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A Quintic Spline Collocation Method for the Fractional Sub-Diffusion Equation with Variable Coefficients

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Abstract

For the time fractional sub-diffusion equation with variable coefficients, a quintic spline method is presented, along the time direction, the recursion formula obtained from the Lagrange interpolation functions is used, along the space direction, the quintic spline interpolation functions, which have high order accuracy when being used to approximate smooth functions and their 1,2,3 order derivatives, are used as the basis functions. Theoretical analyses and numerical examples show that 4 order accuracy in space can be achieved for this scheme.

Key words

Sub-diffusion equation, quintic spline collocation method, fractional

1. Introduction

When studying diffusion phenomena in highly inhomogeneous media, ones often find that traditional integer-order diffusion models always lead to heavy tail phenomena. By comparison, corresponding fractional models behave better when describing probability density. Therefore, in recent years, the fractional diffusion equation with various boundary conditions and nonlinear terms has become more and more important in many fields, such as biology, medicine, fluid mechanics, thermodynamics and electrochemical reaction. More and More researchers are focusing on the numerical solution and simulation of fractional models subject to various conditions. And many excellent research achievements have been presented, such as the finite difference method, the finite element method, the spectral method, the finite volume method and so on. However, in these existing literatures we can find most of the numerical methods are efficient only for fractional equations with constant coefficients. Therefore, the numerical solution of various fractional diffusion equations with variable coefficients remains an important area to be studied.

In this paper, using the quintic spline interpolation functions, which have high order accuracy when being used to approximate smooth functions and their derivatives, we study the numerical solution of the following fractional diffusion equations with variable coefficients

$$\begin{cases} {}_cD_{0,t}^{\beta}u = k(x,t)\dfrac{\partial^2 u}{\partial x^2} + f(x,t), (x,t) \in I \times (0,T]; \\ u(x,0) = \phi(x), x \in I = [a,b]; \\ \alpha_1 u(a,t) + \beta_1 \dfrac{\partial u(x,t)}{\partial x}|_{x=a} = \varphi(t), t \in (0,T]; \\ \alpha_2 u(b,t) + \beta_2 \dfrac{\partial u(x,t)}{\partial x}|_{x=b} = \psi(t), t \in (0,T], \end{cases} \tag{1}$$

where $0 < \beta < 1$， $k(x,t) > 0$ are diffusion coefficient function， $\phi(x), \varphi(t), \psi(t), f(x,t)$ are given smooth functions. ${}_cD_{0,t}^{\beta}u$ is the Caputo derivative of the form

$${}_cD_{0,t}^{\beta}u = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\frac{\partial u}{\partial s}ds \tag{2}$$

With $\Gamma(\bullet)$ the Gamma function. For Equ.(1), we use the quintic spline interpolation functions as basis function in space, present a collocation method combining with the L1 recursion formula in time, analyze the theoretical accuracy, and illustrate the efficiency by some numerical examples.

2. Preparation

Define respectively $\rho_h = \{x_i\}_{i=0}^{N_h}$ and $\rho_t = \{t_i\}_{i=0}^{N_t}$ as uniform partitions of the interval $[a,b]$ and $[0,T]$ with

$$x_i = a + ih, i = 0,1,\cdots,N_h, h = \frac{b-a}{N_h}, t_j = j\tau, j = 0,1,\cdots,N_t, \tau = \frac{T}{N_t}.$$

Let $S_5[a,b]=\{v: v\in C^4[0,1], v|_{\delta_i}\in P_5, 0\le i\le N_h-1\}$ be the space of quintic spline functions, where P_5 is the set of polynomials of degree ≤ 5, and $\delta_i=[x_i,x_{i+1}]$.

For convenience of using quintic spline function as basis function in space, several auxiliary nodes $x_i=a+ih(i=-5,\cdots,-1,N_{h+1},\cdots,N_h+5)$ are added to the meshes, which leads to a new interval $\bar{I}=[a-5h,b+5h]$.

By the results in, we can immediately obtain the expression of quintic spline function and some characteristics as follows:

For $i=-2,-1,\cdots,N_h+1,N_h+2$, the quintic spline functions are defined as

$$B_i(x)=\frac{1}{120h^2}\begin{cases}(x-x_i+3h)^5, \quad x\in[x_{i-3},x_{i-2}];\\ (x-x_i+3h)^5-6(x-x_i+2h)^5, x\in[x_{i-2},x_{i-1}];\\ (x-x_i+3h)^5-6(x-x_i+2h)^5+15(x-x_i+h)^5, \quad x\in[x_{i-1},x_i];\\ (-x+x_i+3h)^5-6(-x+x_i+2h)^5+15(-x+x_i+h)^5, x\in[x_i,x_{i+1}];\\ (-x+x_i+3h)^5-6(-x+x_i+2h)^5, x\in[x_{i+1},x_{i+2}];\\ (-x+x_i+3h)^5, x\in[x_{i+2},x_{i+3}].\end{cases} \tag{3}$$

And for enough smooth function $u(x)$, there is unique quintic spline function $u_s(x)=\sum_{i=-2}^{N_h+2}u_iB_i(x)$ satisfying the following interpolation result.

Lemma 1 Suppose $u(x)\in C^4[a,b]$,and $u_s(x)\in S_5$ is the quintic spline function defined above satisfying

$$u_s(x_i)=u(x_i), i=0,1,\cdots,N_h, u_s'(x_i)=u'(x_i), u_s''(x_i)=u''(x_i), i=0,N_h \tag{4}$$

then, for all nodes $x_i, i=0,1,\cdots,N_h$, it holds

$$\begin{cases}u(x_i)=u_s(x_i)=\frac{1}{120}(u_{i-2}+26u_{i-1}+66u_j+26u_{j+1}+u_{j+2}),\\ u'(x_i)=u_s(x_i)+O(h^6)=\frac{1}{24h}(-u_{i-2}-10u_{i-1}+10u_{i+1}+u_{i+2})+O(h^6),\\ u''(x_i)=u_s''(x_i)+O(h^4)=\frac{1}{6h^2}(u_{i-2}+2u_{i-1}-6u_i+2u_{i+1}+u_{i+2})+O(h^4),\\ u^{(3)}(x_i)=u^{(3)}(x_i)+O(h^4)=\frac{1}{2h^3}(-u_{i-2}+2u_{i-1}-2u_{i+1}+u_{i+2})+O(h^4).\end{cases} \tag{5}$$

In order to deal with the Caputo derivative operator (2), denote by $a_l = l^{1-\beta} - (l-1)^{1-\beta}, l = 1, 2, \cdots, N+1$, and let

$$c_1 = \frac{-a_n \tau^{-\beta}}{\Gamma(2-\beta)}, c_k = \frac{-(a_{n-k+1} - a_{n-k+2})\tau^{-\beta}}{\Gamma(2-\beta)} (k = 2, \cdots, n), c_{n+1} = \frac{a_1 \tau^{-\beta}}{\Gamma(2-\beta)}, \tag{6}$$

Then, the use of piece-wise linear Lagrange interpolation function will result in the following recursion formula :

Lemma 2 When $u \in C^2[0,T]$, for the Caputo derivative operator (2), there is a recursion formula of the form

$${}_cD_{0,t}^{\beta} u(t) |_{t=t_{n+1}} = \sum_{k=1}^{n+1} c_k u(t_k) + R(u(t_{n+1})), n = 1, \cdots, N, \tag{7}$$

here the remainder term $R(u(t_{n+1}))$ satisfies

$$| R(u(t_{n+1})) | \le O(\tau^{2-\beta}). \tag{8}$$

3. Quintic spline collocation method

For convenience, let $D_x^k u = \frac{\partial^k u}{\partial x^k}$. For the $n+1$th level $(0 \le n \le N_t)$, substituting (7) into the first equation of (1), there is

$$D_x^2 u(x, t_{n+1}) - \frac{c_{n+1}}{k(x, t_{n+1})} u(x, t_{n+1}) = \frac{1}{k(x, t_{n+1})} \sum_{j=1}^{n} c_j u(t_j, x) - \frac{f(x, t_{n+1})}{k(x, t_{n+1})} + R(u(t_{n+1})).$$

Denote by $r(x) = \frac{1}{k(x, t_{n+1})} \sum_{j=1}^{n} c_j u(t_j, x) - \frac{f(x, t_{n+1})}{k(x, t_{n+1})}$, dropping the error term $R(u(t_{n+1}))$, we obtain

$$D_x^2 u(x, t_{n+1}) - \frac{c_{n+1}}{k(x, t_{n+1})} u(x, t_{n+1}) = r(x). \tag{9}$$

In (9), the first level $u(x, t_0)$ can be obtained by calculating the initial condition $u(x, 0) = \phi(x)$ in (1).

Let $u_s(x, t_{n+1})$ be the obtained approximation solution. First, substituting all nodes $x_0, x_1, \cdots, x_{N_h}$ into (9), and considering (5) we can get:

$$\frac{u_{i-2}+2u_{i-1}-6u_i+2u_{i+1}+u_{i+2}}{6h^2}-\frac{c_{n+1}}{k(x_i,t_{n+1})}\frac{u_{i-2}+26u_{i-1}+66u_i+26u_{i+1}+u_{i+2}}{120}=r(x_i)+O(h^4) \quad (10)$$

Second, by the two boundary conditions in (1), the use of first two equations of (5) leads to

$$\begin{aligned}&\beta_1(-u_{-2}-10u_{-1}+10u_1+u_2)+\frac{\alpha_1 h}{5}(u_{-2}+26u_{-1}+66u_0+26u_1+u_2)=24\varphi(t_{n+1})h,\\&\beta_2(-u_{N_h-2}-10u_{N_h-1}+10u_{N_h+1}+u_{N_h+2})+\frac{\alpha_2 h}{5}(u_{N_h-2}+26u_{N_h-1}+66u_{N_h}+26u_{N_h+1}+u_{N_h+2})=24\psi(t_{n+1})h.\end{aligned} \quad (11)$$

For obtaining enough linear equations, we take derivative with respect to x in (9) and have

$$D_x^3u(x,t_{n+1})-(\frac{c_{n+1}}{k(x,t_{n+1})})'u(x,t_{n+1})-\frac{c_{n+1}}{k(x,t_{n+1})}D_xu(x,t_{n+1})=r'(x)\,. \quad (12)$$

Taking $x=a, x=b$, and dropping the error term $O(h^4)$, (12) reads:

$$\begin{aligned}&\frac{-u_{-2}+2u_{-1}-2u_1+u_2}{2h^3}+\alpha_a\frac{u_{-2}+26u_{-1}+66u_0+26u_1+u_2}{120}+\beta_a\frac{-u_{-2}-10u_{-1}+10u_1+u_2}{24h}=r'(a),\\&\frac{-u_{N_h-2}+2u_{N_h-1}-2u_{N_h+1}+u_{N_h+2}}{2h^3}+\alpha_a\frac{u_{N_h-2}+26u_{N_h-1}+66u_{N_h}+26u_{N_h+1}+u_{N_h+2}}{120}\\&+\beta_a\frac{-u_{N_h-2}-10u_{N_h-1}+10u_{N_h+1}+u_{N_h+2}}{24h}=r'(b),\end{aligned} \quad (13)$$

where $\alpha_a=-c_{n+1}(\frac{1}{k(x,t_{n+1})})'|_{x=a},\beta_a=\frac{-c_{n+1}}{k(a,t_{n+1})},\alpha_b=-c_{n+1}(\frac{1}{k(x,t_{n+1})})'|_{x=b},\beta_b=\frac{-c_{n+1}}{k(b,t_{n+1})}.$

By (10)-(13), we can obtain the collocation equations

$$(A+\frac{1}{20}h^2QB)U=R \quad (14)$$

Here A,Q,B are all $(N_h+5)\times(N_h+5)$ -dimensional matrices:

$$A=\begin{bmatrix}\theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & & & \\ -1 & 2 & 0 & -2 & 1 & & & \\ 1 & 2 & -6 & 2 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 2 & -6 & 2 & 1 \\ & & & -1 & 2 & 0 & -2 & 1 \\ & & & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5\end{bmatrix}, Q=\begin{bmatrix}0 & & & & & & \\ & 1 & & & & & \\ & & -\frac{c_{n+1}}{k(x_0,t_{n+1})} & & & & \\ & & & \ddots & & & \\ & & & & -\frac{c_{n+1}}{k(x_{N_h},t_{n+1})} & & \\ & & & & & 1 & \\ & & & & & & 0\end{bmatrix},$$

$$B=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & & & \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ & & & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

with

$$\theta_1=-\beta_1+\frac{1}{5}\alpha_1 h,\theta_2=-10\beta_1+\frac{26}{5}\alpha_1 h,\theta_3=\frac{66}{5}\alpha_1 h,\theta_4=10\beta_1+\frac{26}{5}\alpha_1 h,\theta_5=\beta_1+\frac{1}{5}\alpha_1 h,$$

$$\rho_1=-\beta_2+\frac{1}{5}\alpha_2 h,\rho_2=-10\beta_2+\frac{26}{5}\alpha_2 h,\rho_3=\frac{66}{5}\alpha_2 h,\rho_4=10\beta_2+\frac{26}{5}\alpha_2 h,\rho_5=\beta_2+\frac{1}{5}\alpha_2 h,$$

$$\eta_1=\frac{\alpha_a h-5\beta_a}{3},\eta_2=\frac{26\alpha_a h-50\beta_a}{3},\eta_3=22\alpha_a h,\eta_4=\frac{26\alpha_a h+50\beta_a}{3},\eta_5=\frac{\alpha_a h+5\beta_a}{3},$$

$$\lambda_1=\frac{\alpha_b h-5\beta_b}{3},\lambda_2=\frac{26\alpha_b h-50\beta_b}{3},\lambda_3=22\alpha_b h,\lambda_4=\frac{26\alpha_b h+50\beta_b}{3},\lambda_5=\frac{\alpha_b h+5\beta_b}{3}.$$

In addition， the right term R and unknown U are respectively

$$R=[24\varphi(t_{n+1}),2r'(a),6r(x_0)h^2,\cdots,6r(x_{N_h})h^2,2r'(b),24\psi(t_{n+1})]^T，\quad U=[u_{-2},u_{-1},\cdots,u_{N_h+1},u_{N_h+2}]^T.$$

For the collocation system (14), we have

Theorem 1 Suppose model (1) has unique solution, when $u(x,t)\in C_{x,t}^{4,2}([a,b]\times[0,T])$， the approximation solution in (14) satisfies the errors:

$$\|u(x,t)-u_s(x,t)\|_\infty=O(h^4+\tau^{2-\beta}). \tag{15}$$

PROOF When system (1) has unique solution, there is Green function $G(x,s)$ at every time level. In Level $n+1$ ($n=0,\cdots,N_t-1$), for the exact solution $u(x,t_{n+1})$ and approximation solution $u_s(x,t_{n+1})$, denote by $D_x^2 u(x,t_{n+1})=\xi(x)$, $D_x^2 u_s(x,t_{n+1})=\omega(x)$, then there is

$$D_x^m u(x,t_{n+1})=\int_a^b \frac{\partial^m G(x,s)}{\partial x^m}\xi(s)ds, D_x^m u_s(x,t_{n+1})=\int_a^b \frac{\partial^m G(x,s)}{\partial x^m}\omega(s)ds,\ m=0,1.$$

For convenience, let $F(x,u)=\dfrac{c_{n+1}}{k(x,t_{n+1})}u(x,t_{n+1})+r(x),$ defining the following operator:

$$D_{N_h}: S_5[a,b]\to R^{N_h+1}, D_{N_h}(g(x))=(g(x_0),g(x_1),\cdots g(x_{N_h}))^T$$

$$M_{N_h}: R^{N_h+1}\to S_5[a,b],\quad K: C[a,b]\to C[a,b],\ K(g(x))=F(x,\int_a^b G(x,s)g(s)ds).$$

Based on the above denotations, (9) can be written as

$$D_x^2 u - F(x,u)=\xi(x)-K(\xi(x))=(I-K)(\xi(x))=0, \tag{16}$$

(1) has unique solution, so operator $I-K$ is reversible. And the reversible is a bounded operator.

Based on Lemma 1, it has truncation error $O(h^4)$ in the process of (10)-(14), hence, there is

$$(I-K)(\omega(x))=O(h^4). \tag{17}$$

Based on(16) and (17), we obtain

$$(I-K)(\xi(x)-\omega(x))=O(h^4),$$

Again based on the bounded reversibility of Operator $I-K$, there is

$$\|\xi(x)-\omega(x)\|_\infty=O(h^4),$$

Therefore, we get the equality

$$\|u(x,t_{n+1})-u_s(x,t_{n+1})\|_\infty=\|\int_a^b G(x,s)(\xi(x)-\omega(x))ds\|=O(h^4).$$

By combining with the Lemma 2, we complete the proof of the theorem 1.

4. Numerical examples

In order to investigate the theoretical analysis results about the efficiency of the presented quintic spline collocation method, in this section, by using the Matlab R2010, some numerical examples are provided. In all given results, Table 1 lists the corresponding precision with every space step and fixed time step. We take the fractional order as $\beta = 0.01$ to minimize the impacts from the time direction. Table 2 and Table 3 list the corresponding precision with every time step and fixed space step. Here we take the fractional order $\beta = 0.01, 0.5, 0.99$ to check the impact of the fractional order on the precision. In all tables, the errors are calculated in terms of infinite norm, we use E_c to represents the errors at all collocation nodes. Here the computational formula of the convergence rate is:

$$Rate = \log(E_c(N/2)/E_c(N))/\log(2).$$

Example 1. $u(x,t) = t^3(\cos x + e^x), k(x,t) = e^{x+t}, T = 1, a = 0, b = 1$, and

$$f(x,t) = \frac{\Gamma(4)}{\Gamma(4-\beta)} t^{3-\beta}(\cos x + e^x) - t^3(e^x - \cos x)e^{x+t}.$$

Example 2. $u(x,t) = t^{2.1}x^{4.1}, k(x,t) = \cos(x+t), T = 1, a = -1, b = 0$, and

$$f(x,t) = \frac{\Gamma(3.1)}{\Gamma(3.1-\beta)} t^{2.1-\beta} x^{4.1} - 12.71 t^{2.1} x^{2.1} \cos(x+t).$$

Table 1. Errors ($\tau = 1/10000, \beta = 0.01$) along the space direction

M	Example 1			Example 2		
	E_c	Rate	CPUtime (second)	E_c	Rate	CPUtime (second)
8	5.1240e-7		2.1345	1.5812e-6		2.1109
16	3.2854e-8	3.9631	4.2521	1.0216e-7	3.9521	4.2385
32	2.1225e-9	3.9522	8.5219	6.5361e-9	3.9663	8.4326
64	1.3670e-10	3.9567	16.9823	4.2886e-10	3.9299	16.9327

Table 2. Errors ($h = 1/64$) along the time direction in EXAMPLE 1

N	$\beta = 0.01$		$\beta = 0.5$		$\beta = 0.99$	
	E_c	Rate	E_c	Rate	E_c	Rate
200	2.1335e-5		2.5355e-4		5.7342e-3	
400	5.5157e-6	1.9516	9.0358e-5	1.4885	2.9032e-3	0.9819
800	1.4257e-6	1.9519	3.2181e-5	1.4894	1.4672e-3	0.9846
1600	3.6638e-7	1.9603	1.1450e-5	1.4909	7.4035e-4	0.9868
3200	9.4166e-8	1.9601	4.0721e-6	1.4915	3.7235e-4	0.9915

Table 3. Errors ($h = 1/64$) along the time direction in EXAMPLE 2

N	$\beta = 0.01$		$\beta = 0.5$		$\beta = 0.99$	
	E_c	Rate	E_c	Rate	E_c	Rate
200	5.6360e-5		2.8473e-4		6.2834e-3	
400	1.4693e-5	1.9395	1.0215e-4	1.4789	3.2201e-3	0.9644

800	3.7891e-6	1.9552	3.6449e-5	1.4867	1.6373e-3	0.9758
1600	9.7496e-7	1.9584	1.2993e-5	1.4881	8.2921e-4	0.9815
3200	2.5061e-7	1.9599	4.6292e-6	1.4889	4.1897e-4	0.9849

The experimental results from Table 1, Table 2 and Table 3 show that, when solution function $u(x,t)\in C_{x,t}^{4,2}([a,b]\times[0,T])$, quintic spline collocation scheme (14) achieves accuracy of $O(h^4)$ and $O(\tau^{2-\beta})$ along the time and space direction respectively. These numerical results are in agreement with the theoretical results of Theorem 1.

5. Conclusion

By combing with the L1 recursion formula in time and quintic spline functions in space, we have introduced a collocation method for the the fractional sub-diffusion equation with variable coefficients. Theoretical analysis and numerical experiments show this method can achieve the precision $O(h^4+\tau^{2-\beta})$ under certain smooth conditions with the theoretical analysis.

References

1. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep., 339 (2000), 1-77.
2. C. Ingo, R. Magin, T. Parrish, New insights into the fractional order diffusion equation using entropy and kurtosis, Entropy 16 (2014) 5838–5852.
3. I. Podlubny, Fractional Differential Equations, Academic Press, California, USA, 1999.
4. S. Das, R. Kumar, Fractional diffusion equations in the presence of reaction terms, J. Comput. Complex. Appl. 1 (1) (2015) 15-21.
5. S.B. Yuste, L. Acedo, K. Lindenberg, Reaction front in an $A+B\to C$ reaction-subdiffusion process, Phys. Rev. E, 69 (2004) 036126:1-10.
6. G.H. Gao, Z.Z. Sun, H.W. Zhang. A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications. J. Comput. Phys., 259(2)(2014), 33-50.
7. G.-H. Gao, Z.-Z. Sun, A compact finite difference scheme for the fractional sub-diffusion equations, J. Comput. Phys., 230 (2011), 586-595.
8. C.-C. Ji, Z.-Z. Sun, A high-order compact finite difference scheme for the fractional sub-diffusion equation, J. Sci. Comput., 64 (3) (2015), 959-985.
9. B. Jin, R. Lazarov, Z. Zhou, Error estimates for a semidiscrete finite element method for fractional order parabolic equations, SIAM J. Numer. Anal., 51 (1) (2013), 445-466.

10. F. Zeng, C. Li, F. Liu, I. Turner, The use of finite difference/element approaches for solving the time-fractional subdiffusion equation, SIAM J. Sci. Comput., 35 (6) (2013), A2976-A3000.
11. F. Zeng, C. Li, F. Liu, I. Turner, Numerical algorithms for time-fractional subdiffusion equation with second-order accuracy, SIAM J. Sci. Comput., 37 (2015), A55-A78.
12. F. Chen, Q. Xu, J.S. Hesthaven, A multi-domain spectral method for time-fractional differential equations, J. Comput. Phys., 293 (2013), 157-172.
13. M. Zheng, F.Liu, I.Turner, V. Anh, A novel high order space-time spectral method for the time fractional fokker--planck equation, SIAM J. Sci. Comput., 37 (2015), A701-A724.
14. F. Liu, P. Zhuang, I. Turner, K. Burrage, V. Anh, A new fractional finite volume method for solving the fractional diffusion equation, Appl. Math. Model., 38 (2014), 3871-3878.
15. I.J. Schoenberg, Contribution to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math. 4 (1946), 112–141.
16. C. De Boor, Practical Guide to Splines, Springer-Verlag, Berlin, New York, 1978.
17. M.I. Ellina, E.N. Houstis, An O(h6) quintic spline collocation method for fourth order two-point boundary value problems, BIT 28 (1988), 288–301.