Quasi-periodic solutions for a nonlinear non-autonomous Hamiltonian system

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ABSTRACT. The quasi-periodic solutions of nonlinear Hamiltonian system are important tools to enhance our understanding of dynamic behaviours. Therefore, this paper probes into a popular, completely resonant nonlinear beam equation. In the equation model, there is a nonlinear term periodic in the space variable and quasi-periodic in the time variable. The external frequency vector is 2-dimensional. Using the Kolmogorov-Arnold-Moser (KAM) method and the normal form, the author proved the existence of quasi-periodic solutions with two frequencies and gave the analytical expression. The solutions are small amplitude and linearly stable. The research findings shed new light on measurement estimation and normal form technique, provide new insights into the dynamics of beam equation, and promote the application of the KAM method.

RÉSUMÉ. Les solutions quasi périodiques du système hamiltonien nonlinéaire sont des outils importants pour améliorer notre compréhension des comportements dynamiques. Par conséquent, cet article explore une équation de faisceau non linéaire très populaire et complètement résonante. Dans le modèle d'équation, il existe un terme nonlinéaire périodique dans la variable d'espace et quasi périodique dans la variable de temps. Le vecteur de fréquence externe est bidimensionnel. En utilisant la méthode de Kolmogorov-Arnold-Moser (KAM) et la forme normale, l'auteur a prouvé l'existence de solutions quasi-périodiques à deux fréquences et a donné l'expression analytique. Les solutions sont de faible amplitude et linéairement stables. Les résultats de la recherche jète un nouvel éclairage sur l'estimation des mesures et la technique de la forme normale, donne une nouvelle perception de la dynamique de l'équation du faisceau et promouvoir l'application de la méthode KAM.

KEYWORDS: kolmogorov-arnold-moser (KAM) method, hamiltonian, beam equation, quasi-periodic solution, normal form.

MOTS-CLÉS: mthode de kolmogorov-arnold-moser (KAM), hamiltonien, équation du faisceau, solution quasi periodique, forme normale.

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1. Introduction

This paper discusses the existence of quasi-periodic solutions for a beam equation

$$u_{tt} + u_{xxxx} + f(\omega t, x, u) = 0, x \in [0, \pi] \tag{1.1}$$

under the hinged boundary conditions

$$u(t, 0) = u_{xx}(t, 0) = u(t, \pi) = u_{xx}(t, \pi) = 0, \tag{1.2}$$

where $\omega = (\omega_1, \omega_2) \in [\eta, 2\eta]^2$ is a frequency vector with $\eta > 0$; $f(\omega t, x, u)$ is a quasi-periodic nonlinear term in the time variable, and $f(\omega t, x, u) = f(\vartheta, x, u)$ ($\vartheta \in \mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$) is a real analytic function in ϑ and x.

The purpose of the discussion is to verify whether the small amplitude quasi-periodic solutions of $u_{tt} + u_{xxxx} = 0$ can persist under perturbation, whether the finite-dimensional tori is linearly stable, and whether quasi-periodic solutions have zero Lyapunov exponents.

For non-autonomous Hamiltonian systems like Equation (1.1), their quasi-periodic solutions are mainly investigated by the Lyapunov-Schmidt decomposition. However, the quasi-periodic solutions thus obtained tend to be global, failing to provide dynamical information around equilibrium points. Here, the Kolmogorov-Arnold-Moser (KAM) method is adopted to solve the problem.

The KAM theory, named after its proposer Kolmogorov, Arnold, and Moser, is one of the most important mathematical achievements in the 20th century. Later, Wayne, Kuksin, and Pöschel developed the infinite-dimensional KAM theory, which produces quasi-periodic solutions with dynamic properties like linear stability (Chen, 2017; Chen *et al.*, 2017; Cao and Yuan, 2017; Si and Si, 2017; Si and Si, 2018) and vanishing Lyapunov exponents.

The KAM method can be applied to examine the Hamiltonian partial differential equations (PDEs) in the following manner: transforming nonlinear equations into infinite-dimensional Hamiltonian systems; constructing canonical transformations that can change the Hamiltonians to suitable Birkhoff normal forms; looking for quasi-periodic solutions though KAM iterations, that is, setting up a KAM theorem.

Much research has been done on the quasi-periodic solutions of nonlinear beam equations (Eliasson *et al.*, 2016; Geng and You, 2003; Geng and You, 2006; Geng and Zhou, 2018; Liang and Geng, 2006; Procesi, 2010). However, there is only a few reports on those of complete resonant equations. Geng and You (2006) tackle a complete resonant beam equation with a nonlinear term u^3. Tuo and Si (2015) probe into a beam equation with nonlinearity $\phi(t)u^5$ (Gao and Liu, 2017; Ge and Geng, 2018).

In this paper, it is assumed that f in Equation (1.1) has a special nonlinear form$f(t, x, u) = \delta\psi(\omega t, x)u^3$, where δ is a small positive parameter and ψ is quasi-periodic in t and periodic in x. This model is very difficult to solve because of the dependence of nonlinearity on time and space variables, but it is widely applicable.

Equation (1.1) is a perturbation of the linear beam equation $u_{tt} + u_{xxxx} = 0$ whose solution can be written as $u(t,x) = \sum_{j\in\mathcal{J}} q_j(t)\phi_j(x), q_j(t) = I_j \cos(j^2 t + \varphi_j^0)$. Here, for $j \in \mathbb{Z}^+$, $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx$; $\mathcal{J}$ is any subset of $\mathbb{Z}^+ := \{1,2,\cdots\}$; amplitudes $I_j \geq 0$ and φ_j^0 are initial phases. The solutions travel on a rotational torus, which is finite- or infinite-dimensional. Every torus is linearly stable. The Lyapunov exponents of all solutions are zero. Unfortunately, not all the invariant manifolds will be preserved under the perturbation of f, owing to the numerous resonances and the strong perturbation of f on solutions of large amplitude.

This paper innovatively transforms the Hamiltonian into its Birkhoff normal form, despite falling short of the zero-momentum conditions. The perturbation was divided into two parts to overcome the difficulty of small divisor. One part satisfies zero-momentum conditions while the other part does not. Thus, several conditions were added in Section 3. Then, it is necessary to estimate the measure of the parameter ω, so as to maximize the number of parameters satisfying these conditions. Fortunately, the growth of eigenvalues is quartic, which is crucial for measure estimation. The KAM iterations here are in the standard form (Liu and Yuan, 2014), and are thus omitted.

The remainder of this paper is organized as follows: Section 2 transforms the equation into an infinite-dimensional Hamiltonian system; Section 3 obtains the Birkhoff normal form; Section 4 relies on the normal form to prove the existence of quasi-periodic solution and give an analytic expression.

2. Hamiltonian setting

Firstly, Equation (1.1) was converted to a Hamiltonian system. Under Equation (1.2), the operator $B = \frac{d^4}{dx^4}$ has eigenvalues $\zeta_j = j^4$ and eigenfunctions $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx$, where $j \in \mathbb{Z}^+ := \{1,2,\cdots\}$. Let $D_1(\sigma_1) := \{\vartheta \mid |\mathrm{Im}\vartheta| < \sigma_1\}$, $|\psi|_{2a} := \sup_{x\in D_2(2a)} |\psi(\vartheta,x)|$ for $\vartheta \in D_1(\sigma_1)$, $D_2(2a) := \{x \mid |\mathrm{Im}x| < 2a\}$, and $|\psi|_{\sigma_1,2a} := \sup_{(\vartheta,x)\in D_1(\sigma_1)\times D_2(2a)} |\psi(\vartheta,x)|$.

It is assumed that $<\cdot,\cdot>$ is the standard inner product in $\mathbb{C}^2$; $(\cdot,\cdot)$ is the scalar product in $L^2[0,\pi]$, the space of real-valued sequences is

$l^{a,s} = l^{a,s}(\mathbb{R}) := \{q = (q_1, q_2, \cdots),\ (\|q\|_{a,s})^2 = \sum_{i\geq 1} |q_i|^2 i^{2s} e^{2ai} < \infty\}$ $\quad (a > 0, s > \frac{1}{2})$;

$\sigma_1 > 0$ and σ_1 has a positive lower bound $\tilde{\sigma}_1$, a is a positive real number, and "measure" refers to the Lebesgue measure. For function ψ, the following assumptions were put forward:

(A1) ψ has a Fourier expansion $\psi(\vartheta, x) = \psi_0 + \sum_{k\in\mathbb{Z}^2\backslash\{0\}}\psi_k(x)e^{i<k,\vartheta>}$ with$0 \neq \psi_0 \in \mathbb{R}$.

(A2) ψ can analytically extend to $D_1(\sigma_1) \times D_2(2a)$ with $|\psi|_{\sigma_1,2a} < \infty$.

(A3) $\partial_x^{2k+1}\psi(\vartheta, 0) = 0, \forall k \in \mathbb{N}$.

By introducing $\partial_t u = v$, $\partial_t v + Bu = -\delta\psi(\omega t, x)u^3$, Equation (1.1) can be changed to a Hamiltonian system $H = \frac{1}{2}(v, v) + \frac{1}{2}(Bu, u) + \frac{\delta}{4}\int_0^\pi \psi(\omega t, x)u^4 dx$. If

$$u(t, x) = \sum_{j\geq 1}\frac{q_j(t)}{\sqrt[4]{\zeta_j}}\phi_j(x), v(t, x) = \sum_{j\geq 1}\sqrt[4]{\zeta_j}\, p_j(t)\phi_j(x),$$

Where $q = (q_1, q_2, \cdots)$,$p = (p_1, p_2, \cdots)$ and $p, q \in l^{a,s}$., then the Hamiltonian can be transformed to $H = \Lambda + \delta G$, and

$$\Lambda = \frac{1}{2}\sum_{j\geq 1}\sqrt{\zeta_j}\left(q_j^2 + p_j^2\right), G = \frac{1}{4}\int_0^\pi \psi(\omega t, x)\left(\sum_{j\geq 1}\frac{q_j(t)}{\sqrt[4]{\zeta_j}}\phi_j(x)\right)^4 dx.$$

Then, the Hamiltonian system can be expressed as:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \sqrt{\zeta_j}p_j, \dot{p}_j = -\frac{\partial H}{\partial q_j} = -\sqrt{\zeta_j}q_j - \delta\frac{\partial G}{\partial q_j}, j \geq 1 \tag{2.1}$$

The corresponding symplectic structure is $\sum d\, q_i \wedge dp_i$ on $l^{a,s} \times l^{a,s}$. According to Geng and You (2003), if the solution of Equation (2.1) $(p(t), q(t))$ is real analytic for $t \in I$, then for $(t, x) \in\times [0, \pi]$,

$$u(t, x) = \sum_{j\geq 1}\frac{q_j(t)}{\sqrt[4]{\zeta_j}}\phi_j(x) \tag{2.2}$$

must be a real analytic classical solution of Equation (1.1), where I is a real interval. Thus, this paper attempts to find a solution with the form (2.2).

Then, action-angle variables $J \in \mathbb{R}^2$ and $\vartheta \in \mathbb{T}^2$ were introduced to obtain an autonomous system. In this way, the Hamiltonian and the Hamiltonian system can be transformed into

$$H =< \omega, J > + \frac{1}{2}\sum_{j\geq 1}\sqrt{\zeta_j}\,(q_j^2 + p_j^2) + \delta G(q, \vartheta) \tag{2.3}$$

and

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \dot{p}_j = -\frac{\partial H}{\partial q_j}, \dot{\vartheta} = \omega, \dot{J} = -\delta\frac{\partial G}{\partial \vartheta} = -\delta\frac{\partial\int_0^\pi \chi dx}{\partial \vartheta}, (j \geq 1)$$

The corresponding symplectic structure becomes $d\vartheta \wedge dJ + \sum d\, q_i \wedge dp_i$. The following lemma can be proved by the method for Lemma 2.3 in Wang *et al.* (2018).

Lemma 2.1 The gradient $\frac{\partial G}{\partial q}$ is real analytic and satisfies $\| \frac{\partial G}{\partial q} \|_{a,s} = \mathcal{O}((\|q\|_{a,s})^3)$.

3. Brikhoff normal form

This section transforms the Hamiltonian into a Birkhoff normal form. The complex coordinates $\boldsymbol{z} = (\boldsymbol{z_1}, \boldsymbol{z_2}, \cdots)$, $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{z}}_1, \bar{\boldsymbol{z}}_2, \cdots)$ and $\boldsymbol{z}, \bar{\boldsymbol{z}} \in \boldsymbol{l^{a,s}}(\mathbb{C})$ were adopted, where $\boldsymbol{z_i} = \frac{\boldsymbol{q_i + ip_i}}{\sqrt{2}}, \bar{\boldsymbol{z}}_i = \frac{q_i - ip_i}{\sqrt{2}}$ for $i \in \mathbb{Z}^+$.Then, a real analytic Hamiltonian can be obtained as:

$$H = \Lambda + \delta G, \tag{3.1}$$

where $\boldsymbol{\Lambda} =< \boldsymbol{\omega}, \boldsymbol{J} > + \sum_j \sqrt{\boldsymbol{\zeta_j}}\, \boldsymbol{z_j} \bar{\boldsymbol{z}}_j,\quad \boldsymbol{G} = \frac{1}{4} \int_0^{\pi} \boldsymbol{\psi}\,(\boldsymbol{\vartheta}, \boldsymbol{x}) \left(\sum_{j \geq 1} \frac{z_j + \bar{z}_j}{\sqrt[4]{4\zeta_j}} \boldsymbol{\phi_j}(\boldsymbol{x}) \right)^{\boldsymbol{4}} \boldsymbol{dx}.$

The corresponding symplectic structure becomes $\boldsymbol{d\vartheta} \wedge \boldsymbol{dJ} + \boldsymbol{i} \sum_i \boldsymbol{d}\, \boldsymbol{z_i} \wedge \boldsymbol{d\bar{z}_i}$. Moreover, we have

$$\boldsymbol{G} = \frac{1}{16} \sum_{\iota_1, \iota_2, \iota_3, \iota_4 \in \mathbb{Z}^+,\ \iota_1 \pm \iota_2 \pm \iota_3 \pm \iota_4 = 0} \frac{G_{\iota_1 \iota_2 \iota_3 \iota_4}}{\iota_1 \iota_2 \iota_3 \iota_4} (z_{\iota_1} + \bar{z}_{\iota_1})(z_{\iota_2} + \bar{z}_{\iota_2})(z_{\iota_3} + \bar{z}_{\iota_3})(z_{\iota_4} + \bar{z}_{\iota_4}) + \frac{1}{16} \sum_{|k| \geq 1, \iota_1, \iota_2, \iota_3, \iota_4 \in \mathbb{Z}^+} \frac{G_{k, \iota_1 \iota_2 \iota_3 \iota_4}}{\iota_1 \iota_2 \iota_3 \iota_4} e^{i<k,\vartheta>} (z_{\iota_1} + \bar{z}_{\iota_1})(z_{\iota_2} + \bar{z}_{\iota_2})(z_{\iota_3} + \bar{z}_{\iota_3})(z_{\iota_4} + \bar{z}_{\iota_4}),$$

where

$$G_{\iota_1 \iota_2 \iota_3 \iota_4} = \psi_0 \int_0^{\pi} \phi_{\iota_1} \phi_{\iota_2} \phi_{\iota_3} \phi_{\iota_4} dx \tag{3.2}$$

and

$$G_{k, \iota_1 \iota_2 \iota_3 \iota_4} = \int_0^{\pi} \psi_k\,(x) \phi_{\iota_1} \phi_{\iota_2} \phi_{\iota_3} \phi_{\iota_4} dx, |k| \geq 1. \tag{3.3}$$

It is obvious that $G_{\iota_1 \iota_2 \iota_3 \iota_4} = 0$. unless $\iota_1 \pm \iota_2 \pm \iota_3 \pm \iota_4 = 0$ and

$$G_{\iota_1 \iota_2 \iota_1 \iota_2} = \frac{\psi_0}{2\pi} (2 + \delta_{\iota_1 \iota_2}), \tag{3.4}$$

where $\delta_{\iota_1 \iota_2} = 1$ for $\iota_1 = \iota_2$, and $\delta_{\iota_1 \iota_2} = 0$ for $\iota_1 \neq \iota_2$.

Next, an admissible set $\mathcal{J} := \{(n_1, n_2) | n_1, n_2\ are\ odd,\ n_1 \neq 7\,mod(14),\ n_2 > 6n_1^2\}$ was selected, which satisfies the Definition 3.1 in Liang and Geng (2006). Three sets of vectors $(\iota_1, \iota_2, \iota_3, \iota_4) \in \mathbb{Z}_+^4$. were defined for each index set $\mathcal{J}$: the set Δ_0 that $\iota_1, \iota_2, \iota_3$, and ι_4 are all in $\mathcal{J}$, the set Δ_j that $4 - j$ of $\iota_1, \iota_2, \iota_3$, and ι_4 are in $\mathcal{J}$ for j=1,2, and the set Δ_3 that none or only one of $\iota_1, \iota_2, \iota_3$, and ι_4 is in $\mathcal{J}$. Let $\mathcal{J}_0 = \{(\iota_1, \iota_2, \iota_3, \iota_4) \equiv (\iota_1, \iota_2, \iota_1, \iota_2)\}$, $\mathcal{E}_1 = \mathcal{J}_0 \cap \Delta_0$ and $\mathcal{E}_2 = \mathcal{J}_0 \cap \Delta_2$.

The next step is to construct a Hamiltonian S and use its vector-field X_S to transform Equation (3.1) into a fourth-order partial Birkhoff form, so as to study it as a perturbation of a nonlinear integrable system. However, the divisors have to be assumed as nonzero because of the small divisors of the Hamiltonian. Thus, it is necessary to estimate the measure of parameters which make the divisors zero. For this purpose, notations $\bar{\Delta} := \Delta_0 \cup \Delta_1 \cup \Delta_2$ and $\bar{\bar{\Delta}} := (\Delta_0 \backslash \mathscr{E}_1) \cup \Delta_1 \cup (\Delta_2 \backslash \mathscr{E}_2)$ were introduced.

Lemma 3.1 If $(\iota_1, \iota_2, \iota_3, \iota_4) \in \bar{\Delta}$ and k are not zero and δ is sufficiently small, then there is a set $\bar{\Omega} \subset [\eta, 2\eta]^2$ satisfying that, for each $\omega \in \bar{\Omega}$ and$\pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}} \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} \neq 0$,

$$\left|\pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}} \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} + < k, \omega >\right| \geq \frac{\eta\delta^{\frac{1}{3}}}{|k|^4}. \tag{3.5}$$

Moreover, $meas\bar{\Omega} \geq \eta^2 \left(1 - C_2\delta^{\frac{1}{3}}\right)$, where C_2 is a constant depending on η, n_1, and n_2.

Proof Assuming that:

$h_{\iota_1\iota_2\iota_3\iota_4,k} = \pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}} \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} + < k, \omega >$,

$R_{\iota_1\iota_2\iota_3\iota_4,k} = \left\{\omega \in [\eta, 2\eta]^2 : |h_{\iota_1\iota_2\iota_3\iota_4,k}| < \frac{\eta\delta^{\frac{1}{3}}}{|k|^4}\right\}$,

$\Omega^1 = \bigcup_{|k|\geq 1} \bigcup_{(\iota_1,\iota_2,\iota_3,\iota_4)\in\Delta_1} R_{\iota_1\iota_2\iota_3\iota_4,k}$, and $\Omega^2 = \bigcup_{|k|\geq 1} \bigcup_{(\iota_1,\iota_2,\iota_3,\iota_4)\in\Delta_2} R_{\iota_1\iota_2\iota_3\iota_4,k}$.

It is evident that $\tilde{\Omega} = \Omega^0 \cup \Omega^1 \cup \Omega^2$. For hyperplanes

$\pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}} \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} + < k, \omega > = \pm \frac{\eta\delta^{\frac{1}{3}}}{|k|^4}$.

We have

$\text{Measure}R_{\iota_1\iota_2\iota_3\iota_4,k} \leq 2|k|^{-1}\sqrt{2}\eta \frac{2\eta\delta^{\frac{1}{3}}}{|k|^4} \leq \frac{4\sqrt{2}\delta^{\frac{1}{3}}}{|k|^5}\eta^2$.

Hence, $\text{Measure}R_{\iota_1\iota_2\iota_3\iota_4,k} \leq C\frac{\eta^2\delta^{\frac{1}{3}}}{|k|^5}$, where C is an absolute constant. Note that

$$\#\left\{k \in \mathbb{Z}^2 : |k| = l\right\} \leq 2^2 l, l \in \mathbb{Z}^+. \tag{3.6}$$

Case I. If $(\iota_1, \iota_2, \iota_3, \iota_4) \in \Delta_0$, then it can be deduced from (3.6) that:

$\text{Measure}\Omega^0 \leq C\sum_{|k|\geq 1}\frac{\eta^2\delta^{\frac{1}{3}}}{|k|^5}(n_2 - n_1 + 1)^4 \leq C(n_2 - n_1 + 1)^4\eta^2\delta^{\frac{1}{3}}\sum_{1\leq|k|=l}\frac{1}{l^5}2^2 l$.

Since $\sum_{l\geq 1}\frac{1}{l^4}$ is convergent, $\text{Measure}\Omega^0 \leq C\eta^2\delta^{\frac{1}{3}}$, and C is dependent of n_1, n_2 and η.

Case II. For $(\iota_1, \iota_2, \iota_3, \iota_4) \in \Delta_1$, it is assumed that $\iota_4 \notin \mathcal{J}$. The other cases can be treated by analogy. Then, it can be derived that

$| < k, \omega > \pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}} \pm \sqrt{\zeta_{\iota_3}}| \leq 2\eta|k| + 3n_2^2.$

If $|\iota_4| > \sqrt{2\eta|k| + 3n_2^2 + 1}$ and δ is sufficiently small,

$|h_{\iota_1\iota_2\iota_3\iota_4,k}| \geq | \pm \sqrt{\zeta_{\iota_4}}| - | < k, \omega > \pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}} \pm \sqrt{\zeta_{\iota_3}}| > 2\eta|k| + 3n_2^2 + 1 - (2\eta|k| + 3n_2^2) = 1 > \frac{\eta^2\delta^{\frac{1}{3}}}{|k|^4}.$

Thus, it only needs to discuss $1 \leq \iota_4 \leq \sqrt{2\eta|k| + 3n_2^2 + 1}$. Therefore,

$\text{Measure}\Omega^1 \leq C \sum_{|k|\geq 1} \frac{\eta^2\delta^{\frac{1}{3}}}{|k|^4|k|} (n_2 - n_1 + 1)^3 \sqrt{2\eta|k| + 3n_2^2 + 1} \leq C\eta^2\delta^{\frac{1}{3}}(n_2 - n_1 + 1)^3 \sum_{|k|\geq 1} \frac{1}{|k|^4} \leq C\eta^2\delta^{\frac{1}{3}}(n_2 - n_1 + 1)^3 \sum_{l\geq 1} 2^2\, l \frac{1}{l^4} \leq C\eta^2\delta^{\frac{1}{3}},$

and C is dependent of n_1, n_2 and η.

Case III. For $(\iota_1, \iota_2, \iota_3, \iota_4) \in \Delta_2$, it is assumed that ι_4 and $\iota_3 \notin \mathcal{J}$ without loss of generality.

Case III-1. If $\pm\sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} = 0$, then $|h_{\iota_1\iota_2\iota_3\iota_4,k}| = | < k, \omega > \pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}}|$. Assuming that

$\Omega^{2,1} = \bigcup_{|k|\geq 1} \bigcup_{\iota_3,\iota_4 \notin \mathcal{J},\ \iota_1,\iota_2 \in \mathcal{I},\ \pm\sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} = 0} R_{\iota_1\iota_2\iota_3\iota_4,k}$

Then, we have

$\text{Measure}\Omega^{2,1} \leq C \sum_{|k|\geq 1} \frac{\eta^2\delta^{\frac{1}{3}}}{|k|^4|k|} (n_2 - n_1 + 1)^2 \leq C\eta^2\delta^{\frac{1}{3}}(n_2 - n_1 + 1)^2 \sum_{|k|\geq 1} \frac{1}{|k|^5} \leq C\eta^2\delta^{\frac{1}{3}}(n_2 - n_1 + 1)^2 \sum_{l\geq 1} 2^2\, l \frac{1}{l^5} \leq C\eta^2\delta^{\frac{1}{3}},$

where the constant C depends on n_1, n_2 and η.

Case III-2. If $\pm\sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} \neq 0$, then $\iota_3 - \iota_4 \neq 0$. Without loss of generality, it is assumed that $\iota_3 > \iota_4$. Thus, $\iota_3 - \iota_4 := t_0 \geq 1$. Moreover,

$$| \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}}| \geq \iota_3^2 - \iota_4^2 = (\iota_3 - \iota_4)(\iota_3 + \iota_4) = t_0(2\iota_4 + t_0). \tag{3.7}$$

If $\iota_4 > \widetilde{K} := \eta|k| + n_2^2$, it can be derived that $| \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}}| \geq 2\iota_4 + t_0 > 2\widetilde{K} + 1$. Since

$$| < k, \omega > \pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}}| \leq 2\eta|k| + 2n_2^2 \tag{3.8}$$

and

$$|h_{\iota_1\iota_2\iota_3\iota_4,k}| \geq | \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}}| - | < k, \omega > \pm\sqrt{\zeta_{\iota_1}} \pm \sqrt{\zeta_{\iota_2}}|, \tag{3.9}$$

$|h_{\iota_1\iota_2\iota_3\iota_4,k}| > 2\widetilde{K} + 1 - 2\eta|k| - 2n_2^2 = 1 > C\frac{\eta^2\delta^{\frac{1}{3}}}{|k|^4}$

is true if δ is sufficiently small. Hence, it only needs to analyse $1 \le \iota_4 \le \widetilde{K}$. If $t_0 > \widetilde{K} + 1$, it can be deduced from (3.7) that

$| \pm \sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}}| \ge 2t_0\iota_4 + t_0^2 > 2t_0 = 2\widetilde{K} + 2.$

It follows from (3.8) and (3.9) that

$|h_{\iota_1\iota_2\iota_3\iota_4,k}| \ge 2\widetilde{K} + 2 - 2\eta|k| - 2n_2^2 = 2\eta|k| + 2n_2^2 + 2 - 2\eta|k| - 2n_2^2 = 2 > \frac{\eta^2\delta^{\frac{1}{3}}}{|k|^4},$

as δ is sufficiently small. Thus, it is necessary to study the case $1 \le t_0 \le \widetilde{K} + 1$. Thus, $1 \le \iota_3 = \iota_4 + t_0 \le 2\widetilde{K} + 2$. Assuming that

$\Omega^{2,2} = \bigcup_{|k|\ge 1} \bigcup_{\iota_4,\iota_3 \notin \mathscr{J},\ \iota_1,\iota_2 \in \mathcal{J},\ \pm\sqrt{\zeta_{\iota_3}} \pm \sqrt{\zeta_{\iota_4}} \ne 0} R_{\iota_1\iota_2\iota_3\iota_4,k},$

then we have

Measure$\Omega^{2,2} \le C\sum_{|k|\ge 1}\frac{\eta^2\delta^{\frac{1}{3}}}{|k|^4|k|}(n_2 - n_1 + 1)^2(\eta|k| + n_2^2)(2\eta|k| + 2n_2^2 + 2) \le C\eta^2\delta^{\frac{1}{3}}(n_2 - n_1 + 1)^2\sum_{|k|\ge 1}\frac{1}{|k|^5}(n_2 - n_1 + 1)^2(\eta|k| + n_2^2)(2\eta|k| + 2n_2^2 + 2) \le C\eta^2\delta^{\frac{1}{3}}(n_2 - n_1 + 1)^2\sum_{l\ge 1}2^2\, l \quad \frac{1}{l^3} \le C\eta^2\delta^{\frac{1}{3}},$

where C is dependent of n_1, n_2 and η. Thus, for $(\iota_1, \iota_2, \iota_3, \iota_4) \in \Delta_2$, there exists a constant C satisfying Measure$\Omega^2 \le C\eta^2\delta^{\frac{1}{3}}$.

To sum up, assuming $\bar{\Omega} = [\eta, 2\eta]^2 \backslash \tilde{\Omega}$, meas$\bar{\Omega} \ge (1 - C_2\delta^{\frac{1}{3}})\eta^2$ is valid, where the constant C_2 is dependent of n_1, n_2 and η.

Proposition 3.1 Concerning the Hamiltonian (3.1), if parameter δ is sufficiently small for each index set $\mathscr{J}$, then there is a subset $\Omega \subset [\eta, 2\eta]^2$ satisfying Measure$\Omega > 0$ such that for each $\omega \in \Omega$, there exists a transformation Υ that changes (3.1) to a normal form,

$H^\circ\Upsilon = \Lambda + \delta\bar{G} + \delta\hat{G} + \delta^2 K,$

where

$$\bar{G}(z, \bar{z}) = \frac{1}{2}\sum_{\iota_1 \in \mathcal{J}\ or\ \iota_2 \in \mathcal{J}} \bar{G}_{\iota_1\iota_2}|z_{\iota_1}|^2|z_{\iota_2}|^2, \tag{3.10}$$

$$\bar{G}_{\iota_1\iota_2} = \begin{cases} \frac{3\psi_0}{2\pi\iota_1^2\iota_2^2}, \iota_1 \ne \iota_2 \\ \frac{9\psi_0}{8\pi\iota_1^2\iota_2^2}, = \iota_2, \end{cases}$$

$$\hat{G} = \sum_{(\iota_1,\iota_2,\iota_3,\iota_4)\in\Delta_3,\ \iota_1\pm\iota_2\pm\iota_3\pm\iota_4=0} \psi_{\iota_1\iota_2\iota_3\iota_4} (z_{\iota_1} + \bar{z}_{\iota_1})(z_{\iota_2} + \bar{z}_{\iota_2})(z_{\iota_3} + \bar{z}_{\iota_3})(z_{\iota_4} + \bar{z}_{\iota_4}) + \sum_{|k|\geq 1} e^{i<k,\vartheta>}$$

$$\sum_{(\iota_1,\iota_2,\iota_3,\iota_4)\in\Delta_3} \psi_{k,\iota_1\iota_2\iota_3\iota_4} (z_{\iota_1} + \bar{z}_{\iota_1})(z_{\iota_2} + \bar{z}_{\iota_2})(z_{\iota_3} + \bar{z}_{\iota_3})(z_{\iota_4} + \bar{z}_{\iota_4}), \quad (3.11)$$

and $\delta^{\frac{1}{3}}|K| = \mathcal{O}((\|z\|_{a,s})^6)$. The transformation is real analytic and canonical. Besides, Υ is well defined in $D_1(\frac{\sigma_1}{2}) \times \mathcal{U}$, with $\mathcal{U} \subset l^{a,s}$ being a neighbourhood of the origin.

Proof Let $z_j = w_j$, $\bar{z}_j = w_{-j}$ $(j \geq 1)$ and $w_0 = 0$. Then, we have

$$H =< \omega, J > + \sum_{j\geq 1} \sqrt{\zeta_j}\, w_j w_{-j} + \delta \sum_{\iota_1,\iota_2,\iota_3,\iota_4,\ |\iota_1|\pm|\iota_2|\pm|\iota_3|\pm|\iota_4|=0} \psi_{\iota_1\iota_2\iota_3\iota_4} w_{\iota_1} w_{\iota_2} w_{\iota_3} w_{\iota_4}$$
$$+\delta \sum_{|k|\geq 1} \sum_{\iota_1,\iota_2,\iota_3,\iota_4} \psi_{k,\iota_1\iota_2\iota_3\iota_4} e^{i<k,\vartheta>} w_{\iota_1} w_{\iota_2} w_{\iota_3} w_{\iota_4},$$

where $\iota_1, \iota_2, \iota_3, \iota_4 \in \mathbb{Z}/\{0\}$ and

$\psi_{\iota_1\iota_2\iota_3\iota_4} := \frac{G_{\iota_1\iota_2\iota_3\iota_4}}{16|\iota_1\iota_2\iota_3\iota_4|}, \psi_{k,\iota_1\iota_2\iota_3\iota_4} := \frac{G_{k,\iota_1\iota_2\iota_3\iota_4}}{16|\iota_1\iota_2\iota_3\iota_4|}.$

Clearly,

$$\psi_{\iota_1\iota_2\iota_3\iota_4} = 0 \text{ unless } |\iota_1| \pm |\iota_2| \pm |\iota_3| \pm |\iota_4| = 0. \quad (3.12)$$

Assuming that

$$\mathcal{S} = \delta S = \delta \sum_{\iota_1,\iota_2,\iota_3,\iota_4} S_{\iota_1\iota_2\iota_3\iota_4} w_{\iota_1} w_{\iota_2} w_{\iota_3} w_{\iota_4} + \delta \sum_{|k|\geq 1} \sum_{\iota_1,\iota_2,\iota_3,\iota_4} S_{k,\iota_1\iota_2\iota_3\iota_4} e^{i<k,\vartheta>} w_{\iota_1} w_{\iota_2} w_{\iota_3} w_{\iota_4}$$

with coefficients

$$iS_{\iota_1\iota_2\iota_3\iota_4} = \begin{cases} \frac{\psi_{\iota_1\iota_2\iota_3\iota_4}}{\zeta_{\iota_1}{}' + \zeta_{\iota_2}{}' + \zeta_{\iota_3}{}' + \zeta_{\iota_4}{}'}, \text{ if } |\iota_1| \pm |\iota_2| \pm |\iota_3| \pm |\iota_4| = 0 \text{ and } |\iota_1|, |\iota_2|, |\iota_3|, |\iota_4| \in \bar{\bar{\Delta}}, \\ 0, otherwise. \end{cases}$$

and for $k \neq 0$,

$$iS_{k,\iota_1\iota_2\iota_3\iota_4} = \begin{cases} \frac{\psi_{k,\iota_1\iota_2\iota_3\iota_4}}{<k,\omega>}, |k| \geq 1, |\iota_1|, |\iota_2|, |\iota_3|, |\iota_4| \in \bar{\Delta}, \text{and} \\ \zeta_{\iota_1}{}' + \zeta_{\iota_2}{}' + \zeta_{\iota_3}{}' + \zeta_{\iota_4}{}' = 0, \\ \frac{\psi_{k,\iota_1\iota_2\iota_3\iota_4}}{\zeta_{\iota_1}{}' + \zeta_{\iota_2}{}' + \zeta_{\iota_3}{}' + \zeta_{\iota_4}{}' + <k,\omega>}, \ |k| \geq 1, \quad |\iota_1|, |\iota_2|, |\iota_3|, |\iota_4| \in \bar{\Delta}, \text{and} \\ \zeta_{\iota_1}{}' + \zeta_{\iota_2}{}' + \zeta_{\iota_3}{}' + \zeta_{\iota_4}{}' \neq 0, \\ 0, \text{otherwise}, \end{cases}$$

where $\zeta_{i'} = \mathrm{sgn} i \cdot |i|^2$.

For $S_{\iota_1\iota_2\iota_3\iota_4}$, the divisor $|\zeta_{\iota_1}{}' + \zeta_{\iota_2}{}' + \zeta_{\iota_3}{}' + \zeta_{\iota_4}{}'| \geq 1$ for all $(|i|, |j|, |d|, |l|) \in \bar{\bar{\Delta}}$. according to the definition of admissible set. Lemma 3.1 in Wang *et al.* (2018) shows that there exists a subset $\underline{\Omega} \subset [\eta, 2\eta]^2$ such that, for all $0 \neq k \in \mathbb{Z}^2$, any $\omega \in \underline{\Omega}$

satisfies $| < k,\omega > | \geq \frac{\eta\delta^{\frac{1}{3}}}{|k|^4}$ and Measure $\underline{\Omega} \geq (1 - C_1\delta^{\frac{1}{3}})\eta^2$, where the constant C_1 is an absolute constant. Considering Lemma 3.1, it is assumed that $\Omega = \bar{\Omega} \cap \underline{\Omega}$. Thus, Measure $\Omega \geq \eta^2(1 - C\delta^{\frac{1}{3}})$ is valid and Measure $\Omega > 0$ as δ sufficiently small. Using the same proof for Proposition 3.1 in Wang *et al.* (2018), it can be proved that the vector-field of X_S is real analytic in some complex neighbourhood $\vartheta \in D_1(\frac{\sigma_1}{2})$ of $\mathbb{T}^2$ and a neighbourhood of the origin in $l^{a,s}$. Moreover, $\left\|\frac{\partial S}{\partial w}\right\|_{a,s} \leq \frac{C}{\delta^{\frac{1}{3}}}(\|w\|_{a,s})^3$ is true. Let $\Upsilon = X_S^1$ be the time-one map of vector field, then Υ satisfies the result of this theorem. Other estimates can be found in Proposition 3.1 in Wang *et al.* (2018). Q.E.D.

4. Conclusions

Let $\varsigma_j \in [0,1]$ and $\mathbb{Z}_1 := \mathbb{Z}^+ \backslash \mathcal{J}$. Under the complex coordinates

$$\begin{cases} z_{n_1} = \sqrt{\varsigma_1 + I_1}e^{-i\theta_1}, \\ z_{n_2} = \sqrt{\varsigma_2 + I_2}e^{-i\theta_2}, \\ z_j = w_j, j \in \mathbb{Z}_1, \end{cases} \tag{4.1}$$

the Hamiltonian can be transformed into

$$H = \sum_{1\leq i\leq 2} \omega_i J_i + \sum_{1\leq j\leq 2} \varpi_j I_j + \sum_{l\in\mathbb{Z}_1} \widehat{\Omega}_l w_l \bar{w}_l + P. \tag{4.2}$$

The corresponding symplectic structure becomes $\sum_{1\leq i\leq 2} d\,\vartheta_i \wedge dJ_i + \sum_{1\leq j\leq 2} d\,\theta_j \wedge dI_j + i\sum_{l\in\mathbb{Z}_1} d\,w_l \wedge d\bar{w}_l$, where $P = \delta\breve{G} + \delta\hat{G} + \delta^2 K$, with $\breve{G} = \frac{1}{2}\sum_{1\leq i,j\leq 2} \bar{G}_{n_i n_j}(I_i + \varsigma_i)(I_j + \varsigma_j) + \sum_{l\in\mathbb{Z}_1,\, 1\leq l\leq 2} \bar{G}_{ln_j}(I_j + \varsigma_j)|w_l|^2$, $\varpi = \alpha + \delta\tilde{A}\varsigma$, $\hat{\Omega} = \zeta + \delta B\varsigma$, $\alpha = (\sqrt{\zeta}_{n_1}, \sqrt{\zeta}_{n_2})$, $\zeta = (\sqrt{\zeta_l})_{l\in\mathbb{Z}_1}$, and $\tilde{A} = \begin{pmatrix} \bar{G}_{n_1n_1} & \bar{G}_{n_1n_2} \\ \bar{G}_{n_2n_1} & \bar{G}_{n_2n_2} \end{pmatrix}$, $B = \begin{pmatrix} \bar{G}_{1n_1} & \bar{G}_{1n_2} \\ \bar{G}_{2n_1} & \bar{G}_{2n_2} \\ \vdots & \vdots \\ \bar{G}_{ln_1} & \bar{G}_{ln_2} \\ \vdots & \vdots \end{pmatrix}_{l\in\mathbb{Z}_1}$.

Assuming that $\omega = \omega_- + \delta^{\frac{3}{2}}\bar{\omega}, \bar{\omega} \in [0,1]^2$, where $\omega_- \in \Omega$ is fixed and $\omega \in \bar{\bar{\Omega}} := \{\omega \in \Omega \mid |\omega - \omega_-| \leq \delta^{\frac{3}{2}}\}$., it is clear that $\bar{\bar{\Omega}} \times [0,1]^2 \subset \Omega \times [0,1]^2$.

Then, the variables were scaled by setting $\bar{\omega} = \delta^{\frac{1}{2}}\widetilde{\bar{\omega}}$, $\varsigma = \delta^{\frac{1}{2}}\tilde{\varsigma}$, $w = \delta^{\frac{1}{4}}\tilde{w}$, $\bar{w} = \delta^{\frac{1}{4}}\widetilde{\bar{w}}$, $I = \delta^{\frac{1}{2}}\tilde{I}$, and $J = \delta^{\frac{1}{2}}\tilde{J}$. The scaled Hamiltonian can be expressed as:

$\tilde{H} = \delta^{-\frac{3}{2}} H(\vartheta, \delta^{\frac{1}{2}}\tilde{J}, \theta, \delta^{\frac{1}{2}}\tilde{I}, \delta^{\frac{1}{4}}\tilde{w}, \delta^{\frac{1}{4}}\tilde{\bar{w}}, \delta^{\frac{1}{4}}\tilde{\tilde{\omega}}, \delta^{\frac{1}{4}}\tilde{\varsigma}) = \sum_{1\le i\le 2} \tilde{\omega}_i \tilde{J}_i + \sum_{1\le l\le 2} \tilde{\varpi}_l \tilde{I}_l + \sum_{j\in\mathbb{Z}_1} \tilde{\hat{\Omega}}_j \tilde{w}_j \tilde{\bar{w}}_j + \tilde{P}$,

where $\tilde{\omega}_1(\tilde{\tilde{\omega}}) = \delta^{-1}\omega_{-1} + \delta\tilde{\tilde{\omega}}_1$, $\tilde{\omega}_2(\tilde{\tilde{\omega}}) = \delta^{-1}\omega_{-2} + \delta\tilde{\tilde{\omega}}_2$, $\tilde{\omega}(\tilde{\tilde{\omega}}) = (\tilde{\omega}_1(\tilde{\tilde{\omega}}), \tilde{\omega}_2(\tilde{\tilde{\omega}}))$, $\tilde{\varpi}(\varsigma) = \delta^{-1}\alpha + \delta^{\frac{1}{2}}\tilde{A}\tilde{\varsigma}$, and $\tilde{\hat{\Omega}}(\varsigma) = \delta^{-1}\zeta + \delta^{\frac{1}{2}}B\tilde{\varsigma}$. It is clear that $\omega = \delta\tilde{\omega}$, $\varpi = \delta\tilde{\varpi}$, and $\hat{\Omega} = \delta\tilde{\hat{\Omega}}$.

Then, some notations were introduced as $\hat{\omega}_1 = \tilde{\omega}_1(\tilde{\tilde{\omega}})$, $\hat{\omega}_2 = \tilde{\omega}_2(\tilde{\tilde{\omega}})$, $\tilde{\varpi}_1(\tilde{\varsigma})$, $\tilde{\varpi}_2(\tilde{\varsigma})$; $\hat{\omega} = (\tilde{\omega}(\tilde{\tilde{\omega}}), \tilde{\varpi}(\tilde{\varsigma}))$; $\xi = (\tilde{\tilde{\omega}}, \tilde{\varsigma}) \in \Pi$; $\Pi = [\beta, 2\beta]$; and $\beta = \delta^{-\frac{1}{3}}$. Let $\sigma_0 = \frac{\sigma_1}{2}$ and $r = r_0$, where $0 < r_0 < 1$ and r_0 is a constant. Let $\sigma = min\{\frac{\sigma_0}{2}, \frac{1}{2}\}$, $M_{11} = 1$, $M_{12} = \frac{21|\psi_0|}{32\pi}$, $M_2 = \frac{3|\psi_0|}{2\pi}$, $M = max\{M_{11} + M_{12}, M_2\}$, $\tau > 9$, $\mu = 2\tau + 3$, $\gamma_0 \le \frac{1}{c(M+1)2^{6+4\mu}}$ and $\gamma = \frac{\gamma_0^6}{20^{2(1+\mu)(cM)^6}}$. Note that M does not depend on δ.

For a function Q, the corresponding Hamiltonian vector field can be defined as

$X_Q = (Q_{\tilde{J}}, -Q_\vartheta, Q_{\tilde{I}}, -Q_\theta, iQ_{\tilde{\bar{w}}}, -iQ_{\tilde{w}})^T$,

and the weighted norm for X_Q can be defined as:

$|X_Q|^*_{r,\Pi} = \frac{1}{r^2}\|Q_\vartheta\|^*_\Pi + \frac{1}{r^2}\|Q_\theta\|^*_\Pi + \|Q_{\tilde{I}}\|^*_\Pi + \frac{1}{r}\|Q_{\tilde{w}}\|^*_{a,s,\Pi} + \frac{1}{r}\|Q_{\tilde{\bar{w}}}\|^*_{a,s,\Pi}$,

where

$$\|Q\|^*_\Pi = max\left\{\sup_{\xi\in\Pi}|Q|, \sup_{\xi\in\Pi}\left|\frac{\partial Q}{\partial \xi}\right|\right\}.$$

Taking $\delta \le (\frac{\gamma_0^6}{20^{20(1+\mu)(cM)^6}}\sigma^{2\mu+3})^2$ and using the same proof for Theorem 4.1 in Wang *et al.* (2018), it can be derived that there exists a set $\tilde{\Pi}_\delta \subset \Pi$, a family of torus embedding $\Phi: \mathbb{T}^4 \times \tilde{\Pi}_\delta \to \mathcal{P}^{a,s}$ and a map $\tilde{\tilde{\omega}}_0 = (\tilde{\omega}_0, \tilde{\varpi}_0): \tilde{\Pi}_\delta \to \mathbb{R}^4$. that satisfy the following conditions: for any $\xi \in \tilde{\Pi}_\delta$, the map Φ restricted to $\mathbb{T}^4 \times \{\xi\}$ is a real analytic embedding of a rotational torus, whose frequency vector is $\tilde{\tilde{\omega}}_0(\xi)$ for H at ξ; $\tilde{\Pi}_\delta$ is a Cantor set, and Φ, $\tilde{\tilde{\omega}}_0$ are Whitney smooth; every embedding is real analytic in $|\mathrm{Im}\vartheta| < \frac{\sigma}{2}$ and $|\mathrm{Im}\theta| < \frac{\sigma}{2}$; $|\Phi - \Phi_0|^*_r \le c\delta^{\frac{1}{4}}$ and $\|\tilde{\tilde{\omega}}_0 - \hat{\omega}\|^* \le c\delta^{\frac{1}{2}}$ hold uniformly, where Φ_0 is the trivial embedding.

Returning $\tilde{\Pi}_\delta$ to the subset in $\bar{\tilde{\Omega}} \times [0,1]^2$, namely Σ_δ, there exists a Cantor set $\Sigma_\delta \subset \Omega \times [0,1]^2$ satisfying the result in the following Theorem 4.1. Returning $\tilde{\tilde{\omega}}_0$ to the frequencies in the system with Hamiltonian (4.2), namely $\hat{\omega}_0$, the frequency vector can be estimated. Therefore, the nonlinear equation (1.1) admits a Cantor family of rotational tori, which are 4-dimensional. Their frequency vectors are $\hat{\omega}_0 = (\omega_{01}, \omega_{02}, \varpi_{0n_1}, \varpi_{0n_2})$, where $\omega_{01} = \omega_1 + \mathcal{O}(\delta)$, $\omega_{02} = \omega_2 + \mathcal{O}(\delta)$, $\varpi_{0n_1} = \zeta_{n_1} + \mathcal{O}(\delta)$, and $\varpi_{0n_2} = \zeta_{n_2} + \mathcal{O}(\delta)$. A big part of the family of tori persists under small perturbations and is linearly stable. The quasi-periodic solutions have small amplitudes and zero Lyapunov exponents. In short, the following theorem is valid.

Theorem 4.1 Assuming that (A1), (A2) and (A3) are valid, for every admissible index set $\mathscr{J} := \{(n_1, n_2) | n_1, n_2 \ are\ odd,\ n_1 \neq 7\, mod(14),\ n_2 > 6n_1^2\}$, there exits a δ^* which satisfies the following condition: for $0 < \delta < \delta^*$, there exist sets $\Omega \subset [\eta, 2\eta]^2$ and $\Sigma_\delta \subset \Sigma := \Omega \times [0,1]^2$ such that for any $\xi = (\omega_1, \omega_2, \varsigma_{n_1}, \varsigma_{n_2}) \in \Sigma_\delta$, Equation (1.1) under the conditions (1.2) has a quasi-periodic solution

$$u(t,x) = \frac{2}{n_1}\sqrt{\frac{\varsigma_{n_1} + \mathcal{O}(\delta^{\frac{1}{4}})}{\pi}}\, sin\, n_1\, x\, cos\, \varpi_{0n_1}(\xi)t + \frac{2}{n_2}\sqrt{\frac{\varsigma_{n_2} + \mathcal{O}(\delta^{\frac{1}{4}})}{\pi}}\, sin\, n_2\, x\, cos\, \varpi_{0n_2}(\xi)t + \mathcal{O}(\delta^{\frac{1}{4}}),$$

where $|\varpi_{0j} - j^2| \leq c\delta$, Ω and Σ_δ have positive Lebesgue measures.

Les différentes sections sont numérotées de l’introduction jusqu’à la conclusion. Les remerciements et la bibliographie (ainsi que l’extended abstract) ne sont pas numérotés. Les intertitres sont alignés à gauche sans alinéa, comme suit. Les espaces au-dessus s’annulent quand ils sont précédés par un autre inter.

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