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Some Aspects of Picture Fuzzy Set

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Abstract

In this paper, decomposition theorems of picture fuzzy set (PFS) have been proved. (α,δ,β)-cut of picture fuzzy set have been defined and some of its properties are studied. Later on extension principle for PFS has been defined and studied some of its properties. Finally, picture fuzzy arithmetic based on extension principle has been performed.

Keywords

Fuzzy set, Fuzzy arithmetic, Extension principle, Picture fuzzy set

1. Introduction

Fuzzy set theory developed by Zadeh (1965), plays an important role in decision making under uncertain environment. Various direct/indirect extensions of fuzzy set have been made and successfully applied in most of the problems of real world situation. An important generalization of fuzzy set theory is the theory of intuitionistic fuzzy set (IFS), introduced by Atanassov (1986) ascribing a membership degree and a non-membership degree separately in such a way that sum of the two degrees must not exceed one. It is observed that fuzzy sets are IFSs but converse is not necessarily correct. Later IFS has been applied in different areas by various researchers. It is seen that one of the important concept of neutrality degree is lacking in IFS theory. Concept of neutrality degree can be seen in situations when we face human opinions involving more answers of type: yes, abstain, no, refusal. For example, in a democratic election station, the council issues 500 voting papers for a candidate. The voting results are divided into four groups accompanied with the number of papers namely “vote for” (300), “abstain” (64), “vote against” (115) and "refusal of voting" (21). Group "abstain" means that the voting paper is a whitepaper rejecting both "agree" and "disagree" for the candidate but still takes the vote. Group "refusal of voting" is either invalid voting papers or bypassing the vote. On the other hand, in medical diagnosis degree of neutrality can be considered. E.g., there may not have effect of the symptoms temperature, headache on the diseases stomach and chest problems. Similarly, the symptoms stomach pain and chest pain have neutral effect on the diseases viral fever, malaria, typhoid etc. In this regards, Cuong and Kreinovich (2013) introduced Picture fuzzy set (PFS) which is a direct extension of fuzzy set and Intuitionistic fuzzy set by incorporating the concept of positive, negative and neutral membership degree of an element. Cuong (2014) studied some properties of PFSs and suggested distance measures between PFSs. Cuong and Hai (2015) investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets and also constructed main operations for fuzzy inference processes in picture fuzzy systems. Cuong et al., (2015) presented properties of an involutive picture negator and some corresponding De Morgan fuzzy triples on picture fuzzy sets, Cuong et al., (2016) investigated the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets. Phong et al., (2014) studied some compositions of picture fuzzy relations, Son (2016) proposed a new distance measure between PFSs and applied in fuzzy clustering.

In this paper, an attempt has been made to study decomposition theorem of picture fuzzy set (PFS, (α,δ,β)-cut of picture fuzzy set and some of its properties, extension principle for PFS and some of its properties and finally, picture fuzzy arithmetic based on extension principle.

2. Preliminaries

In this section some basic concept of fuzzy set, intuitionistic fuzzy set and picture fuzzy set has been reviewed.

2.1 Fuzzy Set (Zadeh, 1965)

Fuzzy set is a set in which every element has degree of membership of belonging in it. Mathematically, let X be a universal set. Then the fuzzy subset A of X is defined by its membership function

$$\mu_A : X \to [0,1]$$

Which assign a real number $\mu_A(x)$ in the interval [0, 1], to each element $x \in A$, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A.

2.2 Intuitionistic Fuzzy Set (Atanassov, 1986)

A Intuitionistic fuzzy set *A* on a universe of discourse *X* is of the form

$A = \{(x, \mu_A(x), \nu_A(x) : x \in X\},$

Where $\mu_A(x) \in [0,1]$ is called the "degree of membership of x in A", $\nu_A(x) \in [0,1]$ is called the "degree of non-membership of x in A", and where $\mu_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$0 \le \mu_A(x) + \nu_A(x) \le 1.$

The amount $\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$ is called hesitancy of x which is reflection of lack of commitment or uncertainty associated with the membership or non-membership or both in A.

2.3 Picture Fuzzy Set (Cuong and Kreinovich, 2013)

A Picture Fuzzy Set (PFS) *A* on a universe *X* is an object of the form

$A = \left\{\left(x, \mu_A(x), \eta_A(x), \nu_A(x)\right) \mid x \in X\right\}$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of x in *A*, $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A, $\nu_A(x) \in [0,1]$ is called the degree of negative membership of x in A.

$\mu_A(x), \eta_A(x), \nu_A(x)$ must satisfy the condition $\mu_A(x) + \eta_A(x) + \nu_A(x) \le 1 \quad \forall \quad x \in X$.

Then $\forall \ x \in X$, $1 - \left(\mu_A(x) + \eta_A(x) + \nu_A(x)\right)$ is called the degree of refusal membership of "x" in **A**.

2.4 (α,δ,β)-Cut of picture fuzzy set

Let A be a picture fuzzy set of a universe set *X*. Then (α,δ,β)-cut of A is a crisp subset $C_{\alpha,\delta,\beta}(A)$ of the IFS A is given by

$C_{\alpha,\delta,\beta}(A) = \{x: x \in X \text{ such that } \mu_A(x) \ge \alpha,\ \eta_A(x) \le \delta,\ \nu_A(x) \le \beta\}$,

where $\alpha,\delta,\beta \in [0,1]$ with $\alpha+\delta+\beta \le 1$.

That is, ${}^{\alpha}A_{+} = \left\{x \in X : \mu_A(x) \ge \alpha\right\}$, ${}^{\delta}A_{\pm} = \left\{x \in X : \eta_A(x) \le \delta\right\}$ and ${}^{\beta}A_{-} = \left\{x \in X : \nu_A(x) \le \beta\right\}$ are $\alpha-$cut of MF, NeuMF & NMF of PFS A.

2.5 Strong (α,δ,β)-Cut of picture fuzzy set

Let A be a picture fuzzy set of a universe set X. Then, strong (α,δ,β)-cut of A is a crisp subset $C_{\alpha,\delta,\beta}(A)$ of the IFS A is given by

$C_{\alpha,\delta,\beta}(A) = \{x: x \in X \text{ such that } \mu_A(x) > \alpha,\ \eta_A(x) < \delta,\ \nu_A(x) < \beta\}$,

where $\alpha,\delta,\beta \in [0,1]$ with $\alpha+\delta+\beta \leq 1$.

That is, ${}^{\alpha+}A_{+} = \{x \in X : \mu_A(x) > \alpha\}$, ${}^{\delta+}A_{\pm} = \{x \in X : \eta_A(x) < \delta\}$ and ${}^{\beta+}A_{-} = \{x \in X : \nu_A(x) < \beta\}$ are strong $\alpha-$cut of MF, NeuMF & NMF of PFS A.

3. Proposition (Cuong and Kreinovich, 2013)

If $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ and $B = \{(x, \mu_B(x), \eta_B(x), \nu_B(x)) \mid x \in X\}$ be any two PFS of a set X then

(1) $A \subseteq B$ iff $\forall\ x \in X$, $\mu_A(x) \leq \mu_B(x), \eta_A(x) \geq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

(2) $A = B$ iff $\forall\ x \in X$, $\mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$.

(3) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x) = \nu_B(x))) \mid x \in X\}$

(4) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x) = \nu_B(x))) \mid x \in X\}$.

4. Decomposition theorem for PFS

In this section, decomposition theorems for PFS have been discussed.

Theorem 4.1 (First Decomposition Theorem)

Let X be a universe of discourse. For any GTIFN $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ in X,

$$A = \bigcup_{\substack{\alpha \in [0,w_1],\\ \delta \in [w_2,1],\\ \gamma \in [w_3,1]}} (\alpha,\delta,\gamma) C_{(\alpha,\delta,\gamma)}(A)$$

$$= \begin{cases} \bigcup\limits_{\alpha \in [0,w_1]} \alpha\, A_{+} & \text{for MF} \\ \bigcap\limits_{\alpha \in [w_2,1]} \delta\, A_{\pm} & \text{for NeuMF} \\ \bigcap\limits_{\alpha \in [w_3,1]} \gamma\, A_{-} & \text{for NMF} \end{cases}$$

where $A_+, A_\pm$ & A_- indicates membership function (MF), Neutral membership function (NeuF) & non-membership function (NF) respectively; ${}_\alpha A_+ = \alpha.{}^\alpha A_+$, ${}_\delta A_\pm = \delta.{}^\delta A_\pm$ and ${}_\gamma A_- = \gamma.{}^\gamma A_-$ are special picture fuzzy sets; $\cup$ and $\cap$ are standard fuzzy union and intersection respectively.

Proof: For MF, let for each $x \in X$, $\mu_A(x) = a$ where $a \in [0, w_1]$ which indicates degree of belonging in A.

Then,

$$\begin{aligned} \bigcup_{\alpha\in[0,w_1]} \alpha A_+ &= \operatorname*{Sup}_{\alpha\in[0,w_1]} \alpha\, {}^\alpha \mathrm{A}_+ \\ &= \max[\operatorname*{Sup}_{\alpha\in[0,a]} \alpha\, {}^\alpha A_+, \operatorname*{Sup}_{\alpha\in(a,w_1]} \alpha\, {}^\alpha A_+] \end{aligned} \tag{4.1}$$

If $\alpha \in [0,a]$ then $\alpha \le a = \mu_A(x)$

i.e., $\alpha \in {}^\alpha A_+$ then $\alpha\, {}^\alpha A_+ = \alpha.$

If $\alpha \in (a, w_1]$ then $\alpha > a = \mu_A(x)$

i.e., $\alpha \notin {}^\alpha A_+$ then $\alpha\, {}^\alpha A_+ = 0.$

Hence from (4.1), we have

$$\begin{aligned} \bigcup_{\alpha\in[0,w_1]} \alpha A_+ &= \max[\operatorname*{Sup}_{\alpha\in[0,a]} \alpha, 0] \\ &= a \\ &= \mu_A(x) \end{aligned}$$

For NeuMF, let for each $x \in X$, $\eta_A(x) = b$ where $b \in [w_2, 1]$ which indicates degree of non-belonging in A.

$$\begin{aligned} \bigcap_{\delta\in[w_2,1]} \delta \mathrm{A}_\pm &= \operatorname*{Inf}_{\delta\in[w_2,1]} \delta\, {}^\delta A_\pm \\ &= \min[\operatorname*{Inf}_{\delta\in[w_2,b)} \delta\, {}^\delta A_\pm, \operatorname*{Inf}_{\delta\in[b,1]} \delta\, {}^\delta A_\pm] \end{aligned} \tag{4.2}$$

If $\delta \in [w_2, b)$ then $\delta < b = \eta_A(x)$.

i.e., $\delta \notin {}^\delta A_\pm$ then $\delta\, {}^\delta A_\pm = 1.$

If $\delta \in (b, 1]$ then $\delta \ge b = \eta_A(x)$

i.e., $\delta \in {}^\delta A_\pm$ then $\delta\, {}^\delta A_\pm = \delta.$

Hence from (4.2), we have

$$\bigcap_{\delta\in[w_2,1]} \delta\,{}^{\delta}A_{\pm} = \min[1, \underset{\delta\in[b,1]}{Inf}\,\delta\,]$$

$$= b$$

$$= \eta_A(x)$$

For NMF, let for each $x \in X$, $\nu_A(x) = b$ where $c \in [w_3, 1]$ which indicates degree of non-belonging in *A*.

$$\bigcap_{\gamma\in[w_3,1]} \gamma A\_ = \underset{\gamma\in[w_3,1]}{Inf}\ \gamma\,{}^{\gamma}A\_ \\ = \min[\underset{\gamma\in[w_3,c)}{Inf}\ \gamma\,{}^{\gamma}A\_,\ \underset{\gamma\in[c,1]}{Inf}\ \gamma\,{}^{\gamma}A\_\,] \tag{4.3}$$

If $\gamma \in [w_3, c)$ then $\gamma < c = \nu_A(x)$

i.e., $\gamma \notin {}^{\gamma}A\_$ then $\gamma\,{}^{\gamma}A\_ = 1$.

If $\gamma \in [c,1]$ then $\alpha \geq c = \nu_A(x)$

i.e., $\gamma \in {}^{\gamma}A\_$ then $\gamma\,{}^{\gamma}A\_ = \gamma$.

Hence from (4.3), we have

$$\bigcap_{\gamma\in[w_3,1]} \gamma\,{}^{\gamma}A\_ = \min[1, \underset{\gamma\in[c,1]}{Inf}\,\gamma\,]$$

$$= c$$

$$= \nu_A(x)$$

Theorem 4.2 (Second Decomposition Theorem)

Let *X* be a universe of discourse. For any GTIFN $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ in *X*,

$$A = \bigcup_{\substack{\alpha\in[0,w_1],\\ \delta\in[w_2,1],\\ \gamma\in[w_3,1]}} (\alpha,\delta,\gamma) + C_{(\alpha,\delta,\gamma)}(A)$$

$$= \begin{cases} \bigcup_{\alpha\in[0,w_1]} \alpha + {}^{A}+ & \text{for MF} \\ \bigcap_{\alpha\in[w_2,1]} \delta + {}^{A}\pm & \text{for NeuMF} \\ \bigcap_{\alpha\in[w_3,1]} \gamma + {}^{A}- & \text{for NMF} \end{cases}$$

where $A_+, A_\pm$ & A_- indicates membership function (MF), Neutral membership function (NeuF) & non-membership function (NF) respectively; ${}_{\alpha+}A_+ = \alpha.{}^{\alpha+}A_+$, ${}_{\delta+}A_\pm = \delta.{}^{\delta+}A_\pm$ and ${}_{\gamma+}A_- = \gamma.{}^{\gamma+}A_-$ are special picture fuzzy sets; $\cup$ and $\cap$ are standard fuzzy union and intersection respectively.

Proof:

For MF, let for each $x \in X$, $\mu_A(x) = a$ where $a \in [0, w_1]$ which indicates degree of belonging in A.

Then,

$$\begin{aligned} \bigcup_{\alpha \in [0,w_1]} \alpha A_+ &= \sup_{\alpha \in [0,w_1]} \alpha^{\alpha+} A_+ \\ &= \max[\sup_{\alpha \in [0,a]} \alpha^{\alpha+} A_+, \sup_{\alpha \in (a,w_1]} \alpha^{\alpha+} A_+] \end{aligned} \tag{4.4}$$

If $\alpha \in [0, a)$ then $\alpha < a = \mu_A(x)$

i.e., $\alpha \in {}^{\alpha+}A_+$ then $\alpha^{\alpha+}A_+ = \alpha.$

If $\alpha \in [a, w_1]$ then $\alpha \geq a = \mu_A(x)$

i.e., $\alpha \notin {}^{\alpha+}A_+$ then $\alpha^{\alpha+}A_+ = 0.$

Hence from (4.4), we have

$$\begin{aligned} \bigcup_{\alpha \in [0,w_1]} \alpha + A_+ &= \max[\sup_{\alpha \in [0,a)} \alpha, 0] \\ &= a \\ &= \mu_A(x) \end{aligned}$$

For NeuMF, let for each $x \in X$, $\eta_A(x) = b$ where $b \in [w_2, 1]$ which indicates degree of non-belonging in A.

$$\begin{aligned} \bigcap_{\delta \in [w_2,1]} \delta + A_\pm &= \inf_{\delta \in [w_2,1]} \delta^{\delta+} A_\pm \\ &= \min[\inf_{\delta \in [w_2,b)} \delta^{\delta+} A_\pm, \inf_{\delta \in [b,1]} \delta^{\delta+} A_\pm] \end{aligned} \tag{4.5}$$

If $\delta \in [w_2, b]$ then $\delta \leq b = \eta_A(x)$.

i.e., $\delta \notin {}^{\delta+}A_\pm$ then $\delta^{\delta+}A_\pm = 1.$

If $\delta \in (b, 1]$ then $\delta > b = \eta_A(x)$

i.e., $\delta \in {}^{\delta+}A_\pm$ then $\delta^{\delta+}A_\pm = \delta.$

Hence from (4.5), we have

$$\bigcap_{\delta\in[w_2,1]} \delta\,{}^{\delta}A_{\pm} = \min[1, \underset{\delta\in[b,1]}{Inf}\,\delta\ ]$$

$$= b$$

$$= \eta_A(x)$$

For NMF, let for each $x \in X$, $\nu_A(x) = b$ where $c \in [w_3, 1]$ which indicates degree of non-belonging in *A*.

$$\begin{aligned}\bigcap_{\gamma\in[w_3,1]} \gamma + \mathrm{A}_{-} &= \underset{\gamma\in[w_3,1]}{Inf}\ \gamma\,{}^{\gamma+}A_{-} \\ &= \min[\ \underset{\gamma\in[w_3,c)}{Inf}\ \gamma\,{}^{\gamma+}A_{-},\ \underset{\gamma\in[c,1]}{Inf}\ \gamma\,{}^{\gamma+}A_{-}\ ]\end{aligned} \tag{4.6}$$

If $\gamma \in [w_3, c]$ then $\gamma \le c = \nu_A(x)$

i.e., $\gamma \notin {}^{\gamma+}A_{-}$ then $\gamma\,{}^{\gamma+}A_{-} = 1.$

If $\gamma \in (c,1]$ then $\alpha > c = \nu_A(x)$

i.e., $\gamma \in {}^{\gamma+}A_{-}$ then $\gamma\,{}^{\gamma+}A_{-} = \gamma.$

Hence from (4.6), we have

$$\bigcap_{\gamma\in[w_3,1]} \gamma\,{}^{\gamma+}A_{-} = \min[1, \underset{\gamma\in[c,1]}{Inf}\,\gamma\ ]$$

$$= c$$

$$= \nu_A(x)$$

Theorem 4.3 (Third Decomposition Theorem)

Let *X* be a universe of discourse. For any GTIFN $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \mid x \in X\}$ in *X*,

$$A = \bigcup_{\substack{\alpha\in\Lambda(A),\\ \delta\in\Lambda(A),\\ \gamma\in\Lambda(A)}} (\alpha,\delta,\gamma)C_{(\alpha,\delta,\gamma)}(A)$$

$$= \begin{cases} \bigcup_{\alpha\in\Lambda(A)} \alpha\,{}^{A}{}_{+} & \text{for MF} \\ \bigcap_{\alpha\in\Lambda(A)} \delta\,{}^{A}{}_{\pm} & \text{for NeuMF} \\ \bigcap_{\alpha\in\Lambda(A)} \gamma\,{}^{A}{}_{-} & \text{for NMF} \end{cases}$$

where $A_+, A_\pm$ & A_- indicates membership function (MF), Neutral membership function (NeuF) & non-membership function (NF) respectively; ${}_\alpha A_+ = \alpha.{}^\alpha A_+$, ${}_\delta A_\pm = \delta.{}^\delta A_\pm$ and ${}_\gamma A_- = \gamma.{}^\gamma A_-$ are special picture fuzzy sets; $\cup$ and $\cap$ are standard fuzzy union and intersection respectively and $\Lambda(A)$ is the level set of *A*.

Proof is straight forward as above.

5. Properties of Picture Fuzzy Set

In this section, some properties of (α,δ,β)-Cut of picture fuzzy set (PFS).

Theorem 5.1 If *A* and *B* be two PFS's of a universe set *X*, then the following holds

I) $C_{\alpha,\delta,\beta}(A) \subseteq C_{r,\theta,\phi}(A)$ if $\alpha \geq r$, $\delta \leq \theta$, $\beta \leq \phi$

II) $C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$

III) $A \subseteq B$ implies $C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(B)$

IV) $C_{\alpha,\delta,\beta}(A \cap B) = C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$

V) $C_{\alpha,\delta,\beta}(A \cup B) \supseteq C_{\alpha,\delta,\beta}(A) \cup C_{\alpha,\delta,\beta}(B)$

VI) $C_{\alpha,\delta,\beta}(\cap A_i) = \cap C_{\alpha,\delta,\beta}(A_i)$

VII) $C_{1,0,0}(A) = X$.

Proof:

I) $C_{\alpha,\delta,\beta}(A) \subseteq C_{r,\theta,\phi}(A)$ if $\alpha \geq r$, $\delta \leq \theta$, $\beta \leq \phi$.

Let $x \in C_{\alpha,\delta,\beta}(A)$

$\Rightarrow \mu_A(x) \geq \alpha$, $\eta_A(x) \leq \delta$, $\nu_A(x) \leq \beta$

Since, we have $\alpha \geq r$, $\delta \leq \theta$, $\beta \leq \phi$

$\Rightarrow \mu_A(x) \geq \alpha \geq r$, $\eta_A(x) \leq \delta \leq \theta$, $\nu_A(x) \leq \beta \leq \phi$

$\Rightarrow \mu_A(x) \geq r$, $\eta_A(x) \leq \theta$, $\nu_A(x) \leq \phi$

$\Rightarrow x \in C_{r,,\phi}(A)$

$\Rightarrow C_{\alpha,\delta,\beta}(A) \subseteq C_{r,,\phi}(A)$.

II) $C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$

Since $\alpha+\delta+\beta \leq 1$ implies that $1-\delta-\beta \geq \alpha$ and $\delta \leq \delta$, $\beta \leq \beta$.

Therefore, from (I) we get $C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A)$ (5.1)

Again $\alpha \geq \alpha$, $\delta \leq 1-\alpha-\beta$ and $\beta \leq \beta$

Therefore, from (I) we get $C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$ (5.2)

From (5.1) and (5.2) we get

$C_{1-\delta-\beta,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,1-\alpha-\beta,\beta}(A)$

III) $A \subseteq B$ implies $C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(B)$

Let $x \in C_{\alpha,\delta,\beta}(A)$

$\Rightarrow \mu_A(x) \geq \alpha,\ \eta_A(x) \leq \delta,\ \nu_A(x) \leq \beta$

As $B \supseteq A$

$\Rightarrow\ \mu_B(x) \geq \mu_A(x) \geq \alpha, \eta_B(x) \leq \eta_A(x) \leq \delta, \nu_B(x) \leq \nu_A(x) \leq \beta$

$\Rightarrow \mu_B(x) \geq \alpha, \eta_B(x) \leq \delta, \nu_B(x) \leq \beta$

$\Rightarrow x \in\ C_{\alpha,\delta,\beta}(B)$

$\Rightarrow C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(B)$

IV) $C_{\alpha,\delta,\beta}(A \cap B) = C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$

We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Therefore, from (III)

$C_{\alpha,\delta,\beta}(A \cap B) \subseteq C_{\alpha,\delta,\beta}(A)$ and $C_{\alpha,\delta,\beta}(A \cap B) \subseteq C_{\alpha,\delta,\beta}(B)$

$\Rightarrow\ C_{\alpha,\delta,\beta}(A \cap B) \subseteq C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$ (5.3)

Now, let

$x \in C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$

$\Rightarrow x \in C_{\alpha,\delta,\beta}(A)$ and $x \in C_{\alpha,\delta,\beta}(B)$

$\Rightarrow\ \mu_A(x) \geq \alpha\ ;\ \mu_B(x) \geq \alpha \Rightarrow \mu_A(x) \wedge \mu_B(x) \geq \alpha \Rightarrow (\mu_A \cap \mu_B)(x) \geq \alpha$

$\eta_A(x) \leq \delta\ ;\ \eta_B(x) \leq \delta\ \Rightarrow \eta_A(x) \vee \eta_B(x) \leq \delta\ \Rightarrow (\eta_A \cap \eta_B)(x) \leq \delta$

$\nu_A(x) \leq \beta\ ;\ \nu_B(x) \leq \beta\ \Rightarrow \nu_A(x) \vee \nu_B(x) \leq \beta\ \Rightarrow (\nu_A \cap \nu_B)(x) \leq \beta$

$\Rightarrow\ x \in C_{\alpha,\delta,\beta}(A \cap B)$

$\Rightarrow\ C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B) \subseteq\ C_{\alpha,\delta,\beta}(A \cap B)$ (5.4)

From (5.3) and (5.4) we have

$C_{\alpha,\delta,\beta}(A \cap B) = C_{\alpha,\delta,\beta}(A) \cap C_{\alpha,\delta,\beta}(B)$.

V) $C_{\alpha,\delta,\beta}(A \cup B) \supseteq C_{\alpha,\delta,\beta}(A) \cup C_{\alpha,\delta,\beta}(B)$

Again, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

Hence from (III) we get

$C_{\alpha,\delta,\beta}(A) \subseteq C_{\alpha,\delta,\beta}(A \cup B)$ and $C_{\alpha,\delta,\beta}(B) \subseteq C_{\alpha,\delta,\beta}(A \cup B)$

$\Rightarrow\ C_{\alpha,\delta,\beta}(A) \cup C_{\alpha,\delta,\beta}(B) \subseteq C_{\alpha,\delta,\beta}(A \cup B)$

VI) $C_{\alpha,\delta,\beta}(\cap A_i) = \cap\, C_{\alpha,\delta,\beta}(A_i)$

: Let $x \in C_{\alpha,\delta,\beta}(\cap A_i)$

$\Rightarrow (\cap \mu_{A_i})(x) \geq \alpha,\ (\cap \eta_{A_i})(x) \leq \delta,\ (\cap \nu_{A_i})(x) \leq \beta$

$\Rightarrow \wedge \mu_{A_i}(x) \geq \alpha,\ \ \vee \eta_{A_i}(x) \leq \delta,\ \ \vee \nu_{A_i}(x) \leq \beta$

$\Rightarrow x \in C_{\alpha,\delta,\beta}(A_i)$;   for all 'i'

$\Rightarrow x \in \cap C_{\alpha,\delta,\beta}(A_i)$

$\Rightarrow C_{\alpha,\delta,\beta}(\cap A_i) \subseteq \cap C_{\alpha,\delta,\beta}(A_i)$ (5.5)

Let $x \in \cap C_{\alpha,\delta,\beta}(A_i)$

$\Rightarrow x \in C_{\alpha,\delta,\beta}(A_i)\ \forall i$.

$\Rightarrow \wedge\mu_{Ai}(x) \geq \alpha,\ \ \vee\eta_{Ai}(x) \leq \delta,\ \ \vee\nu_{Ai}(x) \leq \beta$

$\Rightarrow (\cap\mu_{Ai})(x) \geq \alpha,\ (\cap\eta_{Ai})(x) \leq \delta,\ (\cap\nu_{Ai})(x) \leq \beta$

$\Rightarrow x \in C_{\alpha,\delta,\beta}(\cap A_i)$

$\Rightarrow \cap C_{\alpha,\delta,\beta}(A_i) \subseteq C_{\alpha,\delta,\beta}(\cap A_i)$ (5.6)

From (5.5) and (5.6) we have

$C_{\alpha,\delta,\beta}(\cap A_i) = \cap\ C_{\alpha,\delta,\beta}(A_i)$.

VII) $C_{1,0,0}(A) = X$

$C_{1,0,0}(A) = \{x : x \in X$ such that $\mu_A(x) \geq 1,\ \eta_A(x) \leq 0,\ \nu_A(x) \leq 0\ \}$

$= X$.

6. Extension Principle of PFS

Definition 6.1: Let X and Y be two non empty sets and $f : X \to Y$ be a mapping. Let A and B be PFS's of X and Y respectively. Then the image of A under the map f is denoted by f(A) and is defined as

$f(A)(y) = (\ \mu_{f(A)}(y)\ ,\ \eta_{f(A)}(y)\ ,\ \nu_{f(A)}(y)\ )$, where

$\mu_{f(A)}(y) = \vee\{\mu_A(x) : x \in f^{-1}(y)\}$;

$\eta_{f(A)}(y) = \wedge\{\eta_A(x) : x \in f^{-1}(y)\}$;

$\nu_{f(A)}(y) = \wedge\{\nu_A(x) : x \in f^{-1}(y)\}$;

i.e $f(A)(y) = (\vee\{\mu_A(x) : x \in f^{-1}(y)\}\ ,\ \wedge\{\eta_A(x) : x \in f^{-1}(y)\}\ ,\ \wedge\{\nu_A(x) : x \in f^{-1}(y)\})$.

Also the pre-image of B under 'f' is denoted by $f^{-1}(B)$ and is defined as

$f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x)\ ,\ \eta_{f^{-1}(B)}(x)\ ,\ \nu_{f^{-1}(B)}(x))$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\eta_{f^{-1}(B)}(x) = \eta_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$

i.e $f^{-1}(B)(x) = (\ \mu_B(f(x))\ ,\ \eta_B(f(x))\ ,\ \nu_B(f(x)))$

Theorem 6.1 Let $f : X \to Y$ be a mapping. Then the following holds

(i) $f(C_{\alpha,\delta,\beta}(A)) \subseteq C_{\alpha,\delta,\beta}(f(A))$, for all $A \in PFS(X)$.

(ii) $f^{-1}(C_{\alpha,\delta,\beta}(B)) = C_{\alpha,\delta,\beta}(f^{-1}(B))$, for all $B \in PFS(Y)$.

Proof: (i) Let $y \in f(C_{\alpha,\delta,\beta}(A))$ then there exist a $x \in C_{\alpha,\delta,\beta}(A)$ such that

$f(x) = y$ and $\mu_A(x) \geq \alpha$, $\eta_A(x) \leq \delta$, $\nu_A(x) \leq \beta$

$\Rightarrow \bigvee\{\mu_A(x) : x \in f^{-1}(y)\} \geq \alpha$, $\bigwedge\{\eta_A(x) : x \in f^{-1}(y)\} \leq \delta$, $\bigwedge\{\nu_A(x) : x \in f^{-1}(y)\} \leq \beta$

i.e $\mu_{f(A)}(y) \geq \alpha$, $\eta_{f(A)}(y) \leq \delta$, $\nu_{f(A)}(y) \leq \beta$

i.e $y \in C_{\alpha,\delta,\beta}(f(A))$

Hence $f(C_{\alpha,\delta,\beta}(A)) \subseteq C_{\alpha,\delta,\beta}(f(A))$, for all $A \in IFS(X)$.

(ii) $C_{\alpha,\delta,\beta}(f^{-1}(B))$

$= \{x \in X : \mu_{f^{-1}(B)}(x) \geq \alpha , \eta_{f^{-1}(B)}(x) \leq \delta , \nu_{f^{-1}(B)}(x) \leq \beta \}$

$= \{x \in X : \mu_B(f(x)) \geq \alpha, \eta_B(f(x)) \leq \delta , \nu_B(f(x)) \leq \beta\}$

$= \{x \in X : f(x) \in C_{\alpha,\delta,\beta}(B)\}$

$= \{x \in X : x \in f^{-1}(C_{\alpha,\delta,\beta}(B))\}$

$= f^{-1}(C_{\alpha,\delta,\beta}(B))$

Thus, $C_{\alpha,\delta,\beta}(f^{-1}(B)) = f^{-1}(C_{\alpha,\delta,\beta}(B))$

7. Picture Fuzzy Arithmetic

In this section, picture fuzzy arithmetic operations will be performed based on extension principle with numerical illustrations.

Let A and B be picture fuzzy sets. Then, A*B (where $* \in (+,-,\cdot,/)$) is defined as

$A*B = \{z , \mu_{A*B}(z), \eta_{A*B}(z), \nu_{A*B}(z)\}$

where $\mu_{A*B}(z) = \bigvee[\mu_A(x)\wedge\mu_B(y)]$, $\eta_{A*B}(z) = \bigwedge[\eta_A(x)\vee\eta_B(y)]$, $\nu_{A*B}(z) = \bigwedge[\nu_A(x)\vee\nu_B(y)]$ and $x*y = z$.

Example: Let A and B be two picture fuzzy sets where

A = {(2, 0.4, 0.2, 0.3),(3, 0.7, 0.1, 0.1),(4, 0.6, 0.2, 0.2)} and B = {(1, 0.6, 0.1, 0.2),(2, 0.5, 0.2, 0.1)}.

7.1 Addition of picture fuzzy sets:

Addition of two picture fuzzy sets A and B can be written as:

$A+B = \{z , \mu_{A+B}(z), \eta_{A+B}(z), \nu_{A+B}(z) \}$

where $\mu_{A+B}(z) = \bigvee[\mu_A(x)\wedge\mu_B(y)]$, $\eta_{A+B}(z) = \bigwedge[\eta_A(x)\vee\eta_B(y)]$, $\nu_{A+B}(z) = \bigwedge[\nu_A(x)\vee\nu_B(y)]$ and $x+y = z$.

Then, the addition of the picture fuzzy sets A and B is

A+B = {(2+1, min (0.4,0.6), max (0.2,0.1), max (0.3,0.2)), (2+2, min (0.4,0.5), max (0.2,0.2), max (0.3, 0.1)) (3+1, min (0.7, 0.6), max (0.1, 0.1), max (0.1, 0.2)), (3+2, min (0.7, 0.5), max (0.1, 0.2), max (0.1, 0.1)) (4+1, min (0.6, 0.6), max(0.2, 0.1), max (0.2, 0.2)), (4+2, min (0.6,0.5), max (0.2,0.2), max (0.2, 0.1))}

= {(3,0.4,0.2,0.3), (4,0.4,0.2,0.3), (4, 0.6, 0.1, 0.2), (5, 0.5, 0.2, 0.1), (5, 0.6, 0.2, 0.2), (6, 0.5, 0.2, 0.2)}

= {(3, 0.4, 0.2, 0.3), (4, max (0.4, 0.6), min (0.2, 0.1), min (0.3,0.2)), (5, max (0.5, 0.6), min (0.2, 0.2), min (0.1, 0.2)), (6, 0.5, 0.2, 0.2)}

= { (3,0.4, 0.2, 0.3), (4, 0.6, 0.1, 0.2), (5, 0.6, 0.2, 0.1), (6, 0.5, 0.2, 0.2)}

7.2 Subtraction of picture fuzzy sets:

Subtraction of two picture fuzzy sets A and B can be written as:

$A\text{-}B = \{z, \mu_{A-B}(z), \eta_{A-B}(z), \nu_{A-B}(z)\}$

where $\mu_{A-B}(z) = \vee[\mu_A(x)\wedge\mu_B(y)]$, $\eta_{A-B}(z) = \wedge[\eta_A(x)\vee\eta_B(y)]$, $\nu_{A-B}(z) = \wedge[\nu_A(x)\vee\nu_B(y)]$ and $x\text{-}y = z$.

Then, the subtraction of the picture fuzzy sets A and B is

A-B = {(2-1, min (0.4, 0.6), max (0.2, 0.1), max (0.3, 0.2)), (2-2, min (0.4, 0.5), max (0.2, 0.2), max (0.3, 0.1)), (3-1, min(0.7,0.6), max (0.1,0.1), max (0.1,0.2)), (3-2, min (0.7,0.5), max (0.1, 0.2), max (0.1, 0.1)), (4-1, min (0.6, 0.6), max (0.2, 0.1), max (0.2, 0.2)), (4-2, min (0.6, 0.5), max (0.2, 0.2), max (0.2, 0.1))}

= {(1,0.4, 0.2, 0.3), (0,0.4, 0.2, 0.3), (2, 0.6, 0.1, 0.2), (1, 0.5, 0.2, 0.1), (3, 0.6, 0.2, 0.2), (2, 0.5, 0.2, 0.2)}

= {(0, 0.4, 0.2, 0.3), (1, max (0.4, 0.5), min (0.2, 0.2), min (0.3, 0.1)),(2, max (0.6, 0.5), min (0.1, 0.2), min (0.2, 0.2)), (3, 0.6, 0.2, 0.2)}

= {(0, 0.4, 0.2, 0.3), (1, 0.5, 0.2, 0.1), (2, 0.6, 0.1, 0.2), (3, 0.6, 0.2, 0.2)}

7.3 Multiplication of Picture fuzzy sets:

Multiplication of two fuzzy sets A and B can be written as:

$A\times B = \{z, \mu_{A\times B}(z), \eta_{A\times B}(z), \nu_{A\times B}(z)\}$

where $\mu_{A\times B}(z) = \vee[\mu_A(x)\wedge\mu_B(y)]$, $\eta_{A\times B}(z) = \wedge[\eta_A(x)\vee\eta_B(y)]$, $\nu_{A\times B}(z) = \wedge[\nu_A(x)\vee\nu_B(y)]$ and $x\times y = z$.

Then, the multiplication of the fuzzy sets A and B is

A×B = {(1×2,min (0.2, 0.1), max (0.3, 0.3), max (0.1, 0.3)), (1×3, min (0.2, 0.2), max (0.3, 0.4), max (0.1, 0.4)) (2×2, min (0.1, 0.1), max (0.3, 0.3), max (0.4, 0.3)), (2×3, min (0.1, 0.2), max (0.3, 0.4), max (0.4, 0.4)), (3×2, min (0.5, 0.1), max (0.3, 0.3), max (0.1,0.3)), (3×3, min (0.5, 0.2), max (0.3,0.4), max (0.1, 0.4)) }

= {(2, 0.1, 0.3, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.1, 0.3, 0.4), (6, 0.1, 0.4, 0.4), (6, 0.1, 0.3, 0.3), (9, 0.2, 0.4, 0.4)}

= {(2,0.1,0.3,0.3), (3,0.2,0.4, 0.4), (4, 0.1, 0.3, 0.4), (6, max (0.1, 0.1), min (0.3,0.4), min (0.3, 0.4)), (9, 0.2, 0.4, 0.4)}

= {(2, 0.1, 0.3, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.1, 0.3, 0.4), (6, 0.1, 0.3, 0.3), (9, 0.2, 0.4, 0.4)} .

7.4 Division of picture fuzzy sets:

Division of two fuzzy sets A and B can be written as:

$A/B = \{ z , \mu_{A/B}(z), \eta_{A/B}(z), \nu_{A/B}(z) \}$

where $\mu_{A/B}(z) = \vee[\mu_A(x)\wedge\mu_B(y)]$, $\eta_{A/B}(z) = \wedge[\eta_A(x)\vee\eta_B(y)]$, $\nu_{A/B}(z) = \wedge[\nu_A(x)\vee\nu_B(y)]$ and $x/y = z$. Then, the division of the fuzzy sets A and B is

A/B = {(2/1, min (0.1,.6), max (0.3,0.2), max (0.2,0.2)), (2/2, min (0.1,0.7), max (0.3,0.1), max (0.2,0.2)) (4/1, min (0.1,0.6), max (0.6, 0.2), max (0.2, 0.2)) , (4/2, min (0.1, 0.7), max (0.6, 0.1), max (0.2, 0.2))

= {(2, 0.1, 0.3, 0.2), (1, 0.1, 0.3, 0.2), (4, 0.1, 0.6, 0.2), (2, 0.1, 0.6, 0.2)}

= {(1, 0.1, 0.3, 0.2), (2, max (0.1, 0.1), min (0.3, 0.6), min (0.2, 0.2)), (4, 0.1, 0.6, 0.2)}

= {(1, 0.1, 0.3, 0.2), (2, 0.1, 0.3, 0.2), (4, 0.1, 0.6, 0.2)}.

8. Conclusion

Two Decomposition theorem of picture fuzzy set (PFS) has been proved. New (α,δ,β)-cut of picture fuzzy set have been defined and some of its properties are proved. Extension principle for PFS has been defined and proved two of its properties. Finally, picture fuzzy arithmetic based on extension principle has been performed with examples. In our further study, other properties of extension principle for PFS will be discussed.

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