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CUSUM tests for change points in AR(P*)* models

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ABSTRACT

This paper analyzes the problem of testing for change points in processes by CUSUM test using asymptotic theory. The limiting distribution of test statistic is derived and Monte Carlo simulation is provided. It shown that the empirical power not only depends on the sample size, but is sensitive to the magnitude of location of structural breaks.

Keywords: Change Points, CUSUM Test, Asymptotic Distribution, AR(P) Processes.

1. INTRODUCTION

Among various issues in the parametric shifts problem, estimation and test of mean shifts have no doubt been regarded as one of the core areas of research in econometrics since the underlying model of time series is often subject to change owing to governmental policy and critical social events: for the references, as to the change point test in time series models, see Picard [1], Incla´n and Tiao [2], Andrews and Ploberger [3], Jach and Kokoszka [4]. Hence, one would like to estimate the location of the change point when tests suggest that a parameter change has occurred. Detecting structural change beforehand is an important step, and it can make us interpret better and more accurately forecast the data. Moreover, it is well known that ignoring such a parameter change can lead to some false conclusion in statistical analysis. See the examples in Hamilton [5]. Therefore, for correct inference, it is imperative to figure out whether the parameters continue to be constant during the whole series or not. Parametric shifts estimation methods have been extensively studied by Bai and Perron [6], Nunes et al. [7], Su and Xiao [8] among others; see also Busetti and Taylor [9], and the references therein.

In this article, special attention is paid to the ergodic stationary processes including linear autoregressive (LAR) time series since they accommodate important linear time series models, such as *AR(P)*, which have been central to the analysis of data with linear characteristics (cf. [10,11,12]). For *AR(P)* models, there are *p+1* parameters, the variance of the white noise and the p autoregressive parameters. Change in any of these over time is a sign of disturbance that is important to detect. Gombay [13] used maximum likelihood function to test for changes in any one of these parameters separately. Lee and Park used the cusum of squares test to test for variance changes in infinite order moving average processes. Unfortunately, they did not consider testing for structural changes in collection of them. Hence, the goal of this paper is that we focus on the CUSUM testing for *p+1* parameter changes in *AR(P)* models.

These and other issues will be addressed in this paper whose structure is as follows. Section 2 first describes the models considered and the assumptions made on the various components. Section 3 contains the main results. Finally, Section 4 presents brief concluding remarks.

2. ASSUMPTIONS AND MODELS

We consider the following model

$$y_t = \mu + \xi_t$$
$$\xi_t = \alpha_1 \xi_{t-1} + \alpha_2 \xi_{t-2} + \cdots + \alpha_p \xi_{t-p} + \sigma_t \varepsilon_t$$

where ξ_t is an *AR(P)* process, p is a positive integer. ε_t are independent identical distribution random variables with $E\varepsilon_t = 0$ and $Var(\varepsilon_t) = 1$. The objective here is to test the following hypothesis:

$H_0 : \beta = (\alpha_1, \alpha_2, \cdots, \alpha_p, \sigma_t)$ remains the same for the whole series;

$$H_1 : Not\ H_0$$

The following assumptions are needed to prove asymptotic validity of our approach.

Assumption 2.1 $E|\varepsilon_t|^{4+\delta} < \infty$, for some $\delta > 0$.

Assumption 2.2 All of the roots of

$1-\alpha_1 z-\alpha_2 z^2-\cdots-\alpha_p z^p=0$ lie out of the unit circle.

Remark 2.3 For the problem of detecting structural changes, Assumption 2.1 may be quite standard in these econometric literature and states that the ε_t's moment of order 4 exists. The Assumption 2.2 can ensure $\xi_t=\sum_{j=0}^{\infty}\varphi_j\varepsilon_{t-j}$. Beveridge and Nelson [13] decomposed that $T^{-1}\sum_{t=1}^{T}\xi_t=T^{-1}\varphi(1)\sum_{t=1}^{T}\varepsilon_t+o_p(1)$, where $\varphi(1)=\sum_{j=0}^{\infty}\varphi_j<\infty$ This showsthat the rate of convergence for $T^{-1}\sum_{t=1}^{T}\xi_t$ and $T^{-1}\sum_{t=1}^{T}\varepsilon_t$ are the same.

The preparatory lemma collects fundamental result required to derive these asymptotic distributions as the sample size approaches to infinity.

Lemma 2.4 If Assumption 2.1 and 2.2 hold, then

$\sqrt{T}\left(\hat{\beta}-\beta\right)$ has a proper, nondegenerated limiting distribution,

where $\hat{\beta}$ is a ordinary least squares estimator.

Remark 2.5 Lemma 2.4 is proved by Kokoszka and Leipus [14] and suggests that $\hat{\mu}-\mu=O_p\left(T^{-1/2}\right)$ and $\hat{\alpha}_i-\alpha_i=O_p\left(T^{-1/2}\right), i=1,\cdots,p$.

3. MAIN RESULTS

In this section, we will derive the CUSUM test statistics under null hypothesis. For some convenience, we denote $\mu_t=\sigma\varepsilon_t$ under H_0.

Theorem 3.1 Consider following statistic test that

$$\Xi=T^{1/2}\frac{\hat{\lambda}}{\hat{\tau}}\max_{1\le k\le T}\left|\frac{\sum_{t=1}^{k}\hat{u}_t^2}{\sum_{t=1}^{T}\hat{u}_t^2}-\frac{k}{T}\right|$$

where $\hat{\lambda}=T^{-1}\sum_{t=1}^{T}\hat{u}_t^{2}$ and $\hat{\tau}_t^{2}=T^{-1}\sum_{t=1}^{T}\hat{u}_t^{4}-\hat{\lambda}^{2}$. $\hat{u}_t$ is the residual of y_t regressing on aconstant. If Assumption 2.1 and 2.2 hold, then under H_0, we have

$$\Xi\Rightarrow\sup_{v\in[0,1]}\left|BB(v)\right|$$

where $BB(v)$ is a standard Brownian bridge.

Proof Observe that , for $\upsilon=k/T$

$$\Xi_T=\frac{\hat{\lambda}}{\hat{\tau}}\max_{0\le v\le 1}\frac{\frac{1}{\sqrt{T}}\left|\sum_{t=1}^{[T_v]}\hat{u}_t^2-v\sum_{t=1}^{t}\hat{u}_t^2\right|}{\frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2}$$

$$=\max_{0\le v\le 1}\frac{1}{\sqrt{T}\hat{\tau}}\left|\sum_{t=1}^{[T_v]}\hat{u}_t^2-v\sum_{t=1}^{T}\hat{u}_t^2\right|$$

Split $\hat{u}_t^2$ into $u_t^2+\sum_{i=1}^{5}\omega_{i,T}$

$$\hat{\xi}_t=\xi_t+\left(\mu-\hat{\mu}\right)=\mu_t+\left(\mu-\hat{\mu}\right)+\alpha_1\xi_{t-1}+\cdots+\alpha_p\xi_{t-p}$$

$$\hat{u}_t^2=\left(\hat{\xi}_t-\hat{\alpha}_1\hat{\xi}_{t-1}-\cdots-\hat{\alpha}_p\hat{\xi}_{t-p}\right)^2$$

$$=u_t^2+\left(\mu-\hat{\mu}\right)^2+D_T^2+2u_t\left(\mu-\hat{\mu}\right)+2D_Tu_t+2D_T\left(\mu-\hat{\mu}\right)=u_t^2+\sum_{i=1}^{5}\omega_{i,T}$$

where $D_T=\sum_{l=1}^{p}\alpha_l\xi_{t-l}-\sum_{l=1}^{p}\hat{\alpha}_l\hat{\xi}_{t-l}$ and $\hat{\mu}=T^{-1}\sum_{t=1}^{T}y_t$

$$\Xi_{i,T}:=\frac{1}{\sqrt{T}}\max_{0\le v\le 1}\left|\sum_{t=1}^{[T_v]}\omega_{i,T}-v\sum_{t=1}^{[T_v]}\omega_{i,T}\right|=o_p(1)\qquad i=1,2,3,4,5.\quad (1)$$

Due to the invariant principle for strong mixing process, it follows that

$$\frac{1}{\sqrt{T}}\max_{0\le v\le 1}\left|\sum_{t=1}^{[Tv]}\xi_t u_t-v\sum_{t=1}^{T}\xi_t u_t\right|=O_p(1).\qquad (2)$$

Now we deal $\Xi_{1,T} and \Xi_{3,T}$ with By Lemma 2, 4, then

$$\hat{\xi}_t-\xi_t=\hat{\mu}-\mu=\frac{1}{T}\sum_{t=1}^{T}\xi_t=\varphi(1)\frac{1}{T}\sum_{t=1}^{T}u_t+o_p(1)=O_p(T^{-1/2})=o_p(1)$$

Which implies that

$$\frac{1}{\sqrt{T}}\max_{0\le v\le 1}\left|\sum_{t=1}^{[Tv]}\omega_{1,T}-v\sum_{t=1}^{T}\omega_{1,T}\right|\le 2\frac{1}{\sqrt{T}}\left|\sum_{t=1}^{T}(\mu-\hat{\mu})^2\right|=O_p(T^{-1/2})=o_p(1),\ (3)$$

and

$$\frac{1}{\sqrt{T}}\max_{0\le v\le 1}\left|\sum_{t=1}^{[Tv]}\omega_{3,T}-v\sum_{t=1}^{T}\omega_{3,T}\right|\le 2\frac{1}{\sqrt{T}}(\mu-\hat{\mu})\max_{0\le v\le 1}\left|\sum_{t=1}^{[Tv]}\varepsilon_t\right|=O_p(T^{-1/2})=o_p(1),\ (4)$$

Note that

$$D_T=\sum_{l=1}^{p}\alpha_l\xi_{t-l}-\sum_{l=1}^{p}\hat{\alpha}_l\hat{\xi}_{t-l}$$

$$=\sum_{l=1}^{p}\left(\alpha_l-\hat{\alpha}_l\right)\xi_{t-l}+\sum_{l=1}^{p}\hat{\alpha}_l\left(\xi_{t-l}-\hat{\xi}_{t-l}\right)=D_{1,T}-D_{2,T}$$

To show $\Xi_{4,T}=o_p(1)$, it is sufficed to show

$$\frac{1}{\sqrt{T}}\max_{0\le\upsilon\le 1}\left|\sum_{t=1}^{[T\upsilon]}u_tD_{i,T}-\upsilon\sum_{t=1}^{T}u_tD_{i,T}\right|=o_p(1),\quad i=1,2.$$

Together with Lemma 2.4 and (2), we get

$$\frac{1}{\sqrt{T}}\max_{0\le\upsilon\le 1}\left|\sum_{t=1}^{[T\upsilon]}u_tD_{1,T}-\upsilon\sum_{t=1}^{T}u_tD_{1,T}\right|$$

$$= \frac{1}{\sqrt{T}} \max_{0\le\upsilon\le1} \left| \sum_{l=1}^{p} (\alpha_l - \hat{\alpha}_l) \left(\sum_{t=1}^{[T\upsilon]} u_t \xi_{t-l} - \upsilon \sum_{t=1}^{T} u_t \xi_{t-l} \right) \right|$$

$$= O_p\left(T^{-1/2} \cdot p\right) = o_p(1) \quad (5)$$

and

$$\frac{1}{\sqrt{T}} \max_{0\le\upsilon\le1} \left| \sum_{t=1}^{[T\upsilon]} u_t W_{2,T} - \upsilon \sum_{t=1}^{T} u_t W_{2,T} \right|$$

$$= \frac{1}{\sqrt{T}} \max_{0\le\upsilon\le1} \left| \sum_{l=1}^{p} \hat{\alpha}_l \left(\sum_{t=1}^{[T\upsilon]} u_t \left(\xi_{t-l} - \hat{\xi}_{t-l}\right) - \upsilon \sum_{t=q+1}^{T} u_t \left(\xi_{t-l} - \hat{\xi}_{t-l}\right) \right) \right|$$

$$\le 2 \frac{1}{\sqrt{T}} \max_{0\le\upsilon\le1} \left| \sum_{l=1}^{p} \hat{\alpha}_l \left(\sum_{t=1}^{[T\upsilon]} u_t \left(\xi_{t-l} - \hat{\xi}_{t-l}\right) \right) \right|,$$

$$= O_p\left(T^{-1}\right) \max_{0\le\upsilon\le1} \left| \sum_{l=1}^{p} \alpha_l \left(\sum_{t=1}^{[T\upsilon]} u_t \right) \right| + o_p(1)$$

$$= O_p\left(T^{-1/2} \cdot p\right) = o_p(1) \quad (6)$$

Thus $\Xi_{4,T} = o_p(1)$ is proved. Furthermore, the proof of $\Xi_{5,T} = o_p(1)$ essential is the same as $R_{4,T} = o_p(1)$ and omitted for brevity.

Finally, we proved $\Xi_{2,T} = o_p(1)$. Due to $D_T = D_{1,T} + D_{2,T}$, it ensures that

$$\frac{1}{\sqrt{T}} \max_{0\le\upsilon\le1} \left| \sum_{t=1}^{[T\upsilon]} u_t D_{1,T}^2 - \upsilon \sum_{t=1}^{T} u_t D_{1,T}^2 \right|$$

$$= \frac{1}{\sqrt{T}} \max_{0\le\upsilon\le1} \left| \left(\sum_{l=1}^{p} \alpha_l - \hat{\alpha}_l \right)^2 \left(\sum_{t=1}^{[T\upsilon]} \xi_{t-l}^2 - \upsilon \sum_{l=1}^{T} \xi_{t-l}^2 \right) \right|$$

$$= O_p\left(T^{-1/2}\right) \cdot O_p\left(T^{-1} \cdot p^2\right) O_p\left(T^{1/2}\right) = o_p(1).$$

Similarly, we can get

$$\frac{1}{\sqrt{T}} \max_{0\le v\le1} \left| \sum_{t=q+1}^{[Tv]} W_{2,T}^2 - v \sum_{t=q+1}^{T} W_{2,T}^2 \right| = op(1)$$

Combining these results, we have

$$\frac{1}{\sqrt{T}} \max_{0\le v\le1} \left| \sum_{t=q+1}^{[Tv]} \hat{u}_t^2 - v \sum_{t=q+1}^{T} \hat{u}_t^2 \right| = \frac{1}{\sqrt{T}} \max_{0\le v\le1} \left| \sum_{t=q+1}^{[Tv]} u_t^2 - v \sum_{t=q+1}^{T} u_t^2 \right| + op(1) \quad (7)$$

And

$$\frac{1}{T} \sum_{t=1}^{T} \hat{u}_t^2 \xrightarrow{P} \frac{1}{T} \sum_{t=1}^{T} u_t^2 = Eu_1^2 = \sigma^2 \quad (8)$$

Finally, we need to show $T^{-1}\sum_{t=1}^{\infty} \hat{u}_t^4 \xrightarrow{P} T^{-1}\sum_{t=1}^{\infty} u_t^4$. According to the previous proof, it guarantees $\hat{u}_t^2 - u_t^2 = \sum_{i=1}^{5} \Xi_{i,T}$ and

$$\frac{1}{T-q} \sum_{i=q+1}^{T} \Xi_{i,T} = op(1) \quad \frac{1}{T-q} \sum_{i=q+1}^{T} \Xi_{i,T}^2 = op(1)$$

It is obvious that

$$\frac{1}{T} \sum_{t=1}^{T} (\hat{u}_t^2 - u_t^2)^2 = \frac{1}{T} \sum_{t=1}^{T} \left(\sum_{i=1}^{5} \Xi_{i,T}\right)^2 \le \frac{5}{T} \sum_{t=1}^{T} \sum_{i=1}^{5} \Xi_{i,T}^2 = op(1) \quad (9)$$

and

$$\frac{1}{T} \sum_{t=1}^{T} (\hat{u}_t^2 + u_t^2)^2 = \frac{1}{T} \sum_{t=1}^{T} (\hat{u}_t^2 - u_t^2)^2 + \frac{4}{T} \sum_{t=1}^{T} u_t^2 \hat{u}_t^2$$

$$\le \frac{2}{T} \sum_{t=1}^{T} (\hat{u}_t^2 - u_t^2)^2 + \frac{8}{T} \sum_{t=1}^{T} u_t^4 = Op(1) \quad (10)$$

Hence

$$\left| \frac{1}{T-q} \sum_{t=q+1}^{T} \hat{\varepsilon}_t^4 - \frac{1}{T-q} \sum_{t=q+1}^{T} \varepsilon_t^4 \right| \le \left(\frac{1}{T-q} \sum_{t=q+1}^{T} (\hat{\varepsilon}_t^2 - \varepsilon_t^2)^2\right)^{\frac{1}{2}} \left(\frac{1}{T-q} \sum_{t=q+1}^{T} (\hat{\varepsilon}_t^2 + \varepsilon_t^2)^2\right)^{\frac{1}{2}}$$

$$= op(1) \cdot Op(1) = op(1)$$

Which implies that

$$\frac{1}{T-q} \sum_{t=q+1}^{T} \hat{\varepsilon}_t^4 \xrightarrow{P} \frac{1}{T-q} \sum_{t=q+1}^{T} \varepsilon_t^4 = E\varepsilon_1^4 \quad (11)$$

In view of (8) and (11), then $\hat{\tau}^2 \xrightarrow{P} \tau^2$. Because of the central limiting theorem, we directly have $\frac{1}{\sqrt{T}\tau} \sum_{t=1}^{T} (u_t^2 - Eu_t^2) \xrightarrow{P} vW(1)$ and $\Xi_T \xrightarrow{d} \sup_{0\le v\le1} |W(v)|$. Hence, the proof is completed.

4. SIMULATIONS

Table 1. Empirical size and power, $\beta = (0.3, 0.3, 1)$

β^*	$T = 500$	$T = 800$	$T = 1000$
(0.3, 0.3, 1)	0.042	0.047	.0.051
(0.2, 0.3, 1)	0.381	0.653	0.838
(0.1, 0.3, 1)	0.425	0.756	0.921
(0.2, 0.3, 1)	0.452	0.756	0.921
(0.1, 0.3, 1)	0.478	0.793	0.946
(0.2, 0.3, 1)	0.499	0.838	0.954
(0.1, 0.3, 1)	0.523	0.874	0.963

In the section, we use Monte Carlo simulation methods to investigate finite sample size and power properties of the CUSUM test statistics. They ε are independent identically distribution standard normal random variables. The test requires experiments are programmed using 5000 replications.

All results refer to the test run at 0.05 nominal asymptotic level, for samples of size T = 500, 800, 1000. Now we consider the problem of test following hypothesis: the parameters change from $\beta = (\alpha_1, \alpha_2, \sigma^2)$ to $\beta^* = (\alpha_1^*, \alpha_2^*, \sigma^{*2})$ at k=0.5T.

Table 2. Empirical size and power, $\beta = (0.3, 0.3, 2)$

β^*	$T=500$	$T=800$	$T=1000$
(0.3, 0.3, 2)	0.048	0.053	0.050
(0.2, 0.3, 4)	0.552	0.731	0.921
(0.1, 0.3, 4)	0.597	0.778	0.965
(0.2, 0.2, 4)	0.629	0.820	0.994
(0.1, 0.2, 4)	0.648	0.877	0.997
(0.2, 0.1, 4)	0.683	0.912	1.000
(0.1, 0.1, 4)	0.715	0.943	1.000

We now discuss the main conclusions that can be drawn from our simulation. The results summarized in Tables show that the test produces good sizes and the powers increase as either the difference between β and β^* or T increase as might be anticipated. Tables 1-2 also indicate that test for changes in $\alpha_1, \alpha_2, \sigma^2$ simultaneously is generally more powerful test for changes only in α_1, σ^2 or α_2, σ^2. The factor may be that, the more change β is, the more probability a series will contain 'oscillations', and the easier the series contains a structural change. In a word, the simulation evidence is intensely in favors of

using CUSUM test to detect parameter change for *AR(p)* models.

5. CONCLUDING REMARKS

In this paper, the CUSUM test for variance changes in autoregressive processes including *AR(p)* is proposed. We have derived the asymptotical distribution of the RCUSQ test statistic is the function of a standard Brownian bridges. The results in simulation show that the new CUSUM test produce good sizes and powers. In concluding, the CUSUM test constitute a functional tool for testing for variance changes in autoregressive time series.

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