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The First Kind Fredholm Integral Equation of Regularization Algorithm Research

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ABSTRACT

As a result of the general operator equation can be transformed into integral equation method for solving, as this paper tell us, we use Tikhonov regularization method to solve the first kind of Fredholm integral equations of the numerical calculation method, and analyze the regularization parameter α and the number of iterations n of the influence of the error δ, the iterative regularization method iteration steps m and different δ how to affect the error.

Keywords: Regularization method, First kind of Fredholm integral equations, Numerical calculation method.

1. INTRODUCTION

A variety of methods has been developed to solve the inverse problems of partial differential equations, such as Pulse Spectrum Technology (PST), the best perturbation method, the Monte Carlo method all kinds of optimization of regularization method. Based on the integral equation method is one of the main approaches to the study of the inverse problem, its basic idea is to attribute the inverse problem solving becomes an integral equation to solve the problem. Inverse spectrum method, Borg approximation method and the method based on the far field pattern is typical of this kind of method.

2. Ill-POSEDANALYSIS

Let's analyze the ill-posed properties of the first kind Fredholm integral equation. We will consider the following form of the first kind Fredholm integral equation:

$$\int_a^b k(x,t)z(t)dt = u(x), c \le x \le d \,. \tag{1}$$

We assume $k(x,t)$ is continuous and have continuous partial derivatives $\frac{\partial k}{\partial x}$ in interval $[c,d]\times[a,b]$. $z(t)$ is unkonwn function in solution space Y , $u(x)$ is given in space X . The distance of space X and Y is defined as

$$\rho_X = (u_1,u_2) = \left(\int_c^d \left[u_1(x)-u_2(x)\right]^2 dx\right)^{1/2},$$

$$\rho_Y(z_1,z_2) = \max_{t\in[a,b]} \left|z_1(t)-z_2(t)\right|$$

respectively.

Supposing the solution of the equation is $z_1(t)$ when the right hand side of the equation is $u = u_1(x)$, i. e.

$$\int_a^b k(x,t)z_1(t)dt = u_1(x), c \le x \le d \,.$$

It is not difficult to verify, for all real ω, N ,

$z_2(t) = z_1(t) + N\sin(\omega t)$ is the solution when the right hand side of the equation is

$$u_2(x) = u_1(x) + N\int_a^b k(x,t)\sin(wt)dt \,.$$

Clearly, for any N , when ω is sufficiently large,

$$\rho_Y(u_1,u_2) = |N|\sqrt{\int_c^d [\int_a^b k(x,t)\sin(\omega t)dt]^2 dx}$$

can be arbitrarily small, as the same time, the distance of $z_1(t)$ and $z_2(t)$

$$\rho_X(z_1,z_2)=\max_{t\in[a,b]}|N\sin(\omega t)|=|N|$$

may be arbitrarily large. This shows that the solution of the first kind Fredholm integral equation does not depend continuously on the right hand side, is ill-posed problem that does not have the stability.

3. REGULARIZATION METHOD THEORY

3.1 Tikhonov regularization method

Aiming at the solution of the ill-posed problems, the Tikhonov regularization method of this paper plus of penalty after the objective function $\|Kx-y\|$, and planning to come to find this new sub-minimum value from a theoretical point of view optimization problem is posedo or come to find this new sub-minimization values satisfy the equation in terms of the theory of integral equations of the second kind integral equation, i. e: For general bounded linear operator

$K:X\to Y$, for any $y\in Y$, we need solve $x^\alpha\in X$ such that the function

$$J_\alpha(x)=\|Kx-y\|_Y^2+\alpha\|x\|_X^2 \tag{2}$$

can reach to the minimum., where $\alpha>0$ is the regularization parameter.

Theorem 1: If space X,Y are Hilbert space, the operator $K:X\to Y$ is linear and bounded. For any $y\in Y$, there exists $x_0\in X$ satisfy

$$\|Kx_0-y\|\le\|Kx-y\| \tag{3}$$

Necessary and sufficient conditions for all $x\in X$ is x_0 satisfy

$$K^*Kx_0=K^*y \tag{4}$$

where the operator $K^*:Y\to X$ is the self-adjoint operator. Proof: First we prove sufficiency. That is we know $K^*Kx_0=K^*y$ and we need come to $\|Kx_0-y\|\le\|Kx-y\|$. Calculated directly from the meaning of the questions

$$\begin{aligned}&\|Kx-y\|^2-\|Kx_0-y\|^2\\&=2\operatorname{Re}\left(Kx_0-y,K(x-x_0)\right)+\left\|K\left(x-x_0\right)\right\|^2\\&=2\operatorname{Re}\left(K^*\left(Kx_0-y\right),x-x_0\right)+\left\|K\left(x-x_0\right)\right\|^2\ge0,\end{aligned}$$

where $x,x_0\in X$. By assumption x_0 satisfy $K^*Kx_0=K^*y$, according to the formula

$$\|Kx-y\|^2-\|Kx_0-y\|^2=\|K(x-x_0)\|^2,$$

We can come to the conclusion x_0 is the minimizer of $\|Kx-y\|$.

Secondly, we have to prove the necessity, which is knowing $\|Kx_0-y\|\le\|Kx-y\|$, i. e x_0 is the minimizer of $\|Kx-y\|$, we need inferr $K^*Kx_0=K^*y$. By assumption, foa any $k>0,x\in X$, we choose to $x=x_0+kz$, then we have

$$\begin{aligned}&\|Kx-y\|^2-\|Kx_0-y\|^2\\&=2\operatorname{Re}\left(Kx_0-y,K(x-x_0)\right)+\left\|K\left(x-x_0\right)\right\|^2\\&=2\operatorname{Re}(K^*\left(Kx_0-y\right),x-x_0)+\left\|K\left(x-x_0\right)\right\|^2\ge0,\end{aligned}$$

$x=x_0+kz$ into the above formular, we have

$$0\le2k\operatorname{Re}\left(K^*\left(Kx_0-y\right),z\right)\ge0,\forall z\in X.$$

Because of the arbitrary z, $K^*\left(Kx_0-y\right)=0\Rightarrow K^*Kx_0=K^*y$ is holding.

Tikhonov regularization method for solving ill-posed problem is the most universal and theoretically a better method. Overdetermined linear system of finite-dimensional $Kx=y$, but for infinite dimensional systems, solving minimization problem is still sick, so we should relax the requirement $\|Kx-y\|$ be $J_\alpha(x)=\|Kx-y\|^2+\alpha\|x\|^2$, and then solve this new function's minimizer. Where $J_\alpha(x)$ is called Tikhonov functional, the second is called stable functional. In general case, when confronted with solving linear ill-posed problem, we often take use of Tikhonov regularization method, whose core idea is to use the norm $\|x\|_2$ including a priori information which is relation to the the prior information of solution such as boundary and smoothness and so on. Its essence is determined as follows solutions x_α^δ of the minimization functional:

$$J_\alpha(x)=\left\|Kx-y^\delta\right\|^2+\alpha\|x\|^2.$$

For solving the same problem, select the functional stability is not the only, it depends on the restriction for the appendix solution. Therefore, the regularization Solutions regularization operator will be different.

According to the previous theorem, the Tikhonov regularization method. There is also the following theorem:

Theroem 2: If space X,Y are Hilbert space, the operator $K:X\to Y$ is linear and bounded. Then

(1) There exists a unique minimizer x^α for the minimization functional $J_\alpha(x)=\|Kx-y\|_Y^2+\alpha\|x\|_X^2$ in space X.

(2) By (1), there exists minimizer x^α satisfy

$$\alpha x^\alpha+K^*Kx^\alpha=K^*y \tag{5}$$

See proof in [2].

To discuss the extremum of functional $J_\alpha(x)$, we give the definition of the minimizing sequence.

Definition 1: Let X be a metric space, M is a subset of X. And asuume the functional $J:M\to R^1$ exists previous supremum and infimum. We call the sequence $\{z_n\}\subset M$ is the minimizing sequence of, if $\lim_{n\to\infty} J(z_n) = d$ and $d = \inf_{z\in M} J(z)$ hold on.

In general, the sequence $\{z_n\}$ may not exist limits. Even if its limit z^* exists, $z^* \in M$ and $\lim_{n\to\infty} J(z_n) = J\left(\lim_{n\to\infty} z_n\right) = J(z^*)$ may not hold on. In the regularization operator configuration aspects, Tikhonov had discussed in detail the transformation method based on variational principle of the method and means of integrating its implementation in the image space, he constructed through the introduction of a stable functional regularization operator, as for above functional $\|x\|_X$ is the stable functional.

Tikhonov functional consists of two items: Correspondence preceding the original problem, the latter corresponding to the prior information. Minimization of the functional requirements of the residual norm is small and approximate solutions to better meet the balance between these two a priori information. About regularization operators can take advantage of the existence of Hilbert space Lax-Milgram theorem establishment.

3.2 Regularization parameter selection

L curve criterion is based on log-log scale to describe the contrast curves $\|x^{\alpha,\delta}\|$ and $\|Kx^{\alpha,\delta} - y^\delta\|$, and then on the basis of the comparison results to determine the regularization parameters. As show below the picture. In the vertical part of the L curve, the regularization parameter is samall, $\|Kx^{\alpha,\delta} - y^\delta\|$ is also small, regularization solution is coincide with the data after disturbance, in contrast, $\|x^{\alpha,\delta}\|$ is sensitive to the change of the regularization parameter, Therefore, the vertical part of the state belong to the less regular dissemination of data errors dominate the total error;In the horizontal part of the curve L, the regularization parameter is large, regularization error is main, with the increase of α, $\|Kx^{\alpha,\delta} - y^\delta\|$ is also become larger but not as α with $\|x^{\alpha,\delta}\|$, therefore, the horizontal part belonged to the regular state. We can choose regularization parameter in the upper corner of the L curve, i.e. from the vertical to the horizontal corner to select. For the upper corner, people often choose the point of maximum curvature of the L curve as the upper corner L. Let

$$u(\alpha) = \log\|Kx_\alpha^\delta - y^\delta\|, v(\alpha) = \log\|x_\alpha^\delta\|,$$

then curvature function of the L curve is

$$K(\alpha) = \frac{|u'v'' - u''v'|}{\left|(u')^2 + (v')^2\right|^{\frac{2}{3}}}.$$

If we know the parameters of expression curve, then we can find the maximum value directly to the curvature function to give the corresponding regularization parameter. Otherwise, we can cubic spline interpolation methods to determine the approximate upper corner, so select regularization parameter.

4. NUMERICAL EXAMPLE

Consider the following integral equation:

$$\int_0^1 (1+ts)e^{ts}x(s)ds = e^t, 0 \le t \le 1 \tag{6}$$

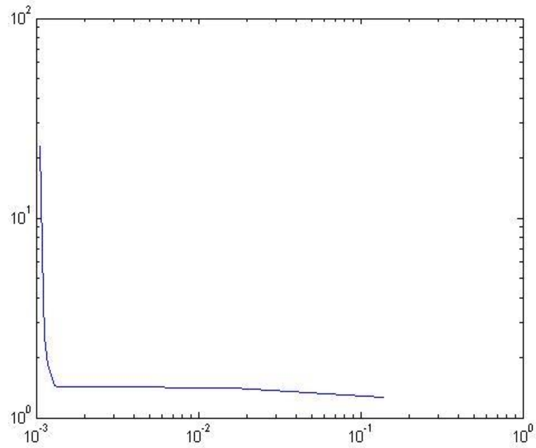

Figure 1. L curve

From the analysis to know, this problem is ill-posed and its unique solution is $x(t) = 1$, we hope to achieve through their regularization method regular solution is approximately equal to the exact solution.

First the operator $K: L^2(0,1) \to L^2(0,1)$ is given by the following formular

$$(Kx)(t) = \int_0^1 (1+ts)e^{ts}x(s)ds \tag{7}$$

and the operator K is adjoint operator, i. e $K^* = K$. We can prove K is one to one and continuous. For the numerical calculation of Kx, the integral equations are discretized in this discussion based on the discrete method of quadrature rules, the following two kinds of method we use Simpson's rule, and $t_i = i/n, i = 0,1,\cdots n$, we replace $(Kx)(t_i)$ by the fololwing formular

$$\sum_{j=0}^{n}\left(1+t_i t_j\right) w_j e^{t_i t_j} x(t_j) \tag{8}$$

where

$$w_j = \begin{cases} \frac{1}{3n} & j = 0 \quad or \quad n \\ \frac{4}{3n} & j = 1,3,\cdots n-1 \\ \frac{2}{3n} & j = 2,4,\cdots n-2 \end{cases} \tag{9}$$

Next, we should use two methods to solve it.

4.1 Tikhonov regularization methods

When writing in a matrix form discrete form we will simply note $Ax = b$, general notice the corresponding coefficient matrix asymmetric, we have the corresponding discrete Tikhonov equation

$$\alpha x_\alpha^\delta + A^2 x_\alpha^\delta = Ay^\delta \qquad (10)$$

where $y^\delta = \left(y_i^\delta\right) \in R^{n+1}$ is the disturbed right hand side. $y_i = \exp\left(i/n\right)$ uniformly distributed random vector and satisfy

$$\left|y - y^\delta\right|_2 := \sqrt{\frac{1}{n+1}\sum_{i=0}^{n}\left(y_i - y_i^\delta\right)^2} \le \delta .$$

Below we will give the general Tikhonov regularization method is applied to get ten group the average results of numerical calculation, and we will give the discrete norm of the error between exact solution $x\left(t\right)=1$ and Tikhonov approximate solution $x^{\alpha,\delta}$. In figure 2, we choose $\delta = 0$, only for small enough α , Simpson rules of discrete error can lead to bigger, and when $\alpha \le 10^{-8}$, for different discrete parameter $n=8$ and $n=16$, the error is also different.

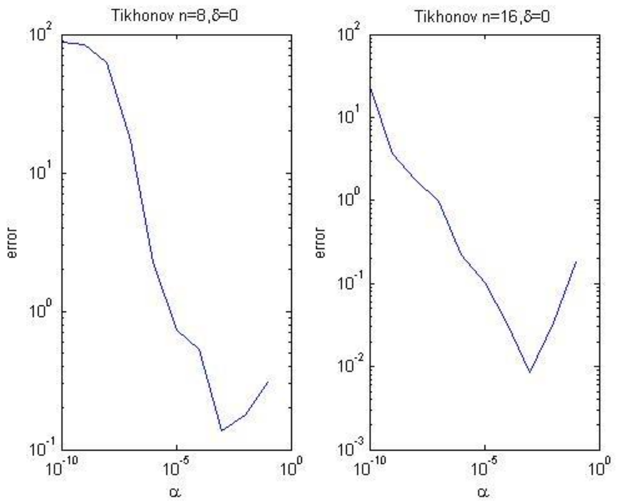

Figure 2. $\delta = 0$ error figure

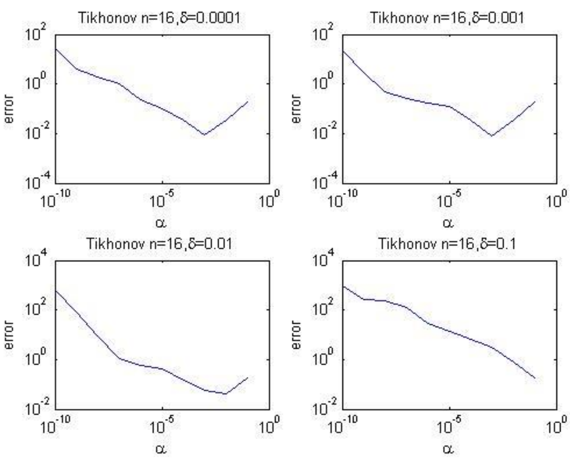

Figure 3. n=16 Tikhonov error figure

Below we give the residual norm reconciliation between the picture norm.

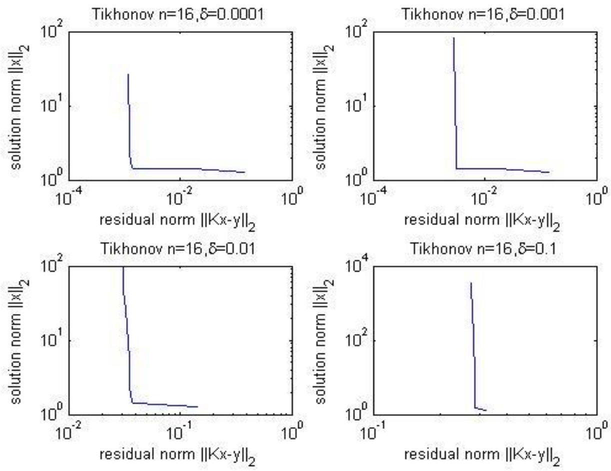

Figure 4. n=16 residual norm and solution norm figure

In figure 3, let $n=16$, and we note that the total error is reduced to a minimum and then increases with the increasing α . From the figure 4, the essence of the regularization method is to solve the original problem of conversion between the fit (the first term in the above formula) norm reconciliation (or semi-norm) Minimum (second term in the above formula) in the data achieve some compromise issue, which is in a regular problem at the heart of the main idea of Tikhonov regularization method, selecting α adequately is playing a important role in regularization methods, when α is large, it equals to join a large amount regular, can make the desires of the solution of the norm or half norm is small, but it at the expanse of the cost of residual norm $\left\|Kx-y\right\|_2$. When α is small, it equals to add a small amount of regular, the obtained solution so that the residual is small, but the norm so that the solution cannot be minimized. One theory suggests that, the selection range of α is $\sigma_n \le \alpha \le \sigma_1$, where $\sigma_1 \cdots \sigma_n$ is singular values. Finally, we will give the general Tikhonov regualarization methods error figure when we use the stopping rule.

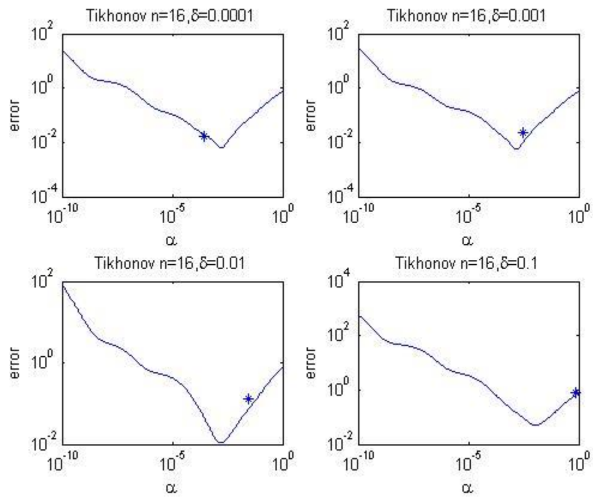

Figure 5. n=16stopping rule error figure

By comparison seen in the posterior Select next parameter stop criterion, the more likely to get the minimum optimal regularization parameters α , which can be obtained by minimizing functions.

4.2 Iteration Tikhonov regularization methods

Since the exact solution has good smoothness, so we may assume smoothness index $r = 2$, so we can use iterative Tikhonov regularization method to solve this equation, the same rule to use Simpson left discretized integral equations obtained discrete iterative equation

$$x_{\alpha,\delta}^{m} = \sum_{k=0}^{m-1} \alpha^k \left[\left(\alpha I + A^2 \right)^{-1} \right]^{k+1} A y^{\delta}, \; x_{\alpha}^{0,\delta} = 0 \qquad (10)$$

where $y^{\delta} := (y_i)^{\delta}$ is disturbed and discreted right hand side and satisfy $y_i = \exp(i/n)$

$$\left\| y - y^{\delta} \right\| = \sqrt{\frac{1}{n+1} \sum_{i=0}^{n} \left(y_i - y_i^{\delta} \right)^2} \le \delta$$

The figure 6 will shows us the error figure when the regularization parameter $\alpha = 0.001$ iteration parameter $m = 1,2,3,4,5,6,7,8,9,100,200,$ 300, 500, and the error $\delta = 0.0001, 0.001, 0.01, 0.1$.

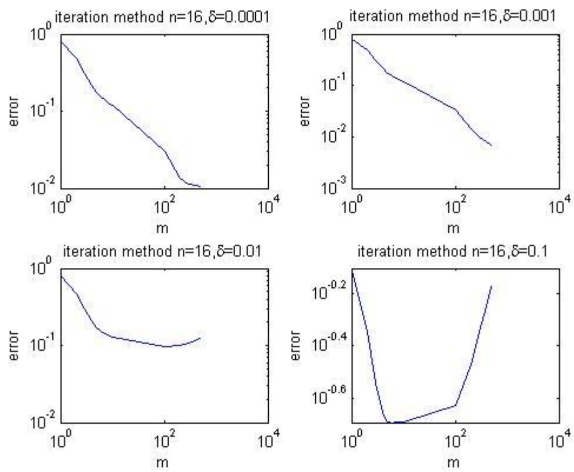

Figure 6. n=16 iteration methods error figure

From the figure 6, when the regularization parameter takes a fixed value, error estimation with the parameter m increases and then decreases to an optimal value and then increases, which fully reflects the positive effect of the iteration parameters. By comparing the obtained optimal convergence precision, iterative Tikhonov regularization method is equal to the higher than normal Tikhonov regularization method. This shows that the iterative Tikhonov regularization method selection parameter is more convenient to calculate faster, more accurate calculation, better stability advantages.

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