



Numerical Simulation for Fractional Percolation Equation

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ABSTRACT

The aims of this paper are to propose approach of explicit finite difference method (EFDM), clarify the problem the mixed fractional derivative in one-dimensional fractional percolation equation (O-DFPE), and the study of consistency, stability, and convergence methods. Use of estimated Grunwald estimation in the analysis of mixed fractional derivatives. However, the given method is successfully applied to the mixed fractional derivative classes with the initial condition (IC) and derivative boundary conditions (DBC). To illustrate the efficiency and validity of the proposed algorithm, examples are given and the results are compared with the exact solution. From the figures shown for the examples in this work, the approximate solution values given by the EFDM for the various grid points are equivalent to the exact solution values with high-precision approximation. To show the effectiveness of the proposed method, where the error between the EFDM and the exact method is zero, the fractional derivative was used with various and random values. Using the package MATLAB and MathCAD 12 Figures were introduced.

1. INTRODUCTION

Seepage hydraulics, groundwater hydraulics, groundwater dynamics, and fluid dynamics in porous media are all modeled using fractional percolation equations (FPEs), which are derived from classical integer order percolation equations [1-6]. Under the hypotheses of continuity and Darcy's law, the regular integer percolation equation for incompressible, single phase percolation flow can be written as:

$$\frac{1}{v} \frac{\partial \theta(\zeta, \mathcal{W})}{\partial \mathcal{W}} = \frac{\partial}{\partial \zeta} \left(M(\zeta) \frac{\partial \theta(\zeta, \mathcal{W})}{\partial \zeta} \right) + f(\zeta, \mathcal{W}), \zeta \in \Omega$$

where the percolation coefficient is $M(\zeta) = M_\zeta$, the pressure is $\theta = \theta(\zeta, \mathcal{W})$, v is the velocity, $f(\zeta, \mathcal{W})$ is the source term. To deal with seepage flow movement in a non-homogeneous porous medium, He [7] first considered a modification of Darcy's law. It's possible to write it as

$$\frac{1}{v} \frac{\partial \theta}{\partial \mathcal{W}} = \frac{\partial^\beta}{\partial \zeta^\beta} \left(M_\zeta \frac{\partial^\alpha \theta}{\partial \zeta^\alpha} \right) + f(\zeta, \mathcal{W}), \zeta \in \Omega$$

This equation is called the FPE equation.

where $0 \leq \beta \leq 1$, $0 < \alpha < 1$, and $q = M(\zeta) \frac{\partial \theta(\zeta, \mathcal{W})}{\partial \zeta}$ is the rate of mass flux in a fluid.

The Riemann-Liouville fractional derivative is the fractional derivative described above [8, 9].

The Riemann-Liouville fractional derivative $\frac{\partial^\gamma \theta(\zeta, \mathcal{W})}{\partial \zeta^\gamma}$ of the order is defined by:

$$\frac{\partial^\gamma \theta(\zeta, \mathcal{W})}{\partial \zeta^\gamma} = \frac{1}{\Gamma(n - \gamma)} \frac{\partial^\gamma}{\partial \zeta^\gamma} \int_0^\zeta \theta(s, \mathcal{W}) (\zeta - s)^{-\gamma + n - 1} ds,$$

where, $\gamma > 0$, and $n - 1 < \gamma \leq n$. If $\beta = 0$, then the FPE can be represented as:

$$\frac{1}{v} \frac{\partial \theta}{\partial \mathcal{W}} = \frac{\partial^0}{\partial \zeta^0} \left(M_\zeta \frac{\partial^\alpha \theta}{\partial \zeta^\alpha} \right) + f(\zeta, \mathcal{W}), \zeta \in \Omega$$

This paper will be interested with the expansion of EFDM where the emphasis would be on a particular type of $v = 1$ and initial- derivative boundary conditions of O-DFPE which discuss unknown pressure function $\theta(\zeta, \mathcal{W})$ satisfying:

$$\frac{\partial \theta}{\partial \mathcal{W}} = \frac{\partial^\beta}{\partial \zeta^\beta} \left(M_\zeta \frac{\partial^\alpha \theta}{\partial \zeta^\alpha} \right) + f(\zeta, \mathcal{W}) \quad (1)$$

the initial condition (IC) with derivative boundary conditions (DBC):

$$\begin{aligned} \theta(\zeta, 0) &= \varphi(\zeta) \\ \theta_\zeta(0, \mathcal{W}) &= \pi_1(\mathcal{W}), 0 < \mathcal{W} \leq T \\ \theta_\zeta(1, \mathcal{W}) &= \pi_2(\mathcal{W}), 0 < \mathcal{W} \leq T \end{aligned} \quad (2)$$

where, $0 \leq \beta \leq 1$, $0 < \alpha < 1$ and the pressure is $\theta = \theta(\zeta, \mathcal{W})$. The percolation coefficient is $M(\zeta) = M_\zeta$, π_1, π_2 are known function of ω . $f(\zeta, \mathcal{W})$ is the source term. The α, β is fractional derivative order shifted Grunwald estimate are defined as:

$$\frac{\partial^\alpha \Theta(\zeta, \mathcal{W})}{\partial \zeta^\alpha} = \frac{1}{(\Delta \zeta)^\alpha} \sum_{p=0}^{i+1} g_{\alpha,p} \Theta_{i-p+1}^s + O(\Delta \zeta)$$

The numerical solutions are examined in this paper of the equation for fractional percolation. Paper structure is as follows. Section 2, we present Literature review. In Section 3, the EFDM for solving O-DFPE is proposed. In Section 4, its convergence is discussed of the EFDM. In Section 5, numerical simulation and comparison are provided to assess the method's efficiency.

2. LITERATURE REVIEW

Since most fractional differential equations do not have explicit analytic solutions, many authors rely on numerical solution strategies based on convergence and stability analysis. [10-18]

Recently, Liu et al. [19] considered numerical simulation with porous media fractional derivatives for the 3D seepage flow. For the one-dimensional FPE, Chen et al. [20] developed a novel implicit finite difference process. For the two-dimensional case of FPE, Chen et al. [21] considered an alternating direction implicit difference procedure. The numerical simulation of variable-order fractional percolation equation in non-homogenous porous media was discussed by Chen et al. [22]. For the two-dimensional FPE, A second order finite difference method was suggested by Guo et al. [23].

In this work we look at the EFDM where will concentrate on a class of $\nu = 1$ and IC with DBC of O-DFPE.

3. THE PROPOSED METHOD

We define EFDM for Eq. (1), and it can be rewritten as the following form:

$$\frac{\partial \Theta}{\partial \mathcal{W}} = \frac{\partial^\beta M_\zeta}{\partial \zeta^\beta} \left(\frac{\partial^\alpha \Theta}{\partial \zeta^\alpha} \right) + M_\zeta \left(\frac{\partial^{\alpha+\beta} \Theta}{\partial \zeta^{\alpha+\beta}} \right) + f(\zeta, \mathcal{W}) \quad (3)$$

First, we create a computational uniform grid for the derivation of the EFDM for O-DFPE with variable coefficients by $\zeta_i = a + i\Delta\zeta$, where $\Delta\zeta = (b - a)/n$ for $i = 0, \dots, n$ and $\mathcal{W}_s = s\Delta\mathcal{W}$, where $\Delta\mathcal{W} = T/M$ for $s = 0, \dots, M$. And $0 \leq \beta \leq 1$, $0 < \alpha < 1$

And using Eq. (2) with applied the forward Euler:

$$\frac{\partial \Theta}{\partial \mathcal{W}} \Big|_{(\zeta_i, \mathcal{W}_s)} \sim \frac{\Theta(\zeta_i, \mathcal{W}_{s+1}) - \Theta(\zeta_i, \mathcal{W}_s)}{\Delta \mathcal{W}} + O(\Delta \mathcal{W})$$

Moreover, it is possible to define the mixed fractional derivatives used in Eq. (3):

$$\begin{aligned} & \frac{\partial^\beta M_\zeta}{\partial \zeta^\beta} \left(\frac{\partial^\alpha \Theta}{\partial \zeta^\alpha} \right) \Big|_{(\zeta_i, \mathcal{W}_s)} + M_\zeta \left(\frac{\partial^{\beta+\alpha} \Theta}{\partial \zeta^{\beta+\alpha}} \right) \Big|_{(\zeta_i, \mathcal{W}_s)} \\ &= \dot{M}_\zeta \left(\frac{1}{\Delta \zeta^\alpha} \sum_{p=0}^{i+1} g_{\alpha,p} \Theta_{i-p+1}^s \right) \\ &+ \frac{M_\zeta}{\Delta \zeta^{\beta+\alpha}} \sum_{p=0}^{i+1} g_{\beta+\alpha,p} \Theta_{i-p+1}^s \end{aligned}$$

Now we use explicit Euler method to even get Θ_i^{s+1} :

$$\begin{aligned} \frac{\Theta_i^{s+1} - \Theta_i^s}{\Delta \mathcal{W}} &= \dot{M}_\zeta \left(\frac{1}{\Delta \zeta^\alpha} \sum_{p=0}^{i+1} g_{\alpha,p} \Theta_{i-p+1}^s \right) \\ &+ M_\zeta \left(\frac{1}{\Delta \zeta^{\beta+\alpha}} \sum_{p=0}^{i+1} g_{\beta+\alpha,p} \Theta_{i-p+1}^s \right) \\ &+ f_i^s \\ \Theta_i^{s+1} &= \dot{M}_\zeta \left(\frac{\Delta \mathcal{W}}{\Delta \zeta^\alpha} \sum_{p=0}^{i+1} g_{\alpha,p} \Theta_{i-p+1}^s \right) \\ &+ M_\zeta \left(\frac{\Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \sum_{p=0}^{i+1} g_{\beta+\alpha,p} \Theta_{i-p+1}^s \right) \\ &+ \Theta_i^s + \Delta \mathcal{W} f_i^s \end{aligned} \quad (4)$$

where:

$$\begin{aligned} g_{\alpha,r} &= (-1)^r \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!}, \\ g_{\beta+\alpha,r} &= (-1)^r \frac{(\beta+\alpha)(\beta+\alpha-1)\dots(\beta+\alpha-r+1)}{r!}, \quad r = \\ &0, 1, 2, \dots \end{aligned} \quad (5)$$

4. CONVERGENCE OF THE EFDM

To prove the convergence of the proposed method, we will need proof of conditionally stable and consistent according to the equivalence theorem of Lax.

4.1 Stability of EFDM

Now, we must prove that O-DFPE by EFDM is conditionally stable:

Theorem: The Eq. (4) with $0 \leq \beta \leq 1$, $0 < \alpha < 1$ is conditionally stable if

$$\frac{\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^\alpha} \leq 1 + \frac{\beta M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} + \frac{\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}}$$

Proof:

The equations system defined by (4) can be written in the form of $\underline{\Theta}^{s+1} = \underline{\rho} \underline{\Theta}^s + \Delta \mathcal{W} \underline{Q}^s$ where

$$\underline{\Theta}^{s+1} = [\Theta_1^{s+1}, \Theta_2^{s+1}, \dots, \Theta_{n-1}^{s+1}]^T$$

$$\Delta \mathcal{W} \underline{F}^s = [\Delta \mathcal{W} f_1^s, \Delta \mathcal{W} f_2^s, \dots, \Delta \mathcal{W} f_{n-1}^s]^T$$

and

$$\rho_{i,j} = \begin{cases} 1 + \mu_i g_{\alpha,1} + N_i g_{\beta+\alpha,1} & \text{for } j = i \\ \mu_i g_{\alpha,0} + N_i g_{\beta+\alpha,0} & \text{for } j = i - 1 \\ \mu_i g_{\alpha,2} + N_i g_{\beta+\alpha,2} & \text{for } j = i + 1 \\ \mu_i g_{\alpha,j+1} + N_i g_{\beta+\alpha,j+1} & \text{for } j < i + 1 \end{cases}$$

matrix entries are $\rho_{i,j}$ for $i = 1, 2, \dots, n - 1$ and $j = 1, \dots, n - 1$.

where the coefficients

$$\mu_i = \frac{M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} \quad \text{and} \quad N_i = \frac{M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}}$$

To get to:

$$\begin{aligned} \Theta_1^{s+1} &= (\mu_1 g_{\alpha,0} + N_1 g_{\beta+\alpha,0}) \Theta_0^s \\ &\quad + (1 + \mu_1 g_{\alpha,1} + N_1 g_{\beta+\alpha,1}) \Theta_1^s \\ &\quad + (\mu_1 g_{\alpha,2} + N_1 g_{\beta+\alpha,2}) \Theta_2^s + \dots \\ &\quad + (\mu_1 g_{\alpha,n} + N_1 g_{\beta+\alpha,n}) \Theta_n^s \\ &\quad + \Delta \mathcal{W} f_1^s \end{aligned}$$

$$\begin{aligned} \Theta_2^{s+1} &= (\mu_2 g_{\alpha,0} + N_2 g_{\beta+\alpha,0}) \Theta_0^s \\ &\quad + (1 + \mu_2 g_{\alpha,1} + N_2 g_{\beta+\alpha,1}) \Theta_1^s \\ &\quad + (\mu_2 g_{\alpha,2} + N_2 g_{\beta+\alpha,2}) \Theta_2^s \\ &\quad + (\mu_2 g_{\alpha,3} + N_2 g_{\beta+\alpha,3}) \Theta_3^s + \dots \\ &\quad + (\mu_2 g_{\alpha,n} + N_2 g_{\beta+\alpha,n}) \Theta_n^s \\ &\quad + \Delta \mathcal{W} f_2^s. \end{aligned}$$

$$\begin{aligned} \Theta_{n-1}^{s+1} &= (\mu_{n-1} g_{\alpha,0} + N_{n-1} g_{\beta+\alpha,0}) \Theta_0^{s/3} \\ &\quad + (1 + \mu_{n-1} g_{\alpha,1} \\ &\quad + N_{n-1} g_{\beta+\alpha,1}) \Theta_1^s \\ &\quad + (\mu_{n-1} g_{\alpha,2} + N_{n-1} g_{\beta+\alpha,2}) \Theta_2^s \\ &\quad + \dots \\ &\quad + (\mu_{n-1} g_{\alpha,n} + N_{n-1} g_{\beta+\alpha,n}) \Theta_n^{s/3} \\ &\quad + \Delta \mathcal{W} f_{n-1}^s \end{aligned}$$

According to the Greshgorin theorem [24],

$$\begin{aligned} \varrho_{i,j} &= 1 + \left(\frac{M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} g_{\alpha,1} + \frac{\beta M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} g_{\beta+\alpha,1} \right) \\ &= 1 - \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} - \frac{\beta M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} - \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \end{aligned}$$

and

$$\begin{aligned} \vartheta_i &= \sum_{l=0}^n \varrho_{i,l} = \frac{M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} \sum_{l=0}^n g_{\alpha,i-j+1} \\ &\quad + \frac{M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \sum_{l=0}^n g_{\beta+\alpha,i-j+1} \\ &\leq \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} + \frac{\beta M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} + \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \end{aligned}$$

Now

$$\vartheta_i = \sum_{l=0}^n \varrho_{i,l}.$$

and therefore $\varrho_{i,j} + \vartheta_i \leq 1$. we also have

$$\begin{aligned} \varrho_{i,j} - \vartheta_i &\geq \left(1 - \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} - \frac{\beta M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} - \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \right) \\ &\quad - \left(\frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} + \frac{\beta M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \right) \\ &\quad + \frac{\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \end{aligned}$$

$$\begin{aligned} &= 1 - \frac{2\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^\alpha} - \frac{2\beta M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} - \frac{2\alpha M_\zeta \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \\ &\geq 1 - \frac{2\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^\alpha} - \frac{2\beta M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \\ &\quad - \frac{2\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \end{aligned}$$

Then:

$$\begin{aligned} &1 - \frac{2\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^\alpha} - \frac{2\beta M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} - \frac{2\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \\ &\geq -1 \\ &\rightarrow -1 + \frac{2\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^\alpha} + \frac{2\beta M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \\ &\quad + \frac{2\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \leq 1 \\ &\rightarrow \frac{\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^\alpha} \leq 1 + \frac{\beta M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} + \frac{\alpha M_{\zeta_{max}} \Delta \mathcal{W}}{\Delta \zeta^{\beta+\alpha}} \end{aligned}$$

4.2 Consistency of EFDM

Notice that the time difference operator in (4) has a local truncation error of order $O(\Delta \omega)$ and the space difference operators in (4) have local truncation errors of orders $O(\Delta \zeta)$. Similar to Lemma 2.1 in [11], we can obtain the consistency of the O-DFPE is:

$$O(\Delta \zeta) + O(\Delta \mathcal{W})$$

Now we can apply the Laxs equivalence theorem [25] after we obtain consistency and conditionally stable. Where it converges at the rate of $O(\Delta \zeta + \Delta \mathcal{W})$.

5. NUMERICAL SIMULATION AND COMPARISON

Some numerical results are discussed in this section to help our theoretical analysis to illustrate the efficacy of method use.

Example 1: Consider the following O-DFPE:

$$\begin{aligned} \frac{\partial \Theta}{\partial \mathcal{W}} &= 5.3\zeta^{0.5} e^{-\mathcal{W}} - \zeta^2 e^{-\mathcal{W}} - 6.8\zeta^{0.5} e^{-\mathcal{W}} \\ &\quad + \frac{\partial}{\partial \zeta} \left((30 - \zeta^2) \frac{\partial^{0.5} \Theta}{\partial \zeta^{0.5}} \right) \end{aligned}$$

Subjects to the IC and DBC:

$$\Theta(\zeta, 0) = \zeta^2$$

$$\Theta_\zeta(0, \mathcal{W}) = 0, \quad 0 < \mathcal{W} \leq T$$

$$\Theta_\zeta(1, \mathcal{W}) = 2e^{-\mathcal{W}}, \quad 0 < \mathcal{W} \leq T$$

That the EXS to this problem is:

$$\Theta(\zeta, \mathcal{W}) = e^{-\mathcal{W}} u \zeta^2$$

Figure 1 indicates the solution obtained using EFDM versus the exact solution at $\zeta = 0.2, 0.4, 0.6, 0.8, 1$ and $\Delta\mathcal{W} = 0.0125$ when $\beta = 1, \alpha = 0.5$. Figure 1 clearly demonstrates that EFDM's numerical solutions are excellent and consistent with exact solutions, illustrating the utility of the proposed zero-error method.

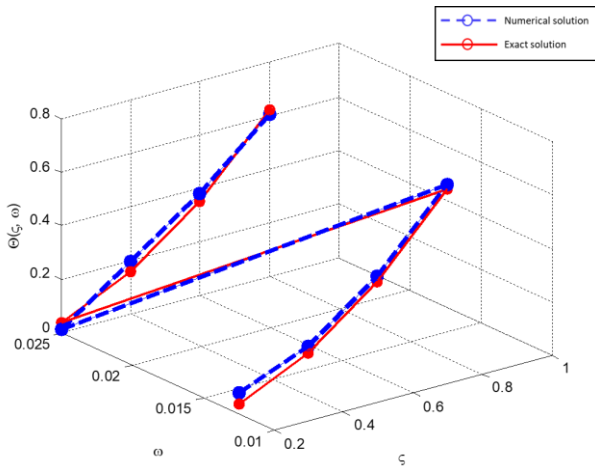


Figure 1. The numerical solution and EXS of example 1

Example 2: Consider the following O-DFPE:

$$\frac{\partial\theta}{\partial\mathcal{W}} = -(\zeta^2 + 2.3\zeta^{0.3})e^{-\mathcal{W}} + \frac{\partial^{0.9}}{\partial\zeta^{0.9}}\left(\frac{\partial^{0.8}\theta}{\partial\zeta^{0.8}}\right),$$

Subjects to the IC and DBC:

$$\theta(\zeta, 0) = \zeta^2$$

$$\theta_\zeta(0, \mathcal{W}) = 0, 0 < \mathcal{W} \leq T$$

$$\theta_\zeta(1, \mathcal{W}) = 2e^{-\mathcal{W}}, 0 < \mathcal{W} \leq T$$

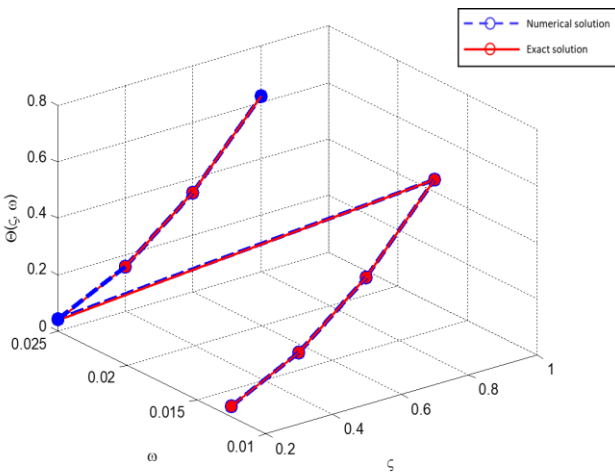


Figure 2. The numerical solution and EXS of example 2

That the EXS to this problem is: $\theta(\zeta, \mathcal{W}) = e^{-\mathcal{W}}\zeta^2$

Shown in Figure 2 Comparison between exact solution and EFDM at $\zeta = 0.2, 0.4, 0.6, 0.8, 1$ and $\Delta\mathcal{W} = 0.0125$ and when $\beta = 0.9, \alpha = 0.8$. Clearly shown that the numerical solution of EFDM is identical to the exact solution, indicating the efficacy of the proposed approach with a zero error rate.

Example 3: Consider the following O-DFPE:

$$\frac{\partial\theta}{\partial\mathcal{W}} = 4.01\zeta^{2.5}e^{-\mathcal{W}} - \zeta^2e^{-\mathcal{W}} - 3.96\zeta^{0.5}e^{-\mathcal{W}} + \frac{\partial^{0.8}}{\partial\zeta^{0.8}}\left(\left(-\zeta^2 + 20\right)\frac{\partial^{0.7}\theta}{\partial\zeta^{0.7}}\right),$$

$$\zeta \in \Omega, \mathcal{W} > 0,$$

Subjects to the IC and DBC:

$$\theta(\zeta, 0) = \zeta^2$$

$$\theta_\zeta(0, \mathcal{W}) = 0, 0 < \mathcal{W} \leq T$$

$$\theta_\zeta(1, \mathcal{W}) = 2e^{-\mathcal{W}}, 0 < \mathcal{W} \leq T$$

That the EXS to this problem is: $\theta(\zeta, \mathcal{W}) = e^{-\mathcal{W}}\zeta^2$.

Figure 3 shows a comparison between the exact solution and the EFDM with $\zeta = 0.2, 0.4, 0.6, 0.8, 1$ and $\Delta\mathcal{W} = 0.0125$ and $\beta = 0.8, \alpha = 0.7$, with an error equal to zero. Figure 3 clearly shows that the numerical solutions of EFDM are excellent and compatible with the exact solutions, demonstrating the usefulness of the proposed approach with zero error.

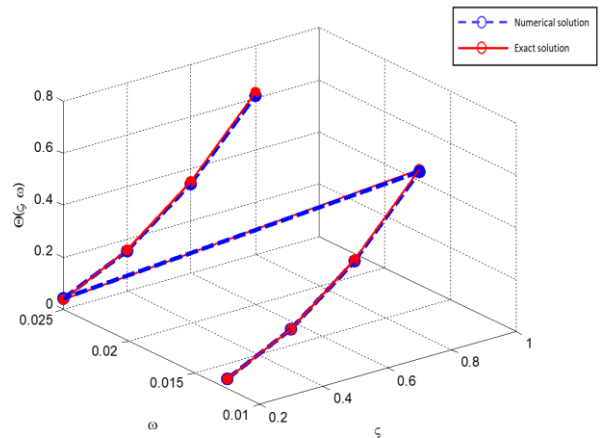


Figure 3. The numerical solution and EXS of example 3

Example 4: Consider the following O-DFPE:

$$\frac{\partial\theta}{\partial\mathcal{W}} = (4.2\zeta^{2.5} + 0.3\zeta^{0.5} + 0.6\zeta^{-1.5})e^{-\mathcal{W}} + \frac{\partial^{0.7}}{\partial\zeta^{0.7}}\left(\left(-\zeta^2 + 2\right)\frac{\partial^{0.8}\theta}{\partial\zeta^{0.8}}\right) - (\zeta^2 + 1 + 4.4\zeta^{0.5})e^{-\mathcal{W}},$$

$$\zeta \in \Omega, \mathcal{W} > 0,$$

Subjects to the IC and DBC:

$$\theta(\zeta, 0) = \zeta^2$$

$$\theta_\zeta(0, \mathcal{W}) = 0, \quad 0 < \mathcal{W} \leq T$$

$$\theta_\zeta(1, \mathcal{W}) = 2e^{-\mathcal{W}}, \quad 0 < \mathcal{W} \leq T$$

That the EXS to this problem is: $\theta(\zeta, \mathcal{W}) = e^{-\mathcal{W}}\zeta^2$.

Figure 4 demonstrates the relation between the numerical solution of EFDM and the exact solution for the numerical scheme at $\zeta = 0.2, 0.4, 0.6, 0.8, 1$ and $\Delta\mathcal{W} = 0.0125$ and when $\beta = 0.7, \alpha = 0.8$. The approach to the numerical solution of EFDM can be seen to be similar to the exact solution.

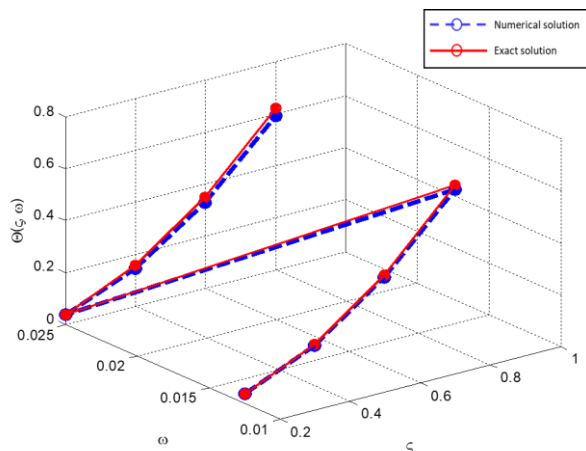


Figure 4. The numerical solution and EXS of example 4

6. CONCLUSIONS

To model percolation properties, the O-DFPE was introduced. We suggested EFDM with the IC and DBC in this paper. The EFDM's stability and convergence with convergence order $O(\Delta \zeta + \Delta \mathcal{W})$ were demonstrated. Furthermore, the proposed method has been successfully extended to the solution of various O-DFPE with mixed fractional derivative groups. This algorithm is quick and straightforward to implement. We also proposed a numerical experiment to help the theoretical analysis and show the practicality of the number scheme in order to demonstrate the efficacy of EFDM. However, a comparison of exact and approximate solution values reveals that the approximate solution values obtained by EFDM are equivalent to the exact solution values obtained with high precision approximation for different grid points. It's also worth remembering that as the approximation order grows, the accuracy rises with it.

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NOMENCLATURE

DBC	Derivative boundary conditions
EFDM	Explicit finite difference method
FPE	Fractional percolation equation
O-DFPE	One-dimensional fractional percolation equation
IC	Initial condition
T	Constant

Greek symbols

$q_{i,j}$	Matrix entries
$f(\zeta, \mathcal{W})$	The source term
$g_{\alpha,r}, g_{\beta+\alpha,r}$	The normalized Grunwald weights
M_ζ	Derivative of the percolation coefficient
M_ζ	The percolation coefficient
$\vartheta_i = \sum_{l=0}^n q_{i,l}$	Radius
$\Theta(\zeta, \mathcal{W})$	Function of pressure
$\Theta(\zeta, 0)$	Initial condition
$\Theta_\zeta(0, \mathcal{W}), \Theta_\zeta(1, \mathcal{W})$	Derivative boundary conditions
α, β	Derivatives in fractional order
$\varphi(\zeta)$	a well-known function ζ
$\pi_1(\mathcal{W}), \pi_1(\mathcal{W})$	a well-known function \mathcal{W}