Advances in Modelling and Analysis A
Vol. 55, No. 2, June, 2018, pp. 57-61
Journal homepage: http://iieta.org/Journals/AMA/AMA_A

Study on existence of solution for some fractional integro differential equations via the iterative process

Somayyeh Dadsetadi, Kazem Nouri*

Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P.O. Box 35195-363, Semnan, Iran

Corresponding Author Email: knouri@semnan.ac.ir

https://doi.org/10.18280/ama_a.550203

Received: 22 April 2018
Accepted: 8 June 2018

Keywords:
fractional Hammerstein integro-differential equations, Caputo fractional derivative, iterative method

ABSTRACT

We study the existence and uniqueness of solution of nonlinear fractional integro-differential equations of the Hammerstein type, using the iterative method under some suitable conditions in the Banach space. At the end, an example is given to illustrate the theory.

1. INTRODUCTION

Although the fractional calculus is not a new topic, but in the recent years, it has significant growth due to its applications in many areas of concepts of physics, mathematics and engineering, the interested reader can refer to the numerous recent works [7, 10-15] and references therein.

Investigations on the existence and uniqueness of solution for fractional differential and integral equations have been recently presented in several literatures [1-5, 8-9, 16-17]. Using fixed point theorems is a basic technique in studying various equations [1, 6, 8, 16]. Anguraj et al. [3] studied the existence and uniqueness theorem for the nonlinear fractional mixed Volterra-Fredholm integro-differential equation with nonlocal initial condition. Balachandran et al. [5] discussed the existence of solutions of first order nonlinear impulsive fractional integro-differential equations in the Banach spaces. In [8], Gautam and Dabas established the existence and uniqueness results of solutions for class of an abstract fractional functional integro-differential equations with state dependent delay subject to not instantaneous impulse using the fixed point theorems. In [17], Wang proved the existence and uniqueness of solutions for a class of the nonlinear fractional differential equation with initial condition and investigated the dependence of the solution on the order of the differential equation and on the initial condition.

In this work, we will use an iterative method to investigation the existence and uniqueness of solution of the following nonlinear fractional Volterra integro-differential equations of the Hammerstein type

$$\left({}^{C}D^{\beta} v\right)(\tau)=g(\tau)+\int_0^{\tau} k(\tau,\eta)\, G\left({}^{C}D^{\gamma} v(\eta)\right)d\eta, \quad \tau\in[0,a], \tag{1}$$

subject to the initial conditions

$$v^{(j)}(0)=v_j, \quad j=0,1,\ldots,m-1, \tag{2}$$

where for $m,n\in\mathbb{N}$, $m-1<\beta<m$, $n-1<\gamma<n$, $\gamma<\beta$ and the fractional derivatives are considered in the Caputo sense. Also, G is an increasing linear transformation on the Banach space *x*.

2. PRELIMINARIES

In this section, we recall some basic definitions and necessary facts of the fractional calculus (for more details see [4] and [10]). Throughout of this paper, we consider the complete metric space (χ,d) which

$$d(h,g)=\max_{\tau\in[0,a]}|h(\tau)-g(\tau)|,$$

for all $h,g\in\chi$.

The Riemann-Liouville fractional integral and the Caputo fractional derivative play main roles in fractional calculus, thus the definition of them will be expressed.

Definition 1. The Riemann-Liouville fractional integral of order $\beta>0$ of a function $v(\tau)$, is defined as

$$(I^{\beta} v)(\tau)=\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\mu)^{\beta-1} v(\mu)\,d\mu, \quad \tau>0, \tag{3}$$

where Γ denotes the Gamma function.

Definition 2. The Caputo derivative of fractional order $\beta\geq 0$ for a function $v(\tau)$ is defined by

$$\left({}^{C}D^{\beta} v\right)(\tau)=\frac{1}{\Gamma(m-\beta)}\int_0^{\tau}(\tau-\mu)^{m-\beta-1} v^{(m)}(\mu)d\mu,$$

where $m=[\beta]+1$ and $[\beta]$ denotes integer part of the real number β. If $\beta=m\in\mathbb{N}_0$ and the usual derivative $v^{(m)}(\tau)$ of order m exists, then $({}^{C}D^{m}v)(\tau)$ coincides with $v^{(m)}(\tau)$.

Also, this definition implies that ${}^{C}D^{\beta}v^{(n)}(\tau) = {}^{C}D^{\beta+n}\, v(\tau)$ and ${}^{C}D^{\beta}z = 0$ (z is a constant).

Proposition 1. Let $\beta, \gamma > 0$ and $m = [\beta] + 1$. *If* $v(\tau) \in C^{m}[0,a]$, then

(i) $\left(I^{\beta}\ {}^{C}D^{\beta}\, v\right)(\tau) = v(\tau) - \sum_{j=0}^{m-1} \frac{v^{(j)}(0)}{j!}\tau^{j},$

(ii) $\left({}^{C}D^{\beta}I^{\beta}\, v\right)(\tau) = v(\tau),$

(iii) $\left(I^{\beta}I^{\gamma}\, v\right)(\tau) = \left(I^{\beta+\gamma}\, v\right)(\tau).$

Lemma 1. ([1]) Let $m-1 < \beta < m$, $n-1 < \gamma < n$ and $\gamma < \beta$. For $\tau \in [0,a]$,

(i) if $v(\tau) \in C[0,a]$, then $\left({}^{C}D^{\gamma}I^{\beta}\, v\right)(\tau) = \left(I^{\beta-\gamma}\, v\right)(\tau),$

(ii) if $v \in C^{m-1}[0,a]$ and $\left({}^{C}D^{\beta}\, v\right)(\tau) \in C[0,a]$, then $\left({}^{C}D^{\gamma}v\right)(\tau) \in C[0,a]$.

Proposition 2. Let $\beta, \gamma > 0$ and $m-1 < \beta < m$, then

$$\left(I^{\beta}\left(\eta - a\right)^{\gamma-1}\right)(\tau) = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\beta)}\left(\tau - a\right)^{\gamma+\beta-1},$$

$$\left({}^{C}D^{\beta}\left(\eta - a\right)^{\gamma-1}\right)(\tau) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)}\left(\tau - a\right)^{\gamma-\beta-1},\ \gamma > m, \text{and}$$

$$\left({}^{C}D^{\beta}\left(\eta - a\right)^{j}\right)(\tau) = 0,\quad j = 0,1,\ldots,m-1.$$

The following lemma is a result of Lemma 1.3 in [9] which characterizes the space $C^{n}[0,a]$.

Lemma 2. Let $n \in \mathbb{N}_0$. The space $C^{n}[0,a]$ consists of those and only those functions h which are represented in the form

$$h(\tau) = \frac{1}{(n-1)!}\int_0^{\tau}(\tau-\eta)^{n-1}\varphi(\eta)d\eta + \sum_{j=0}^{n-1}c_j\tau^{j},$$

where $\varphi(\eta) \in C[0,a]$ and c_j ($j = 0,1,\ldots,n-1$) are appropriate constants. Moreover,

$$\varphi(\eta) = h^{(n)}(\eta),\ c_j = \frac{h^{(j)}(0)}{j!}\ (j = 0,1,\ldots,n-1).$$

3. EXPLANATION OF THE PROBLEM

In this section, we prove a theorem to show that the problem (1) - (2) is equivalent to an integral equation of fractional order.

Theorem 1. Let g, k and G be continuous functions, $m-1 < \beta < m,$ $n-1 < \gamma < n$ and $\gamma < \beta.$ Then a function $v \in C^{m-1}[0,a]$ with $\left({}^{C}D^{\beta}\, v\right)(\tau) \in C[0,a]$ is a solution of fractional integro-differential equation (1) if and only if

$$v(\tau) = \sum_{j=0}^{n-1}\frac{v_j}{j!}\tau^{j} + \frac{1}{\Gamma(\gamma)}\int_0^{\tau}\frac{\sigma(s)}{(\tau-s)^{1-\gamma}}ds, \tag{4}$$

satisfies the integral equation

$$\sigma(\tau) = \sum_{j=n}^{m-1}\frac{v_j}{\Gamma(j-\gamma+1)}\tau^{j-\gamma} + I^{\beta-\gamma}g(\tau) + I^{\beta-\gamma}\int_0^{\tau}k(\tau,\eta)\,G(u(\eta))d\eta, \tag{5}$$

where for $n \le m,$ $\sigma \in C[0,a].$

Proof. Let $v \in C^{m-1}[0,a]$ be a solution of (1) which $\left({}^{C}D^{\beta}v\right)(\tau) \in C[0,a].$ Using Lemma 1, we conclude $\left({}^{C}D^{\gamma}v\right)(\tau) \in C[0,a].$ Since g, k, G and $\left({}^{C}D^{\gamma}v\right)(\tau)$ are continuous, we can apply the operator I^{β} to both sides of Eq. (1). Thus using Proposition 1, we obtain

$$v(\tau) = \sum_{j=0}^{m-1}\frac{v^{(j)}(0)}{j!}\tau^{j} + I^{\beta}g(\tau) + I^{\beta}\left(\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right). \tag{6}$$

Putting $\left({}^{C}D^{\gamma}v\right)(\tau) := \sigma(\tau),$ so $\sigma \in C[0,a],$ and we can apply the operator I^{γ} to both sides of this relation and using Proposition 1, we get

$$v(\tau) = \sum_{j=0}^{n-1}\frac{v_j}{j!}\tau^{j} + \frac{1}{\Gamma(\gamma)}\int_0^{\tau}\frac{\sigma(s)}{(\tau-s)^{1-\gamma}}ds.$$

From Eq. (6) and Lemma 1, we have

$${}^{C}D^{\gamma}v(\tau) = {}^{C}D^{\gamma}\left(\sum_{j=0}^{m-1}\frac{v_j}{j!}\tau^{j}\right) + I^{\beta-\gamma}g(\tau) + I^{\beta-\gamma}\left(\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right). \tag{7}$$

Using Proposition 2, we get

$$\sigma(\tau) = \sum_{j=n}^{m-1}\frac{v_j}{\Gamma(j-\gamma+1)}\tau^{j-\gamma} + I^{\beta-\gamma}g(\tau) + I^{\beta-\gamma}\int_0^{\tau}k(\tau,\eta)\,G(\sigma(\eta))d\eta,$$

and for $n = m$ the first term of the right hand of above relation is equal to zero. Conversely, assume that $\sigma \in C[0,a]$ is a solution of Eq. (5), we show that Eq.(4) satisfies in Eq. (1). Since $\sigma \in C[0,a],$ we can apply the operator ${}^{C}D^{\gamma}$ to both sides of Eq. (4), then from Proposition 1 and 2, we obtain

$$\left({}^{C}D^{\gamma}v\right)(\tau) = \sigma(\tau),$$

and hence $\left({}^{C}D^{\gamma}v\right)(\tau) \in C[0,a].$ Applying I^{γ} to both sides of Eq. (5) and using Propositions 1 and 2, we get

$$v(\tau)=\sum_{j=0}^{m-1}\frac{v_j}{j!}\tau^j+I^{\beta}g(\tau)+I^{\beta}\left(\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right), \tag{8}$$

where according to Proposition 2, we obtain

$$I^{\gamma}\left(\sum_{j=n}^{m-1}\frac{v_j}{\Gamma(j-\gamma+1)}\tau^{j-\gamma}\right)=\sum_{j=n}^{m-1}\frac{v_j}{j!}\tau^j.$$

By Propositions 1 and 2, the continuity of $\left({}^{C}D^{\gamma}v\right)(\tau)$, g,k,G and applying ${}^{C}D^{\beta}$ to both sides of Eq. (8), we have

$$\left({}^{C}D^{\beta}v\right)(\tau)=g(\tau)+\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta,$$

and consequently $\left({}^{C}D^{\beta}v\right)(\tau)\in C[0,a]$. Now we show that $v^{(j)}(0)=v_j\ (j=0,1,\ldots,m-1)$. First using the property of the fractional calculus, we obtain

$$|\left(I^{\alpha}v\right)^{(j)}(\tau)|=|\left(D^{j}I^{\beta}v\right)(\tau)|=|\left(I^{\beta-j}v\right)(\tau)|$$

$$=\left|\frac{1}{\Gamma(\beta-j)}\int_0^{\tau}(\tau-s)^{\beta-j-1}v(s)ds\right|$$

$$\le\frac{\|v\|_C}{\Gamma(\beta-j+1)}\tau^{\beta-j},$$

for $j=0,1,\ldots,m-1,$ thus

$$\left(I^{\beta}v\right)^{(j)}(0)=0,\quad j=0,1,\ldots,m-1. \tag{9}$$

Later on, for $m=1$ according to Eq. (8), we have

$$v(\tau)=v_0+I^{\beta}g(\tau)+I^{\beta}\left(\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right).$$

Using the continuity of the operator I^{β} on $C[0,a]$ and using Eq. (9), we find $v(\tau)\in C[0,a]$ and $v(0)=v_0\ (v^{(0)}(\tau)=v(\tau))$.

Now for $m\ge 2$ according to Eq. (8) and using Proposition 1, we have

$$v(\tau)=\sum_{j=0}^{m-2}\frac{v_j}{j!}\tau^j+I^{m-1}\Big[v_{m-1}+I^{\beta-m+1}g(\tau)$$
$$+I^{\beta-m+1}\left(\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right)\Big].$$

Thus from Lemma 1, we have $v(\tau)\in C^{m-1}[0,a]$, $v^{(j)}(0)=v_j$ for $j=0,1,\ldots,m-2,$ and

$$v^{(m-1)}(\tau)=v_{m-1}+I^{\beta-m+1}g(\tau)+I^{\beta-m+1}\left(\int_0^{\tau}k(\tau,\eta)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right).$$

In the same way of Eq. (9), we can show that

$$\left[I^{\beta-m+1}\left(\int_0^{\tau}k(\tau,t)\,G\left({}^{C}D^{\gamma}v(\eta)\right)d\eta\right)\right](0)=0,\qquad I^{\beta-m+1}g(0)=0.$$

Therefore, $v^{(m-1)}(0)=v_{m-1}$ and the proof is complete.

4. EXISTENCE AND UNIQUENESS

In this section, we study the existence and uniqueness theorem for solution of nonlinear fractional integro-differential equation (1). By Theorem 1, it is sufficient to show that Eq. (5) has a solution $\sigma\in C[0,a]$.

According to Definition 2 and by changing the order of integration, we have

$$\sigma(\tau)=h(\tau)+\int_0^{\tau}L(\tau,\eta)\,G(\sigma(\eta))d\eta, \tag{10}$$

where

$$h(\tau)=\sum_{j=n}^{m-1}\frac{v_j}{\Gamma(j-\gamma+1)}\tau^{j-\gamma}+\frac{1}{\Gamma(\beta-\gamma)}\int_0^{\tau}(\tau-\mu)^{\beta-\gamma-1}g(\mu)d\mu,$$

and

$$L(\tau,\eta)=\frac{1}{\Gamma(\beta-\gamma)}\int_{\eta}^{\tau}(\tau-\mu)^{\beta-\gamma-1}k(\mu,\eta)d\mu.$$

We define

$$L(\Omega)(\tau)=\int_0^{\tau}L(\tau,\eta)\,G(\sigma(\eta))d\eta, \tag{11}$$

from Eqs. (10) and (11), we have

$$\Omega=H+L(\Omega),\quad H\in\chi. \tag{12}$$

Now we define the operator $T:\chi\to\chi$ as follows

$$T\Omega=L(\Omega)+H,\quad \Omega,H\in\chi, \tag{13}$$

from Eqs. (12) and (13), we obtain

$$T\Omega=\Omega.$$

So, we can rewrite equation (10) as follows

$$\sigma(\tau)=h(\tau)+\int_0^{\tau}L(\tau,\eta)\,G(\sigma(\eta))d\eta\equiv T\sigma(\tau). \tag{14}$$

Moreover, let ω denotes the class of those functions $\xi:[0,\infty)\to[0,1)$ which satisfies the condition

$$\xi(\eta_n)\to 1,\quad \text{implies}\quad \eta_n\to 0. \tag{15}$$

To prove the existence and uniqueness of solution for Eq. (1), we present the following theorem.

Theorem 2. Consider the nonlinear Volterra integral equation (14) such that:

(i) $g:[0,a]\to\mathbb{R}$ and $k:[0,a]\times[0,a]\to\mathbb{R}$ are continuous,

(ii) $G:\chi\to\chi$ is an increasing linear transformation and $\xi(\tau)=\frac{G(\tau)}{\tau}\in\omega$, $\tau\neq 0$,

(iii) $\sup_{\tau\in[0,a]}\int_0^a L^2(\tau,\eta)d\eta\leq\frac{1}{a}$.

Then, the integral equation (14) has a unique fixed point σ in χ.

Proof. Consider the iterative process

$$\sigma_{n+1}(\tau)=h(\tau)+\int_0^\tau L(\tau,\eta)\,G(\sigma_n(\eta))d\eta \equiv T\sigma_n(\tau),\quad n=0,1,..., \tag{16}$$

where $\sigma_0\in\chi$ is an appropriate initial guess. So,

$$\begin{aligned}|T\sigma_n(\tau)-T\sigma_{n-1}(\tau)|&=\left|\int_0^\tau L(\tau,\eta)\,G(\sigma_n(\eta))d\eta-\int_0^\tau L(\tau,\eta)\,G(\sigma_{n-1}(\eta))d\eta\right|\\&\leq\int_0^\tau L(\tau,\eta)\,G\,|\sigma_n(\eta)-\sigma_{n-1}(\eta)|\,d\eta\\&\leq\left(\int_0^\tau L^2(\tau,\eta)d\eta\right)^{\frac{1}{2}}\left(\int_0^\tau G^2\,|\sigma_n(\eta)-\sigma_{n-1}(\eta)|\,d\eta\right)^{\frac{1}{2}}.\end{aligned}$$

As the function G is increasing then

$$G\left(|\sigma_n(\tau)-\sigma_{n-1}(\tau)|\right)\leq G\left(d(\sigma_n,\sigma_{n-1})\right),$$

so, we obtain

$$\begin{aligned}d^2(\sigma_{n+1},\sigma_n)&\leq\left(\sup_{\tau\in[0,a]}\int_0^a L^2(\tau,\eta)d\eta\right)G^2\left(d(\sigma_n,\sigma_{n-1})\right)a\\&\leq G^2\left(d(\sigma_n,\sigma_{n-1})\right).\end{aligned}$$

Therefore

$$\begin{aligned}d(\sigma_{n+1},\sigma_n)&\leq G\left(d(\sigma_n,\sigma_{n-1})\right)=\frac{G\left(d(\sigma_n,\sigma_{n-1})\right)}{d(\sigma_n,\sigma_{n-1})}d(\sigma_n,\sigma_{n-1})\\&=\xi\left(d(\sigma_n,\sigma_{n-1})\right)d(\sigma_n,\sigma_{n-1}),\end{aligned} \tag{17}$$

and the sequence $\{d(\sigma_{n+1},\sigma_n)\}$ is decreasing and bounded. Thus there exists $\mu\geq 0$ such that $\lim_{n\to\infty}d(\sigma_{n+1},\sigma_n)=\mu$. If $\mu>0$, then according to relation (17),

$$\frac{d(\sigma_{n+1},\sigma_n)}{d(\sigma_n,\sigma_{n-1})}\leq\xi\left(d(\sigma_n,\sigma_{n-1})\right),\quad n=1,2,...,$$

and we conclude that $\xi\notin\omega$, because $\lim_{n\to\infty}\xi\left(d(\sigma_n,\sigma_{n-1})\right)=1$ whereas $\lim_{n\to\infty}\left(d(\sigma_n,\sigma_{n-1})\right)=\mu>0$. So $\mu=0$ and therefore $\lim_{n\to\infty}d(\sigma_{n+1},\sigma_n)=0$. Now we show that $\{\sigma_n\}$ is a Cauchy sequence. Contrariwise, suppose that

$$\limsup_{m,n\to\infty}d(\sigma_n,\sigma_m)>0. \tag{18}$$

By the triangle inequality and relation (17), we have

$$\begin{aligned}d(\sigma_n,\sigma_m)&\leq d(\sigma_n,\sigma_{n+1})+d(\sigma_{n+1},\sigma_{m+1})+d(\sigma_{m+1},\sigma_m)\\&\leq d(\sigma_n,\sigma_{n+1})+\xi\left(d(\sigma_n,\sigma_m)\right)d(\sigma_n,\sigma_m)+d(\sigma_{m+1},\sigma_m),\end{aligned}$$

hence

$$d(\sigma_n,\sigma_m)\left[1-\xi\left(d(\sigma_n,\sigma_m)\right)\right]\leq d(\sigma_n,\sigma_{n+1})+d(\sigma_{m+1},\sigma_m).$$

Thus, we have

$$d(\sigma_n,\sigma_m)\leq\left(1-\xi\left(d(\sigma_n,\sigma_m)\right)\right)^{-1}\left[d(\sigma_n,\sigma_{n+1})+d(\sigma_{m+1},\sigma_m)\right].$$

Since $\limsup_{m,n\to\infty}d(\sigma_n,\sigma_m)>0$ and $\lim_{n\to\infty}d(\sigma_{n+1},\sigma_n)=0$, then

$$\limsup_{m,n\to\infty}\left(1-\xi\left(d(\sigma_n,\sigma_m)\right)\right)^{-1}=+\infty,$$

from the above relation, we conclude $\limsup_{m,n\to\infty}\xi\left(d(\sigma_n,\sigma_m)\right)=1$ and since $\xi\in\omega$, we obtain

$$\limsup_{m,n\to\infty}d(\sigma_n,\sigma_m)=0.$$

This contradicts with (18), shows $\{\sigma_n\}$ is a Cauchy sequence in χ. Since (χ,d) is a complete metric space, then $\{\sigma_n\}$ is a convergent sequence in χ, that is

$$\exists\sigma\in\chi,\quad\lim_{n\to\infty}\sigma_n=\sigma.$$

Now by taking the limit of both sides of (16), we have

$$\begin{aligned}\sigma(\tau)&=\lim_{n\to\infty}\sigma_{n+1}(\tau)=\lim_{n\to\infty}\left(h(\tau)+\int_0^\tau L(\tau,\eta)\,G(\sigma_n(\eta))d\eta\right)\\&=h(\tau)+\int_0^\tau L(\tau,\eta)\,G(\lim_{n\to\infty}\sigma_n(\eta))d\eta\\&=h(\tau)+\int_0^\tau L(\tau,\eta)\,G(\sigma(\eta))\,d\eta\equiv T\sigma(\tau).\end{aligned}$$

Thus, there exists a solution $\sigma\in\chi$ such that $T\sigma=\sigma$. It is clear that the fixed point of T is unique.

5. APPLICATION

In this section, we are going to demonstrate main result contained in Theorem 2 by an example. Consider the following nonlinear fractional integro-differential equation

$$\left({}^{C}D^{1.75}v\right)(\tau)=\ln(\tau^2)+\frac{1}{9}\int_0^\tau\tau\eta\left({}^{C}D^{0.25}v\right)(\eta)\,d\eta,\quad\tau\in[0,1],$$

$$v^{(j)}(0) = v_j, \quad j = 0,1. \tag{19}$$

Observe that the above equation is a special case of Eq. (1) with

$$\beta = 1.75, \gamma = 0.25, m = 2, n = 1, a = 1,$$
$$g(\tau) = \ln(\tau^2), \ k(\tau,\eta) = \tau\eta, \ \tau,\eta \in [0,1],$$

$$(G\sigma)(\eta) = \frac{1}{9}\sigma(\eta), \ \eta \in [0,1], \sigma \in C[0,1].$$

The functions g and k are continuous and G is increasing linear transformation that satisfies in assumption (ii). To check assumption (iii), let's put

$$L(\tau,\eta) = \frac{1}{\Gamma(\frac{3}{2})}\int_{\eta}^{\tau}\mu\eta\sqrt{\tau-\mu d\mu}$$
$$= \frac{2\eta}{\sqrt{\pi}}\left[\frac{2\tau}{3}(\tau-\eta)^{\frac{3}{2}} - \frac{2}{5}(\tau-\eta)^{\frac{5}{2}}\right].$$

Since $\sup_{\tau\in[0,1]}\int_0^1 L^2(\tau,\eta)\,d\eta \le 1$, then applying Theorem 2, we deduce that Eq. (19) has a unique solution in χ.

REFERENCES

[1] Aghajani A, Jalilian Y, Trujillo JJ. (2012). On the existence of solutions of fractional integro-differential equations. Fractional Calculus and Applied Analysis 15: 44-69. http://dx.doi.org/10.2478/s13540-012-0005-4

[2] Alfifi HY, Ben Saad I, Turki S, El-Abidine ZZ. (2017). Existence and asymptotic behavior of positive solutions for a coupled system of semilinear fractional differential equations. Results in Mathematics 71: 705-730. http://dx.doi.org/10.1007/s00025-016-0528-9

[3] Anguraj A, Karthikeyan P, Trujillo JJ. (2011). Existence of solutions to fractional mixed integro-differential equations with nonlocal initial condition. Advances in Difference Equations 690653. https://doi.org/10.1155/2011/690653

[4] Baghani O, Gachpazan M, Baghani H. (2012). Existence, uniqueness and stability of solutions for a class of nonlinear integral equations under generalized Lipschitz condition. Indian Journal of Pure and Applied Mathematics 43: 309-321. http://dx.doi.org/10.1007/s13226-012-0019-y

[5] Balachandran K, Kiruthika S, Trujillo JJ. (2011). Existence results for fractional impulsive integro-differential equations in Banach spaces. Communications in Nonlinear Science and Numerical Simulation 16: 1970-1977. https://doi.org/10.1016/j.cnsns.2010.08.005

[6] Benyettou L. (2016). Performance evaluation of a multi-sensor system using fixed point DSP for water leak detection. Advances in Modelling and Analysis D 21: 78-87.

[7] Das S. (2008). Functional fractional calculus for system identification and controls. Springer-Verlag, Berlin, Heidelberg. https://doi.org/10.1007/978-3-540-72703-3

[8] Gautam GR, Dabas J. (2014). Existence result of fractional functional integro-differential equation with not instantaneous impulse. International Journal of Advances in Applied Mathematics and Mechanics 1: 11-21.

[9] Henderson J, Luca R. (2017). Existence of nonnegative solutions for a fractional integro-differential equation. Results in Mathematics 72: 747-763. http://dx.doi.org/10.1007/s00025-017-0655-y

[10] Kilbas AA, Srivastava HM, Trujillo JJ. (2006). Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam. https://doi.org/10.1016/S0304-0208(06)80001-0

[11] Klafter J, Lim SC, Metzler R. (2011). Fractional Dynamics in Physics: Recent Advances. World Scientific, Singapore. http://dx.doi.org/10.1007/s11012-012-9661-z

[12] Li G. (2017). A quintic spline collocation method for the fractional sub-diffusion equation with variable coefficients. Advances in Modelling and Analysis A 54: 40-49.

[13] Merdan M, Gökdoğan A, Yildirim A. (2013). On numerical solution to fractional non-linear oscillatory equations. Meccanica 48: 1201-1213.

[14] Nouri K, Baleanu D, Torkzadeh L. (2018). Study on application of hybrid functions to fractional differential equations. Iranian Journal of Science and Technology, Transactions A, Science 42: 1343–1350. https://doi.org/10.1007/s40995-017-0224-y

[15] Nouri K, Elahi-Mehr S, Torkzadeh L. (2016). Investigation of the behavior of the fractional Bagley-Torvik and Basset equations via numerical inverse Laplace transform. Romanian Reports in Physics 68: 503-514.

[16] Nouri K, Nazari M, Keramati B. (2017). Existence results for a coupled system of fractional integro-differential equations with time-dependent delay. Journal of Fixed Point Theory and Applications 19: 2927-2943. https://doi.org/10.1007/s11784-017-0463-8

[17] Wang F. (2012). Existence and uniqueness of solutions for a nonlinear fractional differential equation. Journal of Applied Mathematics and Computing 39: 53-67. https://doi.org/10.1007/s12190-011-0509-9