

$$f(z) = z - h \sinh\left(\frac{z}{h}\right) + \frac{4z^3}{h^2} \cosh\left(\frac{1}{2}\right), \quad g(z) = 1 - f'(z) \quad (6)$$

The non-zero linear strains derived from Eq. (5) are

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_b}{\partial x^2} - f(z) \frac{\partial^2 w_s}{\partial x^2} \\ \varepsilon_z &= g'(z) \varphi(x, t) \\ \gamma_{xz} &= g(z) \left(\frac{\partial \varphi}{\partial x} + \frac{\partial w_s}{\partial x} \right) \end{aligned} \quad (7)$$

By assuming that the material of FG beam obeys Hooke's law, the stresses in the beam become

$$\begin{aligned} \sigma_x &= Q_{11}(z) \varepsilon_x + Q_{13}(z) \varepsilon_z \\ \sigma_z &= Q_{13}(z) \varepsilon_x + Q_{33}(z) \varepsilon_z \\ \tau_{xz} &= Q_{55}(z) \gamma_{xz} \end{aligned} \quad (8)$$

where

$$Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{1-\nu^2}, \quad Q_{13}(z) = \nu Q_{11}(z), \quad \text{and} \quad Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \quad (9)$$

2.4 Equations of motion

In order to derive the equations of motion, Hamilton's principle is used

$$\int_0^T (\delta U + \delta V - \delta K) dt = 0 \quad (10)$$

where δU , δK and δV denote the strain energy, kinetic energy and the work done by external forces, respectively.

The variation of the strain energy can be stated as

$$\begin{aligned} \delta U &= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x \delta \varepsilon_x + \sigma_z \delta \varepsilon_z + \tau_{xz} \delta \gamma_{xz}) dz dx \\ &= \int_0^L \left(N_x \frac{\partial \delta u_0}{\partial x} + N_z \delta \phi - M_b \frac{\partial^2 \delta w_b}{\partial x^2} - M_s \frac{\partial^2 \delta w_s}{\partial x^2} + Q \left(\frac{\partial \delta \phi}{\partial x} + \frac{\partial \delta w_b}{\partial x} \right) \right) dx \end{aligned} \quad (11)$$

where the stress resultants N_x , M_b , M_s and Q are given by

$$\begin{aligned} (N_x, M_b, M_s) &= \int_{-h/2}^{h/2} (1, z, f(z)) \sigma_x dz, \\ N_z &= \int_{-h/2}^{h/2} g'(z) \sigma_z dz, \\ Q &= \int_{-h/2}^{h/2} g(z) \tau_{xz} dz, \end{aligned} \quad (12)$$

The variation of work done by externally transverse loads q can be expressed as

$$\delta V = - \int_0^L q \delta (w_b + w_s) dx \quad (13)$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned} \delta K &= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} [\dot{u} \delta \dot{u} + \dot{w} \delta \dot{w}] \rho(z) dz dx \\ &= \int_0^L \left\{ I_0 [\dot{u}_0 \delta \dot{u}_0 + (\dot{w}_b + \dot{w}_s) (\delta \dot{w}_b + \delta \dot{w}_s)] + J_0 (\dot{w}_b + \dot{w}_s) \delta \dot{\phi} \right. \\ &\quad + J_0 \dot{\phi} \delta (\dot{w}_b + \dot{w}_s) - I_1 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial x} \delta \dot{u}_0 \right) + I_2 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} \right) \\ &\quad + J_1 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \delta \dot{u}_0 \right) - J_2 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} \right) \\ &\quad \left. + K_2 \frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_s}{\partial x} + K_0 \dot{\phi} \delta \dot{\phi} \right\} dx \end{aligned} \quad (14)$$

Substituting the expressions for δU , δV , and δK from Eqs. (11), (13), and (14) into Eq. (10) and integrating by parts, and collecting the coefficients of δu_0 , δw_b , δw_s and $\delta \phi$. The following equations of motion of the FG beam are obtained

$$\begin{aligned} \delta u_0 : \frac{\partial N_x}{\partial x} &= I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_b}{\partial x} - J_1 \frac{\partial \ddot{w}_s}{\partial x} \\ \delta w_b : \frac{\partial^2 M_b}{\partial x^2} + q &= I_0 (\ddot{w}_b + \ddot{w}_s) + J_0 \ddot{\phi} + I_1 \frac{\partial \ddot{u}_0}{\partial x} - I_2 \frac{\partial^2 \ddot{w}_b}{\partial x^2} - J_2 \frac{\partial^2 \ddot{w}_s}{\partial x^2} \\ \delta w_s : \frac{\partial^2 M_s}{\partial x^2} + \frac{\partial Q}{\partial x} + q &= I_0 (\ddot{w}_b + \ddot{w}_s) + J_0 \ddot{\phi} + J_1 \frac{\partial \ddot{u}_0}{\partial x} - J_2 \frac{\partial^2 \ddot{w}_b}{\partial x^2} - K_2 \frac{\partial^2 \ddot{w}_s}{\partial x^2} \\ \delta \phi : \frac{\partial Q}{\partial x} - N_z &= J_0 (\ddot{w}_b + \ddot{w}_s) + K_0 \ddot{\phi} \end{aligned} \quad (15)$$

Equations (15) can be expressed in terms of displacements (u_0 , w_b , w_s and ϕ) by using Eqs. (5), (7), (8) and (12) as follows

$$\begin{aligned} \delta u_0 : A_{11} d_{11} u_0 - B_{11} d_{111} w_b - B_{11}^s d_{111} w_s + L d_1 \phi &= I_0 \ddot{u}_0 - I_1 d_1 \ddot{w}_b - J_1 d_1 \ddot{w}_s \\ \delta w_b : B_{11} d_{111} u_0 - D_{11} d_{1111} w_b - D_{11}^s d_{1111} w_s + L^a d_{11} \phi + q &= I_0 (\ddot{w}_b + \ddot{w}_s) + \\ J_0 \ddot{\phi} + I_1 d_{11} \ddot{u}_0 - I_2 d_{11} \ddot{w}_b - J_2 d_{11} \ddot{w}_s & \\ \delta w_s : B_{11}^s d_{111} u_0 - D_{11}^s d_{1111} w_b - H_{11}^s d_{1111} w_s + A_{55}^s d_{11} w_s + (R + A_{55}^s) d_{11} \phi + q &= \\ I_0 (\ddot{w}_b + \ddot{w}_s) + J_0 \ddot{\phi} + J_1 d_{11} \ddot{u}_0 - J_2 d_{11} \ddot{w}_b - K_2 d_{11} \ddot{w}_s & \\ \delta \phi : L d_1 u_0 - L^a d_{11} w_b + (R + A_{55}^s) d_{11} w_s - A_{55}^s d_{11} \phi + R^a \phi &= J_0 (\ddot{w}_b + \ddot{w}_s) + K_0 \ddot{\phi} \end{aligned} \quad (16)$$

where

$$d_i = \frac{\partial}{\partial x_i}, \quad (i = 1)$$

where A_{11} , B_{11} , etc., are the beam stiffness, defined by

$$(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s) = \int_{-h/2}^{h/2} Q_{11}(1, z, z^2, f, zf, f^2) dz$$

$$\begin{aligned}
(L, L^a, R) &= \int_{-h/2}^{h/2} Q_{13}(1, z, f) g' dz, \\
R^a &= \int_{-h/2}^{h/2} Q_{33}(g'(z))^2 dz, \\
A_{55}^s &= \int_{-h/2}^{h/2} Q_{55}(g(z))^2 dz
\end{aligned} \tag{17}$$

3. ANALYTICAL SOLUTIONS

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0 , w_b , w_s and ϕ can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \phi \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i\omega t} \\ W_{bm} \sin(\lambda x) e^{i\omega t} \\ W_{sm} \sin(\lambda x) e^{i\omega t} \\ \Phi_m \sin(\lambda x) e^{i\omega t} \end{Bmatrix} \tag{18}$$

where U_m , W_{bm} , W_{sm} and Φ_m are arbitrary parameters to be determined, ω is the Eigen frequency associated with nth Eigenmode, and $\lambda = m\pi / L$.

For the beam with two ends simply supported, the boundary conditions are given by

$$u = w = M_b = 0 \quad \text{at} \quad x = 0, L \tag{19}$$

Substituting the expansions of u_0 , w_b , w_s and ϕ from Eqs. (18) into the equations of motion Eq. (16). The analytical solutions can be obtained from the following equations

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{pmatrix} - \omega^2 \begin{pmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ 0 & m_{24} & m_{34} & m_{44} \end{pmatrix} \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \Phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

where

$$\begin{aligned}
s_{11} &= \lambda^2 A_{11}, & s_{12} &= -\lambda^3 B_{11}, & s_{23} &= \lambda^4 D_{11}^s, \\
s_{22} &= \lambda^4 D_{11}, & s_{13} &= -\lambda^3 B_{11}^s, & s_{24} &= \lambda^2 L^a, \\
s_{33} &= \lambda^4 H_{11}^s + \lambda^2 A_{55}^s, & s_{14} &= -\lambda L, & s_{34} &= \lambda^2 (A_{55}^s + R), \\
s_{44} &= \lambda^2 A_{55}^s + R^a
\end{aligned} \tag{21}$$

$$\begin{aligned}
m_{11} &= I_0, & m_{12} &= -\lambda I_1, & m_{13} &= -\lambda J_1, \\
m_{22} &= I_0 + \lambda^2 I_2, & m_{23} &= I_0 + \lambda^2 J_2, & m_{24} &= m_{34} = J_0, \\
m_{33} &= I_0 + \lambda^2 K_2, & m_{44} &= K_0
\end{aligned}$$

4. NUMERICAL RESULTS AND DISCUSSION

In the following computation, the material constituents of the FGM beam are considered to be composed by alumina and aluminum with the material properties as follows- Alumina (Al_2O_3): $E_c = 380$ GPa, $\rho_c = 3960$ kg/m³, $\nu = 0.3$ - Aluminum (Al): $E_m = 70$ GPa, $\rho_m = 2702$ kg/m³, $\nu = 0.3$

The dimensionless frequency is defined as

$$\bar{\omega} = (\omega L^2 / h) \sqrt{\rho_m / E_m}$$

To check the accuracy of the method used in this investigation, the non-dimensional frequencies $\bar{\omega}$ computed by the present theory are compared with those of Ould Larbi et al. [13] and classical beam theory (CBT).

Table 1. First three non-dimensional frequencies $\bar{\omega}$ of FG beams

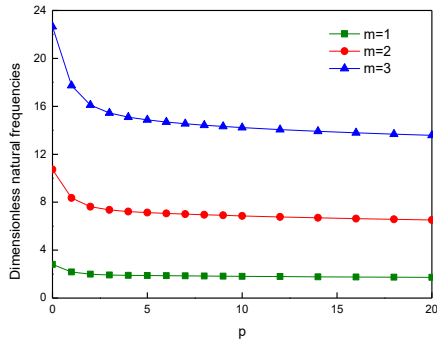
L/h	mode	theory	p					
			0	0.5	1	2	5	10
5	1	CBT	5.395	4.593	4.148	3.779	3.595	3.492
		HSDT [12]	5.153	4.411	3.990	3.626	3.400	3.281
		Present	5.166	4.434	4.026	3.670	3.437	3.304
	2	CBT	20.619	17.541	15.798	14.326	13.588	13.238
		HSDT [12]	17.884	15.461	14.012	12.640	11.535	11.022
		Present	17.998	15.587	14.168	12.813	11.678	11.117
	3	CBT	43.348	36.831	33.028	29.746	28.085	27.475
		HSDT [12]	34.225	29.849	27.108	24.319	21.699	20.555
		Present	34.556	30.168	27.459	24.676	21.985	20.753
20	1	CBT	5.478	4.664	4.216	3.847	3.663	3.554
		HSDT [12]	5.460	4.651	4.205	3.836	3.648	3.539
		Present	5.466	4.669	4.238	3.879	3.685	3.560
	2	CBT	21.844	18.598	16.810	15.333	14.596	14.168
		HSDT [12]	21.573	18.396	16.634	15.162	14.373	13.926
		Present	21.603	18.475	16.769	15.336	14.519	14.013
	3	CBT	48.899	41.633	37.617	34.295	32.636	31.688
		HSDT [12]	47.594	40.653	36.769	33.468	31.572	30.534
		Present	47.686	40.845	37.079	33.863	31.903	30.737

Table 1 presents the first three non-dimensional frequencies of FG beams for different values of power law index k and span-to-depth ratio L/h . Results are in good agreements with the published results of Ould Larbi et al.

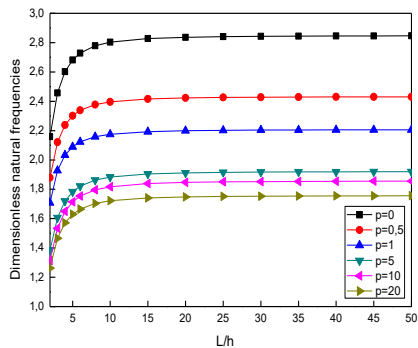
[13]. The small difference observed between the results obtained by the present theory and Ould Larbi et al. [13] is due to the effect of thickness stretching which is omitted this latter. It can be observed also that there is a remarkable

difference between the frequencies of CBT and those of shear deformable beam theories for thicker FG beam.

The variation of natural frequencies in terms of the power-law index and side-to-thickness ratio is plotted in figure 2. It can be seen from this figure that the natural frequencies decrease with the increase of the power-law index. It is due to the fact that a higher value of p corresponds to lower value of volume fraction of the ceramic phase, and thus makes the plates become the softer ones.



(a)



(b)

Figure 2. Effect of the power-law index p and side-to-thickness ratio L/h on the natural frequency ω of Al/Al₂O₃ beam

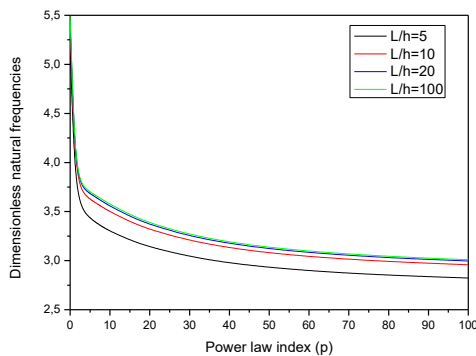


Figure 3. Variation of the non-dimensional fundamental frequency of FG beam with power-law index p and side-to-thickness ratio L/h .

Figure 3 shows the non-dimensional fundamental natural frequency versus the volume fraction exponent p for different values of span-to-depth ratio L/h . It can be seen from this figure that the full ceramic beams ($p=0$) lead to a highest frequency. However, the lowest frequency values are

predicted for the full metal ($p \rightarrow \infty$). This is due to the fact that an increase in the value of the volume fraction exponent results in a decrease of the value of the elasticity modulus. In other words, the beam becomes flexible as the power law exponent increases. Therefore, as also known from mechanical vibrations, natural frequencies decrease as the stiffness of a structure decreases.

5. CONCLUSION

The free vibration response of FGM beams is studied based on the higher-order shear deformation theories. This method considers both the shear deformation and thickness stretching effects by a hyperbolic distribution of all displacements through the thickness and without introducing a shear correction factor. The equations of motion for the functionally graded beams are derived from Hamilton's principle. The dimensionless frequencies are presented for the FGM beams with the material properties varying continuously in the thickness direction according to power-law form. The effects of the slenderness ratio, the material gradient parameters on the frequencies are examined in detail. The results are validated by comparing them with the results of other researcher. The numerical results show that the effects of the shear deformation on the frequencies tend to be more significant when the beams become shorter (or thicker). Furthermore, for a given length-height ratio, the shear deformation effects are more evident for higher-mode frequencies than for lower-mode frequencies.

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NOMENCLATURE

b	Width, mm
C	Stiffness coefficient ,
E	Young's modulus, N.m-2
h	Thickness, mm
I	inertia term
K	kinetic energy, J
L	Length, mm
P	Material property
q	external forces, kN
u	Axial displacement, mm
U	strain energy, J
V	work done, J
w	Transverse displacement, mm

Greek symbols

ε	Normal strain
γ	Shear strain, rad
σ	Normal stress, N.m-2
τ	Shear stress, N.m-2
ω	Frequencie Hz
ρ	Mass density kg, m-3
ν	Coefficient de Poisson
∂	differentiation
δ	variation
∇	Laplacian operator

Superscripts

p	Power law index
i	fluid (pure water)
t	temps
.	differentiation with respect to the time variable

Subscripts

b	Bending
c	ceramic
n	mode
m	metal
s	Shear