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α-Pareto optimal solutions for fuzzy multiple objective optimization problems using MATLAB

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ABSTRACT

I present a computational view to generate α-Pareto optimal solutions for the fuzzy multiple objective optimization problems based on the α-Level sets method and the weighting method using MATLAB® (R2014a) . In this paper, two MATLAB codes based on two hybrid algorithms for solving linear multiple objective programming problems involving fuzzy parameters in: (1) The right hand side of the constraints, and (2) The objective functions are introduced. These fuzzy parameters are characterized as fuzzy numbers. For such problems, the α-Pareto optimality is introduced by extending the ordinary Pareto optimality on the basis of the α-Level sets of fuzzy numbers. Also, two numerical examples are given to clarify the main results developed in the paper. The hand solutions of the numerical examples and the solutions by the MATLAB codes are identical.

1. INTRODUCTION

Multiple objective optimization problems are a famous class in Mathematical programming (MP), [1-3]. It deals with the problems that have a conflict multiple objectives under constraints. This paper presents a numerical approach using MATLAB® , [4-7], to solve linear multiple objective optimization problems involving fuzzy parameters. In real life problems, it is natural to consider that the possible values of these parameters are ambiguously known to experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy numbers, [8-13].

In the following section, basic definitions are given. A hybrid algorithm, an illustrative numerical example and a MATLAB® code to solve linear multiple objective optimization problem involving fuzzy parameters in the right hand side of the constraints based on the α-Level set mrthod and the weighting method are introduced in section (3). In section (4), a hybrid algorithm, an illustrative numerical example and MATLAB® code to solve linear multiple objective optimization problem involving fuzzy parameters in the objective functions based on the α-Level set method and the weighting method are presented.

2. BASIC DEFINITIONS

Let $\mathbb{X}$ be a set. The characteristic function $\mu_{\mathbb{M}}$ of the classical subset $\mathbb{M}$ of $\mathbb{X}$ takes its values in the two-element set $\{0\,,1\}$. A fuzzy set $\mathbb{M}$ has a characteristic function taking its values in the interval $[0\,,1]$. $\mu_{\mathbb{M}}$ is also called a membership function. $\mu_{\mathbb{M}}(x)$ is the grade of membership of $x \in \mathbb{X}$ in $\mathbb{M}$. $\mathbb{M}$ is symbolically denote by $\{(x\,,\mu_{\mathbb{M}}(x)) : x \in \mathbb{X}\}$. A fuzzy set can be denoted by $\widetilde{\mathbb{M}}$, and the ordinary set by $\mathbb{M}$, [8-12].

There are many different definitions for the fuzzy real number. The trapezoidal fuzzy real numbers can be defined as follows, [8-12]:

Definition (1):

A real fuzzy number $\widetilde{m}$ is a fuzzy subset of the real line $\mathbb{R}$ with membership $\mu_{\widetilde{m}}$ which have the following properties:

(I) $\mu_{\widetilde{m}}$ is a continuous mapping from $\mathbb{R}$ to a closed interval $[0\,,1]$,
(II) $\mu_{\widetilde{m}}(x) = 0$, for all $x \in (-\infty\,, m_1]$,
(III) $\mu_{\widetilde{m}}(x)$ is strictly increasing on $[\,m_1, m_2]$,
(IV) $\mu_{\widetilde{m}}(x) = 1$, for all $x \in [\,m_2\,, m_3]$,
(V) $\mu_{\widetilde{m}}(x)$ is strictly decreasing on $[\,m_3, m_4]$,
(VI) $\mu_{\widetilde{m}}(x) = 0$, for all $x \in [m_4\,, +\infty)$,

where $m_1, m_2\,, m_3\,, m_4$ are real numbers.

The fuzzy number can be denoted by $\widetilde{m} = [m_1, m_2\,, m_3\,, m_4]$.

Definition (2):

The fuzzy number $\widetilde{m} = [m_1, m_2\,, m_3\,, m_4]$ is a trapezoidal fuzzy real number, denote by $[m_1, m_2\,, m_3\,, m_4]$, its membership function $\mu_{\widetilde{m}}$ is given by:

$$\mu_{\widetilde{m}}(m) = \begin{cases} 0, & m \le m_1, \\ 1 - \left[\frac{(m-m_2)}{(m_1-m_2)}\right], & m_1 \le m \le m_2, \\ 1, & m_2 \le m \le m_3, \\ 1 - \left[\frac{(m-m_3)}{(m_4-m_3)}\right], & m_3 \le m \le m_4, \\ 0, & m_4 \le m\,. \end{cases} \tag{1}$$

Another form for the membership function $\mu_{\widetilde{m}}$ satisfying assumptions (I to VI) of definition (1) is given by:

$$\mu_{\widetilde{m}}(m) = \begin{cases} 0, & m \le m_1, \\ 1 - \left[\frac{(m-m_2)}{(m_1-m_2)}\right]^2, & m_1 \le m \le m_2, \\ 1, & m_2 \le m \le m_3, \\ 1 - \left[\frac{(m-m_3)}{(m_4-m_3)}\right]^2, & m_3 \le m \le m_4, \\ 0, & m_4 \le m\,. \end{cases} \tag{2}$$

There are many other possible forms for a membership function: linear, exponential, hyperbolic, etc.

Also, the ordinary subset of level α can be defined as follows, [8-12]:

Definition (3):

Let $\alpha \in [0\,,1]$, one can call the ordinary subset of level α of a fuzzy subset $\widetilde{\mathbb{M}}$, the ordinary subset $\mathbb{M}_\alpha = \{x : \mu_{\widetilde{\mathbb{M}}}(x) \geq \alpha\}$.

3. LINEAR MULTIPLE OBJECTIVE OPTIMIZATION PROBLEMS WITH FUZZY PARAMETERS IN THE RIGHT HAND SIDE OF THE CONSTRAINTS

Consider the following linear multiple objective optimization (LMOO) problem involving fuzzy numbers in the right hand sides of the constraints (LMOO) $_{\tilde{b}}$:

(LMOO) $_{\tilde{b}}$:

$$Maximize\big(f_1(X), f_2(X), \ldots, f_k(X)\big) \tag{3}$$

subject to

$$X \in \mathbb{X}(\tilde{b})=\{ X \in \mathbb{R}^n : \textstyle\sum_{j=1}^{n} a_{ij}x_j \leq \tilde{b}_i, i \in \mathbb{V}, \; x_j \geq 0, j \in \mathbb{N}\},$$

where

$\mathbb{V} = \{1,2, \ldots, v\}$,

$\mathbb{N} = \{1,2, \ldots, n\}$,

$\tilde{b}_i$: represent fuzzy numbers involved in constraints, $i \in \mathbb{V}$,

X : is an n-vector of the variables,

$\mathbb{R}^n$: is the set of all n-vectors of real numbers.

It is assumed that, $\tilde{b}_i$, $i \in \mathbb{V}$ in problem (3), are fuzzy numbers whose membership functions are $\mu_{\tilde{b}}(b_i)$, $i \in \mathbb{V}$.

By introducing the concept of α-level set or α-cut of the fuzzy numbers $\tilde{b}_i, i \in \mathbb{V}$, then problem (3), for a certain degree α , can be understood as the following nonfuzzy α-linear multiple objective optimization $(\alpha -$ LMOO$)$ $_b$ problem:

$(\alpha -$ LMOO$)$ $_b$:

$$Maximize\big(f_1(X), f_2(X), \ldots, f_k(X)\big) \tag{4-1}$$

subject to

$$X \in \mathbb{X}(b)=\{ X \in \mathbb{R}^n : \textstyle\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i \in \mathbb{V}, \; x_j \geq 0, j \in \mathbb{N}\}, \tag{4-2}$$

$$b \in L_\alpha(\tilde{b}) \tag{4-3}$$

where $L_\alpha(\tilde{b})$ is the α-level set of the fuzzy numbers, $\tilde{b}_i, i \in \mathbb{V}$.

Based on the definition of the α-level set of the fuzzy numbers, the concept of α-Pareto optimal solution to the α-linear multiple objective optimization problem (4) can be introduced in the following definition, [8-12]:

Definition (4):

A point $X^* \in \mathbb{X}(b)$ is said to be an α-Pareto optimal solution to the $(\alpha -$ LMOO$)$ $_b$ problem (4) , if and only if there does not exist another $X \in \mathbb{X}(b)$, $b^* \in L_\alpha(\tilde{b})$, such that $f_r(X) \geq f_r(X^*), (r = 1,2,\ldots,k)$, with strictly inequality holding for at least one r, where the corresponding value of number b^* is called $\alpha -$ level optimal number.

To find an α-Pareto optimal solution to the $(\alpha -$ LMOO$)$ $_b$ problem (4), a hybrid algorithm based on the α-level set of the fuzzy numbers, [8-12], and the weighting method, [1-3], is introduced as follows:

Hybrid Algorithm (I):

Step (1): Transform problem (3) to the form of problem (4).

Step (2): Use the weighting method and the α-level set method to transform problem (4) to the following form:

$$Maximize \textstyle\sum_{r=1}^{k} w_r f_r(X) \tag{5-1}$$

subject to

$$X \in \mathbb{X}(b)=\{ X \in \mathbb{R}^n : \textstyle\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i \in \mathbb{V}, \; x_j \geq 0, j \in \mathbb{N}\}, \tag{5-2}$$

$$b \in L_\alpha(\tilde{b}) \tag{5-3}$$

where $w_r \geq 0$, $r = 1,2, \ldots, k$, $\sum_{r=1}^{k} w_r = 1$.

Step (3): Let $\alpha = \alpha^*$, and use the membership function $\mu_{\tilde{b}}$ of the form (1) or (2) to transform problem (5) to the following form:

$$Maximize \textstyle\sum_{r=1}^{k} w_r f_r(X) \tag{6-1}$$

subject to

$$X \in \mathbb{X}(b)=\{ X \in \mathbb{R}^n : \textstyle\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i \in \mathbb{V}, \; x_j \geq 0, j \in \mathbb{N}\}, \tag{6-2}$$

$$h_i \leq b_i \leq H_i \, , i \in \mathbb{V} \tag{6-3}$$

where $w_r \geq 0$, $r = 1,2, \ldots, k$, $\sum_{r=1}^{k} w_r = 1$.

Step (4): Let $w_r = w_r^* \geq 0$, then use the simplex (or any other method) to solve problem (6).

Step (5): Let X^* is the α-Pareto optimal solution to problem (6):

(i) If $w_r > 0$, for all r , thus X^* is an efficient solution,

(ii) If $w_r \geq 0$, for all r , and X^* is a unique for problem (6), thus X^* is an efficient solution ,

(iii) If $w_r \geq 0$, for all r , and there are alternative solutions, thus, use the non-inferiority test.

Step (6): Stop.

Example (I):

Consider the following LMOO problem involving fuzzy numbers in the right hand side of the constraints:

$$Maximize\; f_1(x_1, x_2) = 6x_1 + x_2 \,,\; Maximize\; f_2(x_1, x_2) = x_1 + 2x_2,$$

subject to

$$3x_1 + 2x_2 \leq \tilde{\lambda},\; 5x_1 \leq 9,\; x_1, x_2 \geq 0.$$

where $\alpha = 0.33$, $\tilde{\lambda} = (1,6,7,9)$ and membership function $\mu_{\tilde{\lambda}}(\lambda)$ of the form (2).

Use the hybrid algorithm (I):to solve the above problem, (let $w_1^* = w_2^* = 0.5$).

Solution:

$$\mu_{\tilde{\lambda}}(\lambda)=\begin{cases}0, & \lambda\le 1,\\ 1-\left[\frac{(\lambda-6)}{(1-6)}\right]^2, & 1\le\lambda\le 6,\\ 1, & 6\le\lambda\le 7,\\ 1-\left[\frac{(\lambda-7)}{(9-7)}\right]^2, & 7\le\lambda\le 9,\\ 0, & 9\le\lambda\quad .\end{cases}$$

$\mu_{\tilde{\lambda}}(\lambda)\ge 0.33$

$1-\left[\frac{(\lambda-6)}{(1-6)}\right]^2\ge 0.33$ where $1\le\lambda\le 6$,

$\Rightarrow\ 25-(6-\lambda)^2\ge 8.25,\quad\Rightarrow\quad 16.75\ge(6-\lambda)^2,\quad\Rightarrow$ $\pm\sqrt{16.75}\ge(6-\lambda)^2$,

$\sqrt{16.75}\ge 6-\lambda\ \Rightarrow\ \lambda\ge\frac{12-\sqrt{67}}{2}$, Accepted

$-\sqrt{16.75}\ge 6-\lambda\ \Rightarrow\ \lambda\ge\frac{12+\sqrt{67}}{2}$, Rejected.

$1-\left[\frac{(\lambda-7)}{(9-7)}\right]^2\ge 0.33$ where $7\le\lambda\le 9$,

$\Rightarrow\ 4-(\lambda-7)^2\ge 1.32,\quad\Rightarrow\ 2.67\ge(\lambda-7)^2,\quad\Rightarrow$ $\pm\sqrt{2.68}\ge(\lambda-7)^2$,

$\sqrt{2.68}\ge\lambda-7\ \Rightarrow\ \lambda\le\frac{35-\sqrt{67}}{5}$, Accepted

$-\sqrt{2.68}\ge\lambda-7\ \Rightarrow\ \lambda\le\frac{35+\sqrt{67}}{5}$, Rejected.

Thus, $\frac{12-\sqrt{67}}{2}\le\lambda\le\frac{35+\sqrt{67}}{5}$.

Apply the weighting method, follows:

$Maximize\ F=[0.5(6x_1+x_2)+0.5(x_1+2x_2)]$

subject to

$3x_1+2x_2\le\lambda,\ 5x_1\le 9,\ \lambda\ge\frac{12-\sqrt{67}}{2},\lambda\le\frac{35+\sqrt{67}}{5},\ x_1,x_2,\lambda\ge 0.$

By solving the above linear programming problem, the α-Pareto optimal solution is: $F^*\cong 6.93$, $x_1^*\cong 1.8$, $x_2^*\cong 1.62$, $\lambda^*\cong 8.64$.

Also, to find an α-Pareto optimal solution to example (I), A MATLAB code (I) based on hybrid algorithm (I) is introduced as follows:

<u>MATLAB Code (I):</u>

```
%weight of the first objective
w1=0.5;
%weight of the second objective
w2=0.5;
%first objective
obj1=[6, 1 ,0];
%second objective
obj2=[-1, 2 ,0 ];
%LHS of the constraints
LHS_C=[3 2 -1; 5 0 0];
%RHS of the constraints
RHS_C=[0;9];
alpha=0.33;
%intervals of the fuzzy number
fznum1=[1,6,7,9];
syms x;
%The following is solving the
%member ship function using
%the alpha cut in order to get
%the ranges of the fuzzy number
%Array that carries only
%the accepted ranges
 accepted=[];
 %The first part of the
 % membership function
 s=solve((1-((x  -  fznum1(2))/(fznum1(1)-fznum1(2)))^2)>=
alpha,x);
 %converts symbolic answer to double
 interval=double(s(1));
 %I have to make sure that
 %the ranges falls in the
 %interval of the membership
 %function
 if interval(1)>= fznum1(1)&& interval(1) <= fznum1(2)
 %Add to the end of
 %the accepted array
  accepted(end+1)=interval(1);
 end
 if interval(2)>= fznum1(1)&& interval(2) <= fznum1(1)
  accepted(end+1)=interval(2);
 end
 %The second part of the
 % membership function
 s2=solve((1-((x-fznum1(3))/(fznum1(4)-fznum1(3)))^2)>=
alpha,x);
 %Converts symbolic
 %answer to double
 interval2=double(s2(1));
 %I have to make sure
 %that the ranges falls
 %in the interval of the
 %membership function
 if interval2(1)>= fznum1(3)&& interval2(1) <= fznum1(4)
  accepted(end+1)=interval2(1);
 end
 if interval2(2)>= fznum1(3)&& interval2(2) <= fznum1(4)
  accepted(end+1)=interval2(2);
 end
 fprintf('The final range of the fuzzy number #1')
 accepted
%Multiply the objective functions
%by the weights and -1,
% since they are
%maximization, and
%add them together to
%have a single objective
obj=-1*w1*obj1 +-1*w2*obj2;
%add two new constraints
%containing the fuzzy
%numbers ranges
N_LHS_C=cat(1,LHS_C, [0 0 -1],[0 0 1 ]);
N_RHS_C=cat(1,RHS_C,[-1*accepted(1);accepted(2)]);
%solve the new problem
%using linear programming
%function
lb = zeros(3,1);
[x,fval]=linprog(obj,N_LHS_C,N_RHS_C,[],[],lb);
fprintf('The optimal objective function value :')
z=-fval
```

```
fprintf('The optimal values of the decision variables')
x

RUN

The final range of the fuzzy number #1
accepted =
  1.9073  8.6371
Optimization terminated.
The optimal objective function value :
z =
  6.9278
The optimal values of the decision variables
x =
  1.8000
  1.6185
  8.6371
```

4. LINEAR MULTIPLE OBJECTIVE OPTIMIZATION PROBLEMS WITH FUZZY PARAMETERS IN THE OBJECTIVE FUNCTIONS

Consider the following LMOO problem involving fuzzy numbers in the objective functions $(\text{LMOO})_{\tilde{\lambda}}$:

$(\text{LMOO})_{\tilde{\lambda}}$:

$$Maximize\left(f_1(X,\tilde{\lambda}_1), f_2(X,\tilde{\lambda}_2), \dots, f_k(X,\tilde{\lambda}_k)\right) \quad (7)$$

subject to

$$X \in \mathbb{X}=\{ X \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij}x_j \le b_i, i \in \mathbb{V}, \quad x_j \ge d_j > 0, j \in \mathbb{N}\},$$

where

$$f_r(X,\tilde{\lambda}_r) = (C_r + \tilde{\lambda}_r C_r^{/})^t X, \quad r \in \mathbb{K}, \quad (8)$$

$\mathbb{K} := \{1,2,\dots,k\}$,
$\tilde{\lambda}_r$: are n-vector of fuzzy numbers involved in the objective functions, $r \in \mathbb{K}$,
t : denotes transpose,
C_r : is n-vector of the coefficients, $r \in \mathbb{K}$,
$C_r^{/}$: is a diagonal matrix of dimension n.
d_j : are certain lower bounds for the decision $x_j, j \in \mathbb{N}$.

It is assumed that, $\tilde{\lambda}_r$, $r \in \mathbb{K}$ in problem (7), are fuzzy numbers whose membership functions are $\mu_{\tilde{\lambda}}(\lambda_r), r \in \mathbb{K}$.

By introducing the concept of α-level set or α-cut of the fuzzy numbers $\tilde{\lambda}_r$, $r \in \mathbb{K}$, then problem (7), for a certain degree α , can be understood as the following nonfuzzy α-linear multiple objective programming $(\alpha - \text{LMOO})_{\lambda}$ problem:

$(\alpha - \text{LMOO})_{\lambda}$:

$$Maximize\left(f_1(X,\lambda_1), f_2(X,\lambda_2), \dots, f_k(X,\lambda_k)\right) \quad (9\text{-}1)$$

subject to

$$X \in \mathbb{X}(b)=\{ X \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij}x_j \le b_i, i \in \mathbb{V}, \quad x_j \ge d_j > 0, j \in \mathbb{N}\}, \quad (9\text{-}2)$$

$$\lambda \in L_\alpha(\tilde{\lambda}) \quad (9\text{-}3)$$

where $L_\alpha(\tilde{\lambda})$ is the α-level set of the fuzzy numbers, $\tilde{\lambda}_r, r \in \mathbb{K}$.

Based on the definition of the α-level set of the fuzzy numbers, the concept of α-Pareto optimal solution to the α-linear multiple objective optimization problem (9) can be introduced in the following definition, [8-12]:

Definition (5):

A point $X^* \in \mathbb{X}$ is said to be an α-Pareto optimal solution to the $(\alpha - \text{LMOO})_{\lambda}$ problem (9) , if and only if there does not exist another $X \in \mathbb{X}(b)$, $\lambda^* \in L_\alpha(\tilde{\lambda})$, such that $f_r(X,\lambda) \ge f_r(X^*,\lambda^*), (r = 1,2,\dots,k)$, with strictly inequality holding for at least one r , where the corresponding value of number λ^* is called $\alpha -$level optimal number and $L_\alpha(\tilde{\lambda})$ is the $\alpha -$level set of the fuzzy number $\tilde{\lambda}$.

To find an α-Pareto optimal solution to the $(\alpha - \text{LMOO})_{\lambda}$ problem (9), a hybrid algorithm based on the α-level set of the fuzzy numbers and the weighting method is introduced as follows:

Hybrid Algorithm (II):

Step (1): Transform problem (7) to the form of problem (9).

Step (2): Use the weighting method and the α-level set method to transform problem (9) to the following form:

$$Maximize \sum_{r=1}^{k} w_r f_r(X,\lambda_r) \quad (10\text{-}1)$$

subject to

$$X \in \mathbb{X}=\{ X \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij}x_j \le b_i, i \in \mathbb{V}, \quad x_j \ge d_j > 0, j \in \mathbb{N}\}, \quad (10\text{-}2)$$

$$\lambda \in L_\alpha(\tilde{\lambda}) \quad (10\text{-}3)$$

where $w_r \ge 0, \ r = 1,2,\dots,k, \ \sum_{r=1}^{k} w_r = 1$.

Step (3): Let $\alpha = \alpha^*$, and use the membership function $\mu_{\tilde{\lambda}}$ of the form (1) or (2) to transform problem (5) to the following form:

$$Maximize \sum_{r=1}^{k} w_r f_r(X,\lambda_r) \quad (11\text{-}1)$$

subject to

$$X \in \mathbb{X}=\{ X \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij}x_j \le b_i, i \in \mathbb{V}, \quad x_j \ge d_j > 0, \ j \in \mathbb{N}\}, \quad (11\text{-}2)$$

$$l_{rj} \le \lambda_{rj} \le L_{rj}, r \in \mathbb{K}, j \in \mathbb{N} . \quad (11\text{-}3)$$

where $w_r \ge 0, \ r = 1,2,\dots,k, \ \sum_{r=1}^{k} w_r = 1$.

The constraint (10-3) is replaced by the constraint (11-3), where l_{rj} and L_{rj} represent the lower and upper bound on λ_{rj} respectively.

Step (4): (I) The nonlinearity in the objective functions of problem (11) can be treated using the following transformation:

$$y_{rj} = \lambda_{rj} x_j , r \in \mathbb{K}, j \in \mathbb{N}. \quad (12)$$

(II) Consequently, problem (11) becomes:

$$Maximize\ \sum_{r=1}^{k} w_r\ (C_r^t X + C_r^{/t} y_r) \qquad (13\text{-}1)$$

subject to

$$X \in \mathbb{X}=\{ X \in \mathbb{R}^n : \sum_{j=1}^{n} a_{ij}x_j \le b_i, i \in \mathbb{V},$$
$$x_j \ge d_j > 0, j \in \mathbb{N}\}, \qquad (13\text{-}2)$$

$$l_{rj}x_j \le y_{rj} \le L_{rj}x_j\ , r \in \mathbb{K}, j \in \mathbb{N}. \qquad (13\text{-}3)$$

where $w_r \ge 0,\ \ r = 1,2, \ldots, k,\ \ \sum_{r=1}^{k} w_r = 1$.

Step (5): Let $w_r = w_r^* \ge 0$, then use the simplex (or any other method) to solve problem (11).

Step (6): Let X^* is the α-Pareto optimal solution of problem (13) and the corresponding α -level optimal numbers:

$$\lambda_{rj}^* = \frac{y_{rj}^*}{x_{rj}^*}, r \in \mathbb{K}, j \in \mathbb{N}, \qquad (14)$$

(I) If $w_r > 0$, for all r , thus X^* is an efficient solution, go to step (7).

(II) If $w_r \ge 0$, for all r , and X^* is a unique for problem (13), thus X^* is an efficient solution, go to step (7).

(III) If $w_r \ge 0$, for all r , and there are alternative solutions, thus, use the non-inferiority test.

Step (7): Stop.

Example (II):

Consider the following linear multiple objective optimization problem involving fuzzy numbers in the objective functions:

$$Maximize\ f_1(x_1, \tilde{\lambda}_1) = -2x_1 + (3 + \tilde{\lambda}_{12})x_2,$$
$$Maximize\ f_2(x_1, \tilde{\lambda}_2) = (-2 + \tilde{\lambda}_{21})x_1 + x_2,$$

subject to

$$2x_1 \le 7, \quad 4x_2 \le 9, \quad x_1 \ge 1, x_2 \ge 1 .$$

where $\alpha = 0.36$, $\tilde{\lambda}_{12} = 90,1,3,5)$, $\tilde{\lambda}_{21} = (1,6,7,9)$ and membership function $\mu_{\tilde{\lambda}}(\lambda)$ of the form (2).

Use the hybrid algorithm (II) to solve the above problem, (let $w_1^* = w_2^* = 0.5$).

Solution:

$$\mu_{\tilde{\lambda}_{12}}(\lambda_{12}) = \begin{cases} 0, & \lambda_{12} \le 0, \\ 1 - \left[\frac{(\lambda_{12}-1)}{(0-1)}\right]^2, & 0 \le \lambda_{12} \le 1, \\ 1, & 1 \le \lambda_{12} \le 3, \\ 1 - \left[\frac{(\lambda_{12}-3)}{(5-3)}\right]^2, & 3 \le \lambda_{12} \le 5, \\ 0, & 5 \le \lambda_{12} \quad . \end{cases}$$

$\mu_{\tilde{\lambda}}(\lambda) \ge 0.36$

$1 - \left[\frac{(\lambda_{12}-1)}{(0-1)}\right]^2 \ge 0.36$ where $0 \le \lambda \le 1$,

$\Rightarrow\ 1 - (-\lambda_{12} + 1)^2 \ge 0.36, \ \Rightarrow\ 1 - 036 \ge (1 - \lambda_{12})^2, \ \Rightarrow$ $064 \ge (1 - \lambda_{12})^2, \Rightarrow\ \pm 0.8 \ge 1 - \lambda_{12},$

$0.8 \ge 1 - \lambda_{12}\ \Rightarrow\ \lambda_{12} \ge 0.2$, Accepted

$-0.8 \ge 1 - \lambda_{12}\ \Rightarrow\ \lambda_{12} \ge 1.8$, Rejected.

$1 - \left[\frac{(\lambda_{12}-3)}{(5-3)}\right]^2 \ge 0.36$ where $3 \le \lambda \le 5$,

$\Rightarrow\ 4 - (\lambda_{12} - 3)^2 \ge 1.44, \quad \Rightarrow\ 2.56 \ge (\lambda_{12} - 3)^2, \quad \Rightarrow$ $\pm 1.6 \ge \lambda_{12} - 3,$

$1.6 \ge \lambda_{12} - 3 \Rightarrow\ \lambda_{12} \le 4.6$, Accepted

$-1.6 \ge \lambda_{12} - 3\ \Rightarrow\ \lambda_{12} \le 1.4$, Rejected.

Thus, $0.2 \le\ \lambda_{12} \le 4.6$.

$$\mu_{\tilde{\lambda}_{21}}(\lambda_{21}) = \begin{cases} 0, & \lambda_{21} \le 1, \\ 1 - \left[\frac{(\lambda_{21}-6)}{(1-6)}\right]^2, & 1 \le \lambda_{21} \le 6, \\ 1, & 6 \le \lambda_{21} \le 7, \\ 1 - \left[\frac{(\lambda_{21}-7)}{(9-7)}\right]^2, & 7 \le \lambda_{21} \le 9, \\ 0, & 9 \le \lambda_{21} \quad . \end{cases}$$

$1 - \left[\frac{(\lambda_{21}-6)}{(1-6)}\right]^2 \ge 0.36$ where $1 \le \lambda \le 6$,

$\Rightarrow\ 1 - \left[\frac{(\lambda_{21}-6)}{(-5)}\right]^2 \ge 0.36, \quad \Rightarrow \quad 25 - (6 - \lambda_{21})^2 \ge 9, \quad \Rightarrow$ $16 \ge (6 - \lambda_{21})^2, \pm 4 \ge 1 - \lambda_{21},$

$4 \ge 1 - \lambda_{21}\ \Rightarrow\ \lambda_{21} \ge 2$, Accepted.

$-4 \ge 1 - \lambda_{21}\ \Rightarrow\ \lambda_{21} \ge 10$, Rejected.

$1 - \left[\frac{(\lambda_{21}-7)}{(9-7)}\right]^2 \ge 0.36$ where $7 \le\ \lambda_{21} \le 9$,

$\Rightarrow\ 4 - (\lambda_{21} - 7)^2 \ge 1.44, \quad \Rightarrow \quad 2.56 \ge (\lambda_{21} - 7)^2, \quad \Rightarrow$ $\pm 1.6 \ge \lambda_{21} - 7,$

$1.6 \ge \lambda_{21} - 7 \Rightarrow\ \lambda_{21} \le 8.6$, Accepted.

$-1.6 \ge \lambda_{21} - 7\ \Rightarrow\ \lambda_{21} \le 5.4$, Rejected.

Thus, $2 \le\ \lambda_{12} \le 8.6$.

Apply the weighting method, follows:

$$Maximize\ F = [0.5(-2x_1 + (3 + \lambda_{12})x_2) + 0.5((-2 + \lambda_{21})x_1 + x_2)] = \\ = [-2x_1 + 2x_2 + 0.5\lambda_{12}x_2 + 0.5\lambda_{21}x_1]$$

subject to

$2x_1 \le 7, \quad 4x_2 \le 9, \quad x_1 \ge 1, x_2 \ge 1\ , \ \ \lambda_{21}x_1 \ge 2x_1\ ,$ $\lambda_{21}x_1 \le 8.6x_1\ , \lambda_{12}x_2 \ge 0.2x_2\ , \lambda_{12}x_2 \le 4.6x_2,$ $\lambda_{12}, \lambda_{21} \ge 0.$

Let $y_{12} = \lambda_{12}x_2,\ y_{21} = \lambda_{21}x_1$.

$$Maximize\ F = [-2x_1 + 2x_2 + 0.5y_{12} + 0.5y_{21}]$$

subject to

$2x_1 \le 7, \quad 4x_2 \le 9, \quad x_1 \ge 1, x_2 \ge 1\ , y_{21} - 2x_1 \ge 0$, $y_{21} - 8.6x_1 \le 0$, $y_{12} - 0.2x_2 \ge 0$, $y_{12} - 4.6x_2 \le 0$, $y_{12}, y_{21} \ge 0.$

By solving the above linear programming problem, the α-Pareto optimal solution is: $F^* \cong 17.73$, $x_1^* \cong 3.5$, $x_2^* \cong 2.25$, $y_{12}^* \cong 10.35$, $y_{21}^* \cong 30.1$, λ_{12}^*=4.6, λ_{21}^*=8.6.

To find an α-Pareto optimal solution for example (II), A MATLAB code (II) based on hybrid algorithm (II) is introduced as follows:

MATLAB Code (II):

```
%weight of the first objective
w1=0.5;
%weight of the second objective
w2=0.5;
%first objective, max
```

```
obj1=[-2,3,1,0];
%second objective, max
obj2=[-2,1,0,1];
%LHS of the constraints
LHS_C=[2 0 0 0; 0 4 0 0; -1 0 0 0; 0 -1 0 0];
%RHS of the constraints
RHS_C=[7;9;-1;-1];
alpha=0.36;
%intervals of the
%first fuzzy number
fznum1=[0,1,3,5];
%intervals of the
%second fuzzy number
fznum2=[1,6,7,9];
syms x;
%The following is solving
%the member ship function
%using the alpha cut in
%order to get the ranges
%of the fuzzy number
%Array that carries only
%the accepted ranges of
%the first fuzzy number
   accepted=[];
  %The first part of
  %the membership function
  s=solve((1-((x-    fznum1(2))/(fznum1(1)-fznum1(2)))^2)>= alpha,x);
  interval=double(s(1));%converts symbolic answer to double
  %we have to make sure
  %that the ranges falls
  %in the interval of the
  %membership function
  if interval(1)>= fznum1(1)&& interval(1) <= fznum1(2)
    accepted(end+1)=interval(1); %add to the end of the accepted array
  end
  if interval(2)>= fznum1(1)&& interval(2) <= fznum1(1)
    accepted(end+1)=interval(2);
  end
  %the second part of
  %the membership function
  s2=solve((1-((x-  fznum1(3))/(fznum1(4)-fznum1(3)))^2)>= alpha,x);
  %converts symbolic answer to double
  interval2=double(s2(1));
  %we have to make sure that
  %the ranges falls in the
  %interval of the
  %membership function
  if interval2(1)>= fznum1(3)&& interval2(1) <= fznum1(4)
    accepted(end+1)=interval2(1);
  end
  if interval2(2)>= fznum1(3)&& interval2(2) <= fznum1(4)
    accepted(end+1)=interval2(2);
  end
  fprintf('The final range of the fuzzy number #1')
  accepted
  %array that carries only
  %the accepted ranges of
  %the second fuzzy number
  accepted2=[];
  %The first part of the
  %membership function
  s=solve((1-((x-    fznum2(2))/(fznum2(1)-fznum2(2)))^2)>= alpha,x);
  %converts symbolic answer to double
  interval=double(s(1));
  %we have to make sure
  %that the ranges falls
  %in the interval of the
  %membership function
  if interval(1)>= fznum2(1)&& interval(1) <= fznum2(2)
    accepted2(end+1)=interval(1); %add to the end of the accepted array
  end

  if interval(2)>= fznum2(1)&& interval(2) <= fznum2(1)
    accepted2(end+1)=interval(2);
  end
  %the second part of
  %the membership function
  s2=solve((1-((x-  fznum2(3))/(fznum2(4)-fznum2(3)))^2)>= alpha,x);
  %converts symbolic answer to double
  interval2=double(s2(1));
  %we have to make sure that
  %the ranges falls in the interval of the
  %membership function
  if interval2(1)>= fznum2(3)&& interval2(1) <= fznum2(4)
    accepted2(end+1)=interval2(1);
  end
  if interval2(2)>= fznum2(3)&& interval2(2) <= fznum2(4)
    accepted2(end+1)=interval2(2);
  end
  fprintf('The final range of the fuzzy number #2')
  accepted2
%multiply the objective functions
%by the weights and -1,
%since they are
%maximization, and add them
%together to have a single objective
obj=-1*w1*obj1 +-1*w2*obj2;
%add four new constraints containing
%the two fuzzy numbers ranges
N_LHS_C=cat(1,LHS_C, [0 accepted(1) -1 0],[0 -1*accepted(2) 1 0],[ accepted2(1) 0 0 -1],[ -1*accepted2(2) 0 0 1]);
N_RHS_C=cat(1,RHS_C,[0;0;0;0]);
%solve the new problem using
%linear programming function
lb = zeros(4,1);
N_RHS_C
N_LHS_C
[x,fval]=linprog(obj,N_LHS_C,N_RHS_C,[],[],lb);
fprintf('The optimal objective function value :')
z=-fval
fprintf('The optimal values of the decision variables')
x
```

RUN

```
The final range of the fuzzy number #2
accepted2 =
  2.0000  8.6000

N_RHS_C =
   7
```

```
  9
 -1
 -1
  0
  0
  0
  0

N_LHS_C =

  2.0000      0      0      0
       0  4.0000      0      0
 -1.0000      0      0      0
       0 -1.0000      0      0
       0  0.2000 -1.0000      0
       0 -4.6000  1.0000      0
  2.0000      0      0 -1.0000
 -8.6000      0      0  1.0000

Optimization terminated.
The optimal objective function value :
z =
  17.7250
```

5. CONCLUSIONS

The provided MATLAB codes use the α-Level sets method to transform the fuzzy LMOO problems to non-fuzzy LMOO problems and the weighting method to obtain an α-Pareto optimal solution to the non-fuzzy LMOO problems.

The hand solutions of the numerical examples by the hybrid algorithms and the solutions by the MATLAB codes are identical.

The scientists and the engineers can apply the presented codes and the hybrid algorithms to different practical fuzzy LMOO problems to obtain numerical solutions.

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