) - Q(\sigma_1,\sigma_2,\dots,\sigma_n,\tau))d\tau. \tag{10}$$

where, the components can be defined as follows:

$$\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \lim_{n\to\infty}\Omega_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \sum_{i=0}^{\infty} v_i \text{ as } \sigma_i = 1,2,\dots \tag{11}$$

Theorem 1 Odibat [40]: Let H, defined in Eq. (10), be an operator from a Hilbert space H to H. The series solution:

$$\xi(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \lim_{n\to\infty}\xi_n^*(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \sum_{i=0}^{\infty} v_i.$$

If $0 < \delta < 1$ exists such that $\|H[v_0 + v_1 + v_2 + \dots + v_{i+1}]\| \leq \delta\|H[v_0 + v_1 + v_2 + \dots + v_i]\|$, (i.e $\|v_{i+1}\| \leq \delta\|v_i\|$), $\forall i \in N \cup \{0\}$ then the equation (11) convergences.

Theorem 2 Odibat [40] If non-linear problem equation (1) is converged then the series solution $\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta) = \sum_{i=0}^{\infty} v_i$ defined in Eq. (11) is an exact solution.

Theorem 3 Odibat [40] Let the series solution $\sum_{i=0}^{\infty} v_i$ defiend in Eq. (11) is convergent to the solution $\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)$. Then the maximum error, $\mathrm{E}_j(x,t)$, is estimated as:

$$\mathrm{E}_j(x,t) \leq \frac{1}{1-\delta}\delta^{j+1}\|v_0\|.$$

If the series $\sum_{i=0}^{j} v_i$ is used as an approximation to the solution $\Omega(\sigma_1,\sigma_2,\dots,\sigma_n,\zeta)$ of problem Eq. (5).

5. PROBLEM TEST AND DISCUSSION

In the present section, numerical examples are given to explain the effectiveness and accuracy of the proposed numerical method. The programs used here have been coded in MathCAD 12 and Matlab software package. The results obtained using our technique are also introduced for comparison.

Example 1: Consider the following equation:

$$\begin{aligned}\beta_1 D_\zeta \Omega(\sigma_1,\zeta) + \beta_2 D_{+\zeta}^{1-0.5e^{-\sigma_1\zeta}}\Omega(\sigma_1,\zeta) \\ + \beta_3 D_{-\zeta}^{1-0.5e^{-\sigma_1\zeta}}\Omega(\sigma_1,\zeta) \\ + \beta_4 D_{\sigma_1}\Omega(\sigma_1,\zeta) - \beta_5 D_{\sigma_1}^2\Omega(\sigma_1,\zeta) \\ = Q(\sigma_1,\zeta),\end{aligned} \tag{12}$$

Subjects to the IC and N-LBCs:

$$\Omega(\sigma_1,0) = 10{\sigma_1}^2(1-\sigma_1)^2,\ 0 \le \sigma_1 \le 1,$$

$$\Omega(0,\zeta) = \int_0^1 \sigma_1(10{\sigma_1}^2(1-\sigma_1)^2(1+\zeta))d\sigma_1 - 0.167(\zeta+1),$$

$$\Omega(1,\zeta) = \int_0^1 \sigma_1(10{\sigma_1}^2(1-\sigma_1)^2(1+\zeta))d\sigma_1 - \left(\frac{\zeta}{6}+\frac{1}{6}\right), \quad 0 \le \zeta \le J,$$

where:

$$\begin{aligned}Q(\sigma_1,\zeta) = {} & 20\sigma_1(\zeta+1)(1-\sigma_1)^2 - 20{\sigma_1}^2(\zeta+1)(1-\sigma_1) \\ & - 2(10\zeta+10)(1-\sigma_1)^2 - 8\sigma_1(10\zeta \\ & + 10)(1-\sigma_1) + 2(10\zeta+10){\sigma_1}^2 \\ & + 10{\sigma_1}^2(1-\sigma_1)^2\left(\frac{t^{0.5e^{-\sigma_1\zeta}}}{\Gamma(1+0.5e^{-\sigma_1\zeta})}\right) \\ & + 10{\sigma_1}^2(1-\sigma_1)^2\left(\frac{(1-t)^{0.5e^{-\sigma_1\zeta}}}{\Gamma(1+0.5e^{-\sigma_1\zeta})}\right) \\ & + 10{\sigma_1}^2(1-\sigma_1)^2,\end{aligned}$$

And

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 1,$$

The exact solution to this problem is:

$$\Omega(\sigma_1,\zeta) = 10{\sigma_1}^2(1-\sigma_1)^2(1+\zeta), \qquad 0 \le \sigma_1 \le 1.$$

Figure 1 establishes the comparison between the numerical solution and the exact solution for the Numerical Scheme at $\zeta = 0.3, 0.5, 0.7$ when $\gamma = 1 - 0.5e^{-\sigma_1\zeta}$, $\tau = h = \frac{1}{10}$. Table 1 demonstrates the absolute value of the maximum errors (MEs) of the numerical solution at ζ =1,2,3.

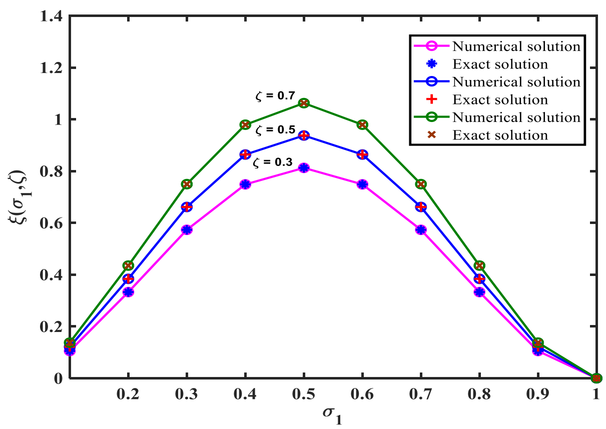

Figure 1. The numerical solution and exact solutions of equation (12) at $\zeta = 0.3, 0.5, 0.7$

Clearly, we can conclude from Figure 1 that the numerical solutions are excellent consistent with the exact solutions and prove the effectiveness of the proposed method with an error equal zero, see Table 1.

Table 1. MEs of our method at $\zeta = 1, 2, 3$ for example 1

h=τ	ME of our method
1/50	0.0000
1/100	0.0000
1/200	0.0000
1/400	0.0000

Example 2: Consider the problem equation (1) with the following N-LBCs and IC:

$$\begin{aligned}& \beta_1 D_\zeta \Omega(\sigma_1,\zeta) + \beta_2 D_{+\zeta}^{0.8+0.005\cos(\sigma_1\zeta)\sin(\sigma_1\zeta)}\Omega(\sigma_1,\zeta) \\ & + \beta_3 D_{-\zeta}^{0.8+0.005\cos(\sigma_1\zeta)\sin(\sigma_1\zeta)}\Omega(\sigma_1,\zeta) \\ & + \beta_4 D_{\sigma_1}\Omega(\sigma_1,\zeta) - \beta_5 D_{\sigma_1}^2\Omega(\sigma_1,\zeta) = Q(\sigma_1,\zeta),\end{aligned} \tag{13}$$

Subjects to the IC and N-LBCs:

$$\Omega(\sigma_1,0) = 5\sigma_1[1-\sigma_1], \qquad 0 \le \sigma_1 \le 1,$$

$$\Omega(0,\zeta) = \int_0^1 \sigma_1 \cdot \zeta[5(\zeta+1)\cdot\sigma_1(1-\sigma_1)]d\sigma_1 - \left[\frac{5\sigma_1\cdot\zeta(\zeta+1)}{6}\right], 0 \le \zeta \le J,$$

$$\Omega(1,\zeta) = \int_0^1 \sigma_1 \cdot \zeta[5(\zeta+1)\cdot\sigma_1(1-\sigma_1)]d\sigma_1 - [0.833\cdot(\sigma_1\cdot\zeta^2 + \sigma_1\cdot\zeta], 0 \le \zeta \le J,$$

where:

$$\begin{aligned}Q(\sigma_1,\zeta) = {} & (5\zeta+5)(1-\sigma_1) - (5\zeta+5)\sigma_1 \\ & - 2(-10\zeta-10) - 5\sigma_1(1-\sigma_1) + 5\sigma_1(1 \\ & - \sigma_1)\left(\frac{\zeta^{0.2-0.5e-2\cos(\sigma_1\zeta)\sin(\sigma_1)}}{\Gamma(2-\zeta^{0.2-0.5e-2\cos(\sigma_1\zeta)\sin(\sigma_1)})}\right) \\ & + 5\sigma_1(1 \\ & - \sigma_1)\left(\frac{(1-\zeta)^{0.2-0.5e-2\cos(\sigma_1\zeta)\sin(\sigma_1)}}{\Gamma(2-\zeta^{0.2-0.5e-2\cos(\sigma_1\zeta)\sin(\sigma_1)})}\right),\end{aligned}$$

And

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 1,$$

The exact solution to this problem is:

$$\Omega(\sigma_1,\zeta) = 5\sigma_1(1-\sigma_1)(\zeta+1), 0 \le \sigma_1 \le 1.$$

The absolute errors of the numerical solutions, at $\zeta = 0.5, 0.6, \tau = h = \frac{1}{10}$ and $\alpha(\sigma_1,\zeta) = 0.8 + 0.005\cos(\sigma_1\zeta)\sin(\sigma_1\zeta)$ are shown in Table 2. a comparison of the numerical solutions is made by the results reported in proposed method and exact solution. Table 3. demonstrates the absolute value of the maximum errors (MEs) of the numerical solution at $\tau = h = \frac{1}{50}, \frac{1}{100}, \frac{1}{200}, \frac{1}{400}$. Also, the results of the presented method at $\zeta = 4, 5, 6$.

Obviously, we can deduce from Figure 2 that the numerical solutions are excellent agreement with the exact solutions and indicate the effectiveness of the proposed method with an error equal zero, as clarify in Table 2 and Table 3.

Table 2. Some of comparison between exact solution and analytical solution when $\boldsymbol{\gamma(\sigma_1,\zeta) = 0.8 + 0.005\cos(\sigma_1\zeta)\sin(\sigma_1\zeta)}$ for example, 2

σ_1	ζ	Exact Solution	Variational Iteration Method	$\lvert\xi_{ex}-\xi_{VIM}\rvert$
0	0.5	0.000	0.000	0.000
0.1	0.5	0.675	0.675	0.000
0.2	0.5	1.200	1.200	0.000
0.3	0.5	1.575	1.575	0.000
0.4	0.5	1.800	1.800	0.000
0.5	0.5	1.875	1.875	0.000
0.6	0.5	1.800	1.800	0.000
0.7	0.5	1.575	1.575	0.000
0.8	0.5	1.200	1.200	0.000
0.9	0.5	0.675	0.675	0.000
1	0.5	0.000	0.000	0.000
0	0.6	0.000	0.000	0.000
0.1	0.6	0.720	0.720	0.000
0.2	0.6	1.280	1.280	0.000
0.3	0.6	1.680	1.680	0.000
0.4	0.6	1.920	1.920	0.000
0.5	0.6	2.000	2.000	0.000
0.6	0.6	1.920	1.920	0.000
0.7	0.6	1.680	1.680	0.000
0.8	0.6	1.280	1.280	0.000
0.9	0.6	0.720	0.720	0.000
1	0.6	0.000	0.000	0.000

Table 3. MEs of our method at $\zeta = 4,5,6$ for example 2

$h=\tau$	MEs of our method
1/50	0.0000
1/100	0.0000
1/200	0.0000
1/400	0.0000

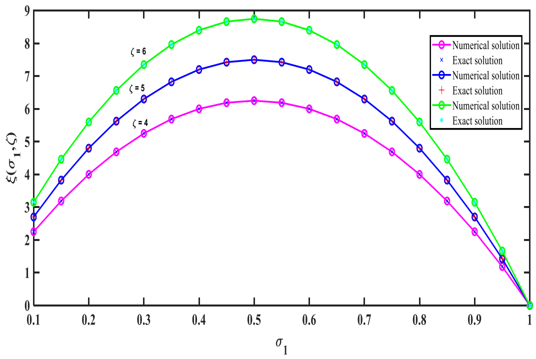

Figure 2. The numerical solution and exact solutions $\Omega(\sigma_1,\zeta)$ of Eq. (13)

Example 3: Consider the problem Eq. (1) with the following IC and N-LBCs:

$$\begin{aligned}\beta_1 D_\zeta \Omega(\sigma_1,\sigma_2,\zeta) + \beta_2 D_{+\zeta}^{1-0.5e^{-\sigma_1,\sigma_2\zeta}}\Omega(\sigma_1,\sigma_2,\zeta) &+ \beta_3 D_{-\zeta}^{1-0.5e^{-\sigma_1,\sigma_2\zeta}}\Omega(\sigma_1,\sigma_2,\zeta)\\ &= -\beta_4 D_{\sigma_1}\Omega(\sigma_1,\sigma_2,\zeta) - \beta_4 D_{\sigma_2}\Omega(\sigma_1,\sigma_2,\zeta)\\ &+ \beta_5 D_{\sigma_1}^2\Omega(\sigma_1,\sigma_2,\zeta) + \beta_5 D_{\sigma_2}^2\Omega(\sigma_1,\sigma_2,\zeta)\\ &+ Q(\sigma_1,\sigma_2,\zeta),\end{aligned} \tag{14}$$

Subjects to the IC and N-LBCs:

$$\Omega(\sigma_1,\sigma_2,0) = 10{\sigma_1}^2{\sigma_2}^2(1-\sigma_1)^2(1-\sigma_2)^2,\quad 0 \le \sigma_1,\sigma_2 \le 1,$$

$$\Omega(0,\sigma_2,\zeta) = \int_0^1 {\sigma_1}^2(10{\sigma_1}^2\cdot\zeta^2(1-\sigma_1)^2(1-\sigma_2)^2(1+\zeta))d\sigma_1 - \left(\frac{2{\sigma_2}^2(1-\sigma_2)^2(1+\zeta)}{21}\right),$$

$$\Omega(\sigma_1,0,\zeta) = \int_0^1 {\sigma_1}^2(10{\sigma_1}^2\cdot\zeta^2(1-\sigma_1)^2(1-\sigma_2)^2(1+\zeta))d\sigma_2 - \left(\frac{\sigma_2(1-\sigma_1)^2(1+\zeta)}{3}\right),$$

$$\Omega(1,\sigma_2,\zeta) = \int_0^1 {\sigma_1}^2\left(10{\sigma_1}^2\zeta^2(1-\sigma_1)^2(1-\sigma_2)^2(1+\zeta)\right)d\sigma_1 - \left(0.095(1-\sigma_2)^2({\sigma_2}^2+\zeta{\sigma_2}^2)\right),$$

$$\Omega(\sigma_1,1,\zeta) = \int_0^1 {\sigma_1}^2\left(\frac{10{\sigma_1}^2\zeta^2(1-\sigma_1)^2(1-\sigma_2)^2}{(1+\zeta)}\right)d\sigma_2 - \left(0.3\sigma_2(1-\sigma_1)^2(1+\zeta)\right), 0 \le \zeta \le J,$$

where:

$$\begin{aligned}Q(\sigma_1,\sigma_2,\zeta) &= 2\sigma_1,{\sigma_2}^2(10\zeta+10)(1-\sigma_1)^2(1-\sigma_2)^2 - 2{\sigma_1}^2{\sigma_2}^2(10\zeta+10)(1-\sigma_1)(1-\sigma_2)^2\\ &- 2{\sigma_2}^2(10+10)(1-\sigma_1)^2(1-\sigma_2)^2 - 8\sigma_1{\sigma_2}^2(10\zeta+10)(1-\sigma_1)(1-\sigma_2)^2\\ &+ 2{\sigma_1}^2{\sigma_2}^2(10\zeta+10)(1-\sigma_2)^2 + 2{\sigma_1}^2\sigma_2(10\zeta+10)(1-\sigma_1)^2(1-\sigma_2)^2\\ &- 2{\sigma_1}^2{\sigma_2}^2(10\zeta+10)(1-\sigma_1)^2(1-\sigma_2) + 2{\sigma_1}^2(10\zeta+10)(1-\sigma_1)^2(1-\sigma_2)^2\\ &+ 2{\sigma_1}^2\sigma_2(10\zeta+10)(1-\sigma_1)^2(1-\sigma_2) + 2{\sigma_1}^2{\sigma_2}^2(10\zeta+10)(1-\sigma_1)^2\\ &+ 10{\sigma_1}^2{\sigma_2}^2(1-\sigma_1)^2(1-\sigma_2)^2 + 10{\sigma_1}^2{\sigma_2}^2(1-\sigma_1)^2(1-\sigma_2)^2\left(\frac{\zeta^{5e^{-\sigma_1,\sigma_2\zeta}}}{\Gamma(1+5e^{-\sigma_1,\sigma_2\zeta})}\right)\\ &+ 10{\sigma_1}^2{\sigma_2}^2(1-\sigma_1)^2(1-\sigma_2)^2\left(\frac{(1-\zeta)^{5e^{-\sigma_1,\sigma_2\zeta}}}{\Gamma(1+5e^{-\sigma_1,\sigma_2\zeta})}\right),\end{aligned}$$

and

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 1,$$

that the exact solution to this problem is:

$$\Omega(\sigma_1, \sigma_2, \zeta) = 10{\sigma_1}^2{\sigma_2}^2(1-\sigma_1)^2(1-\sigma_2)^2(1+\zeta),$$
$$0 \le \sigma_1, \sigma_2 \le 1.$$

Some of comparison between exact solution and proposed method for VFOMADM with 2-D at $\zeta = 4{,}5{,}6$, $\tau = h = \frac{1}{10}$ and $\alpha(\sigma_1, \sigma_2, \zeta) = 1 - 0.5e^{-\sigma_1,\sigma_2\zeta}$, are shown in Table 4 with an error equal zero. In addition to, the absolute values of the MEs of the numerical solution at $\zeta = 1{,}2{,}3$ are shown in Table 5 with an error equal zero.

Table 4. Some of comparison between exact solution and analytical solution when $\alpha(\sigma_1, \sigma_2, \zeta) = 1 - 0.5e^{-\sigma_1,\sigma_2\zeta}$ for example 3

$\sigma_1 = \sigma_2$	ζ	Exact Solution	Variational Iteration Method	$\lvert\xi_{ex} - \xi_{VIM}\rvert$
0	4	0.000	0.000	0.000
0.1	4	6.561×10^{-4}	6.561×10^{-4}	0.000
0.2	4	6.554×10^{-3}	6.554×10^{-3}	0.000
0.3	4	0.019	0.019	0.000
0.4	4	0.033	0.033	0.000
0.5	4	0.039	0.039	0.000
0.6	4	0.033	0.033	0.000
0.7	4	0.019	0.019	0.000
0.8	4	6.554×10^{-3}	6.554×10^{-3}	0.000
0.9	4	6.561×10^{-4}	6.561×10^{-4}	0.000
1	4	0.000	0.000	0.000
0	5	0.000	0.000	0.000
0.1	5	6.561×10^{-4}	6.561×10^{-4}	0.000
0.2	5	6.554×10^{-3}	6.554×10^{-3}	0.000
0.3	5	0.019	0.019	0.000
0.4	5	0.033	0.033	0.000
0.5	5	0.039	0.039	0.000
0.6	5	0.033	0.033	0.000
0.7	5	0.019	0.019	0.000
0.8	5	6.554×10^{-3}	6.554×10^{-3}	0.000
0.9	5	6.561×10^{-4}	6.561×10^{-4}	0.000
1	5	0.000	0.000	0.000
0	6	0.000	0.000	0.000
0.1	6	6.561×10^{-4}	6.561×10^{-4}	0.000
0.2	6	6.554×10^{-3}	6.554×10^{-3}	0.000
0.3	6	0.019	0.019	0.000
0.4	6	0.033	0.033	0.000
0.5	6	0.039	0.039	0.000
0.6	6	0.033	0.033	0.000
0.7	6	0.019	0.019	0.000
0.8	6	6.554×10^{-3}	6.554×10^{-3}	0.000
0.9	6	6.561×10^{-4}	6.561×10^{-4}	0.000
1	6	0.000	0.000	0.000

Table 5. MEs of the numerical solution at ζ =1,2,3 for example 3

h=τ	MEs of our method
1/50	0.0000
1/100	0.0000
1/200	0.0000
1/400	0.0000

Example 4: Consider the problem Eq. (1) with the following N-LBCs and IC:

$$\begin{aligned}&\beta_1 D_\zeta \Omega(\sigma_1, \sigma_2, \zeta)\\ &+ \beta_2 D_{+\zeta}^{0.8+0.005\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)}\Omega(\sigma_1, \sigma_2, \zeta)\\ &+ \beta_3 D_{-\zeta}^{0.8+0.005\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)}\Omega(\sigma_1, \sigma_2, \zeta)\\ &= -\beta_4 D_{\sigma_1}\Omega(\sigma_1, \sigma_2, \zeta) - \beta_4 D_{\sigma_2}\Omega(\sigma_1, \sigma_2, \zeta)\\ &+ \beta_5 D_{\sigma_1}^2\Omega(\sigma_1, \sigma_2, \zeta) + \beta_5 D_{\sigma_2}^2\Omega(\sigma_1, \sigma_2, \zeta)\\ &+ Q(\sigma_1, \sigma_2, \zeta),\end{aligned} \tag{15}$$

Subjects to the IC and N-LBCs:

$$\Omega(\sigma_1, \sigma_2, 0) = 5\sigma_1(1-\sigma_1)(1-\sigma_2), \quad 0 \le \sigma_1, \sigma_2 \le 1,$$
$$\Omega(0, \sigma_2, \zeta) = \int_0^1 {\sigma_1}^2(5\sigma_1\sigma_2\zeta(1-\sigma_1)(1-\sigma_2)(1+\zeta))d\sigma_1 + \left(\frac{\sigma_2(\sigma_2-1)(1+\zeta)}{6}\right),$$
$$\Omega(\sigma_1, 0, \zeta) = \int_0^1 {\sigma_1}^2(5\sigma_1\sigma_2\zeta(1-\sigma_1)(1-\sigma_2)(1+\zeta))d\sigma_2 + \left(\frac{5{\sigma_1}^4(\sigma_1-1)(1+\zeta)}{6}\right),$$
$$\Omega(1, \sigma_2, \zeta) = \int_0^1 {\sigma_1}^2(5\sigma_1\sigma_2\zeta(1-\sigma_1)(1-\sigma_2)(1+\zeta))d\sigma_1 + 0.167(\sigma_2-1)(\sigma_2+\zeta\sigma_2),$$
$$\Omega(\sigma_1, 1, \zeta) = \int_0^1 {\sigma_1}^2\big(5\sigma_1\sigma_2\zeta(1-\sigma_1)(1-\sigma_2)(1+\zeta)\big)d\sigma_2 + 0.833(\sigma_1-1)({\sigma_1}^4+\zeta{\sigma_1}^4),$$
$$0 \le \zeta \le J,$$

where,

$$Q(\sigma_1,\sigma_2,\zeta) = 2\sigma_1\sigma_2(5\zeta+5)(1-\sigma_1)(1-\sigma_2) - \sigma_1{}^2\sigma_2(5\zeta+5)(1-\sigma_2) + \sigma_1{}^2(5\zeta+5)(1-\sigma_1)(1-\sigma_2) - \sigma_1{}^2\sigma_2(5\zeta+5)(1-\sigma_1) - 2\sigma_2(5\zeta+5)(1-\sigma_1)(1-\sigma_2) + 4\sigma_1,\sigma_1(5\zeta+5)(1-\sigma_2) + 2\sigma_1{}^2(5\zeta+5)(1-\sigma_1) + 5\sigma_1{}^2\sigma_2(1-\sigma_1)(1-\sigma_2) + 5\sigma_1,\sigma_2(1-\sigma_1)(1-\sigma_2)\left(\frac{\zeta^{0.2-0.5e-2\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)}}{\Gamma(2-\zeta^{0.2-0.5e-2\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)})}\right) + 5\sigma_1,\sigma_2(1-\sigma_1)(1-\sigma_2)\left(\frac{(1-\zeta)^{0.2-0.5e-2\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)}}{\Gamma(2-\zeta^{0.2-0.5e-2\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)})}\right),$$

And

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 1,$$

The exact solution to this problem is:

$$\Omega(\sigma_1,\sigma_2,\zeta) = 5\sigma_1,\sigma_2(1-\sigma_1)(1-\sigma_2)(\zeta+1), \quad 0 \le \sigma_1,\sigma_2 \le 1.$$

The absolute value of the MEs of the numerical solution at $\zeta = 1,2,3$ with an error equal zero (Table 6). Moreover, Table 7 demonstrates the comparison between the numerical solution and the exact solution at at $\zeta = 4,5,6$, $\tau = h = \frac{1}{10}$ and $\alpha(\sigma_1,\sigma_2,\zeta) = 1 - 0.5e^{-\sigma_1,\sigma_2\zeta}$ with an error equal zero.

Table 6. MEs of the numerical solution at ζ = 4, 5, 6 for example 4

h=τ	MEs of our method
0.200	0.0000
0.100	0.0000
0.050	0.0000
0.025	0.0000

Table 7. Some of comparison between exact solution and analytical solution when $\boldsymbol{\alpha(\sigma_1,\sigma_2,\zeta) = 0.8 + 0.005\cos(\sigma_1,\sigma_2\zeta)\sin(\sigma_1,\sigma_2)}$ for example 4

$\sigma_1 = \sigma_2$	ζ	Exact Solution	Variational Iteration Method	$\lvert\xi_{ex} - \xi_{VIM}\rvert$
0	1	0.0000	0.0000	0.000
0.1	1	0.0081	0.0081	0.000
0.2	1	0.0512	0.0512	0.000
0.3	1	0.1323	0.1323	0.000
0.4	1	0.2304	0.2304	0.000
0.5	1	0.3125	0.3125	0.000
0.6	1	0.3456	0.3456	0.000
0.7	1	0.3087	0.3087	0.000
0.8	1	0.2048	0.2048	0.000
0.9	1	0.0730	0.0730	0.000
1	1	0.0000	0.0000	0.000
0	2	0.0000	0.0000	0.000
0.1	2	0.0120	0.0120	0.000
0.2	2	0.0768	0.0768	0.000
0.3	2	0.1985	0.1985	0.000
0.4	2	0.3456	0.3456	0.000
0.5	2	0.4688	0.4688	0.000
0.6	2	0.5184	0.5184	0.000
0.7	2	0.4631	0.4631	0.000
0.8	2	0.3072	0.3072	0.000
0.9	2	0.1094	0.1094	0.000
1	2	0.0000	0.0000	0.000

6. CONCLUSION

Three main objectives were achieved in this work started from presenting mixed initial and boundary conditions are used to modify the treatment of initial boundary conditions. of TSMDVOF-MIDENLCB then explaining an alternative approach for proposed problems of the variational iteration method. Finally, studying the method of convergence and addressing the condition of sufficient for convergence. In addition, we got for each iteration, a new initial solution and conclude this system was fast to reach the exact value when mixed initial and boundary togethared using fractional variatinal iteration method in terms of execution time. This algorithm was simple and easy to implement. Moreover, successfully applied the given approach for solving various TSMDVOF-MIDENLCB classes. The efficiency and the accuracy of the suggested method clarified from the observation of all simulation results and numerical examples.

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NOMENCLATURE

TSMDVOF-MIDENLCs	Two-sided multi-dimensional variable order fractional mobile/ immobile diffusion equation with non-local conditions
SC	Sufficient conditions
FDEs	Fractional partial differential equations
IC	initial condition
N-LBCs	Non-local boundary conditions
FVIM	Fractional variational iteration method
$E_j(x,t)$	maximum error

Greek symbols

$\varphi(\sigma_1,\sigma_2,\ldots,\sigma_n,\zeta)$, $L_n(\sigma_1,\sigma_2,\ldots,\sigma_n,\zeta)$, $Q(\sigma_1,\sigma_2,\ldots,\sigma_n,\zeta)$, $M_n(\sigma_1,\sigma_2,\ldots,\sigma_n,\zeta)$	Function of known σ_n and ζ
Ω_n^*	New successive initial solution
λ	Multiplier of Lagrangian