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Existence and stability of collinear equilibrium points in elliptic restricted three body problem with radiating primary and triaxial secondary

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ABSTRACT

The present paper studies the motion of an infinitesimal mass around a stellar primary and triaxial secondary moving around each other in the elliptic orbits about their common center of mass in the neighbourhood of Collinear equilibrium points. The location and stability of the collinear points are found to be affected by the radiation pressure and triaxility parameters. The nature of stability, however, remains unchanged. The collinear points $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are unstable in the Lyapunov sense. But, the collinear point $\boldsymbol{L_3}$ shows a stable behavior for small values of radiation and triaxiality parameters.

1. INTRODUCTION

The restricted three body problem has been widely studied by many researchers. This is because of its application to the celestial and stellar bodies. The system consists of two finite bodies, known as primaries, moving about their common center of mass, being attracted by the gravitational attraction of each other. The motion of the third body is influenced by both the primaries. The primaries may describe either a circular or an elliptic path. If the system performs motion about a circular orbit, then it is a circular restricted three body problem otherwise it is an elliptic restricted three body problem. The elliptic restricted three body problem generalizes the circular restricted three body problem. The orbit of the Jupiter around the Sun is a fixed ellipse and the Trojan asteroids are influenced by the gravitational attraction of the Sun and Jupiter. The stability of such systems (ER3BP) moving in elliptic orbits was investigated by many authors, [1-8] and many others.

The bodies in the classical model of the problem were considered as spherical, but many celestial bodies are either oblate spheroids or triaxial or both, and not spheres. For instance, the Mars, Jupiter, Saturn, Neutron stars, Regulus and white dwarfs are oblate spheroids whereas, the Moon and Pluto and its moon Charon are triaxial. The Earth is also oblate and triaxial as well. This oblateness and triaxiality of the primaries cause perturbations in the system.

This inspired many authors to include these characterizations in their study of Restricted/Elliptic Resricted three body problem. The stability of the collinear equilibrium points in the Photogravitational/Generalised Photogravitational Elliptic Restricted three body problem was studied in [9-11]. The linear stability of periodic orbits of the Lagrangian equilibrium points of the ERTBP, was studied in [12] based on some value of the mass ratio. The dynamical properties of the solar sail in Elliptic Restricted three body problem were also analysed and studied in [13-15]. The effect of the solar radiation pressure on the location and stability of the equilibrium points was studied in [16] taking bigger primary as radiating and taking both primaries as radiating in [17]. In both studies the result was that the radiation pressures of the primaries affect the location of the collinear points. The positions and stability of the collinear equilibrium points, $L_{1,2,3}$ of an infinitesimal body in the elliptic restricted three-body problem (ER3BP) when both primaries of the system are luminous and oblate spheroids moving in elliptic orbits around their common center of mass was studied in [18]. The stability of the Lagrangian equilibrium points in the elliptic restricted three body problem with radiating and triaxial primaries was studied based on Averaging method and Floquet's theory respectively in [19] and [20].

There are five equilibrium points in the Elliptic Restricted three body problem. The equilibrium points are the points at which the particle has zero velocity and zero acceleration and are very important for astronautical applications [16]. Three of the equilibrium points are called collinear points as they lie on the x-axis (axis joining the two primaries) and are denoted by $\boldsymbol{L_1}$, $\boldsymbol{L_2}$, $\boldsymbol{L_3}$. The remaining two are called the Lagrangian points and are denoted by $\boldsymbol{L_4}$, $\boldsymbol{L_5}$.

In the present work, we study the effects of solar radiation pressure and the triaxiality of the primaries. The motion of the Collinear equilibrium points has been studied in the frame work of the Elliptic Restricted three body problem. To the best of our knowledge propensities due to radiation of the primary and the triaxiality of the secondary has not been previously studied.

The study of the collinear points is important. This is because the orbits near these points are useful for spacecraft missions. These are suitable to set permanent observatories of the Sun, the magnetosphere of the Earth, links with the hidden part of the Moon, and others [21]. The locations of the collinear points have been found analytically in terms of the mass ratio, solar radiation pressure and the triaxiality of the primaries. The method proposed in [22] and [16] has been used. The stability of the collinear points has also been studied by averaging the system. The numerical calculations and the graphs have been plotted using the *Wolfram Mathematica 10.2* and *Matlab* software, respectively. The locations of the collinear points and the stability has been analysed for the Sun-Earth system.

The present paper is organised as follows: Section 1, which is introduction; Section 2 provides the equations of motion; Section 3 gives the location of the Collinear points; Section 4 focuses on the stability of the different Collinear points. The conclusions of the work are drawn in Section 5.

2. EQUATIONS OF MOTION

The differential equations of motion of the infinitesimal mass in the elliptic restricted three body problem under radiating and triaxial primaries in the barycentric, pulsating and rotating, non-dimensional coordinates are derived in [19] and given in equation (1). The notations in principle are taken from [4] with some minor modifications in the notation being done for adapting to the present problem, presented as :

$$\ddot{x} - 2\dot{y} = \frac{1}{(1 + e\cos v)}\frac{\partial\Omega}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{1}{(1+e\cos v)}\frac{\partial\Omega}{\partial y} \tag{1}$$

where denotes differentiation with respect to v, and

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left[\frac{(1-\mu)q}{r_1} + \frac{\mu}{r_2} + \frac{\mu(2\sigma_1-\sigma_2)}{2r_2^3} - \frac{3\mu(\sigma_1-\sigma_2)}{2r_2^5}\right] \tag{2}$$

Let,

$$k = \frac{1}{n^2}\left[\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu(4\sigma_1-3\sigma_2)}{2r_2^5} - \frac{15\mu(\sigma_1-\sigma_2)y^2}{2r_2^7}\right] \tag{3}$$

Then, (1) can be written as,

$$\ddot{x} - 2\dot{y} = \frac{1}{(1+e\cos v)}\left[x\left(1 - k + \frac{3\mu(\sigma_1-\sigma_2)}{n^2r_2^5}\right) + \frac{\mu(1-\mu)}{n^2}\left(\frac{q}{r_1^3} - \frac{1}{r_2^3} - \frac{3(2\sigma_1-\sigma_2)}{2r_2^5} + \frac{15(\sigma_1-\sigma_2)y^2}{2r_2^7}\right)\right]$$

$$\ddot{y} + 2\dot{x} = \frac{1}{(1+e\cos v)}[1-k]y \tag{4}$$

where

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2); \tag{5}$$

and,

$$\sigma_1 = \frac{a^2-b^2}{5R^2}; \sigma_2 = \frac{b^2-c^2}{5R^2};$$

$$r_1 = (x+\mu)^2 + y^2, r_2 = (x-1+\mu)^2 + y^2 \tag{6}$$

The dimensionless variables are introduced using the distance between the primaries given by:

$$r = \frac{a(1-e^2)}{(1+e\cos v)}$$

Here, m_1 and m_2 are the masses of the bigger and smaller primaries positioned at $(x_i, 0)$, i= 1, 2, q=1-δ the radiation pressure; σ_1, σ_2 are triaxiality parameters, σ_i, (i=1,2)[23]; and *a, b ,c* are semi axes and R is the distance between the primaries; r_i (i=1,2) are the distances of the infinitesimal mass from the bigger and smaller primaries respectively; while e is the eccentricity of the either primary around the other and v is the true anomaly.

3. LOCATION OF COLLINEAR EQUILIBRIUM POINTS

The equilibrium points of the system are the points where the consumption of the resources is the minimum, hence, are given by the equations:

$$\frac{\partial\Omega}{\partial x} = 0; \frac{\partial\Omega}{\partial y} = 0. \tag{7}$$

where, Ω is given by equation (2). But, the collinear points lie on the x-axis; hence, are given by the conditions:

$$\frac{\partial\Omega}{\partial x} = 0; \frac{\partial\Omega}{\partial y} = 0;\ y = 0. \tag{8}$$

Hence, using equations (7) and (8), we get:

$$f(x) = \left[x - \frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)q}{r_1^3} + \frac{\mu(x-1+\mu)}{r_2^3} + \frac{3\mu(x-1+\mu)(2\sigma_1-\sigma_2)}{2r_2^5}\right\}\right] = 0 \tag{9}$$

There are three collinear equilibrium points. These are denoted by L_1, lying between the bigger and the smaller primary (-μ<x<1-μ); L_2, lying to the right of smaller primary (x>1-μ) and L_3, lying to the left of the bigger primary (x<-μ).

3.1 Location of L_1

To find the solution for L_1, substituting $x = x_2 - \rho$, such that $r_2 = \rho$ and $r_1 = 1 - \rho$ into equation (9), we have:

$$1 - \mu - \rho - \frac{1}{n^2}\left\{\frac{(1-\mu)q}{(1-\rho)^2} + \frac{\mu}{\rho^2} + \frac{3\mu(2\sigma_1-\sigma_2)}{2\rho^4}\right\} = 0 \tag{10}$$

Now, rearranging the terms, and simplifying we have:

$$\rho^3\left[1 - \frac{\left\{1-\frac{3}{2}(2\sigma_1-\sigma_2)-2\delta\right\}}{\left\{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}\right\}}\rho + \frac{\left\{\frac{1}{3}-\frac{3}{2}(2\sigma_1-\sigma_2)-\frac{4\delta}{3}\right\}}{\left\{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}\right\}}\rho^2\right] = \left\{\left[\frac{\mu(1+15(2\sigma_1-\sigma_2))}{3(1-\mu)\left\{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}\right\}}\right](1-\rho)^2\left[1 - 30(2\sigma_1-\sigma_2)\rho + \frac{45}{2}(2\sigma_1-\sigma_2)\rho^2 - \{n^2 + 6(2\sigma_1-\sigma_2)\}\rho^3\right]\right\} \tag{11}$$

Now, let,

$$\left[\frac{\mu(1+15(2\sigma_1-\sigma_2))}{3(1-\mu)\left\{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}\right\}}\right]^{\frac{1}{3}} = \lambda$$

Then using series expansion given as:

$$\rho = \lambda(1 + c_1\lambda + c_2\lambda^2 + \cdots) \tag{12}$$

The simplified equation can be written as:

$$\rho = \lambda\left[1 - \frac{1}{3}\left\{\frac{1+\frac{59(2\sigma_1-\sigma_2)}{2}-\frac{2\delta}{3}}{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}}\right\}\lambda - \frac{1}{9}\left\{\frac{1+\frac{387(2\sigma_1-\sigma_2)}{2}-\frac{2\delta}{3}}{\left\{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}\right\}^2}\right\}\lambda^2 + \dots\right] \quad (13)$$

Hence, the solution for L_1 is given by:

$$x = 1 - \mu - \lambda\left[1 - \frac{1}{3}\left\{\frac{1+\frac{59(2\sigma_1-\sigma_2)}{2}-\frac{2\delta}{3}}{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}}\right\}\lambda - \frac{1}{9}\left\{\frac{1+\frac{387(2\sigma_1-\sigma_2)}{2}-\frac{2\delta}{3}}{\left\{1-\frac{(2\sigma_1-\sigma_2)}{2}-\frac{4\delta}{3}\right\}^2}\right\}\lambda^2 + \cdots\right] \quad (14)$$

3.2 Location of L_2

For finding the location of L_2 , substituting $x = x_2 + \rho$ such that $r_2 = \rho, r_1 = 1 + \rho$. Then, substituting, the values in equation (9), we have:

$$\frac{\rho\left[(1+n^2)\left(1+\rho+\frac{\rho^2}{3}\right)+\frac{(n^2-q)}{3\rho}\right]}{(1+\rho)^2} = \frac{\mu}{3(1-\mu)}\left[\frac{1-n^2\rho^3+\frac{3(2\sigma_1-\sigma_2)}{2}\frac{}{\rho^2}}{\rho^2}\right] \quad (15)$$

On simplification, we have:

$$\rho^3\left[1 + \frac{\left\{1-\frac{3}{2}(2\sigma_1-\sigma_2)-2\delta\right\}}{\left\{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}\right\}}\rho + \frac{\left\{\frac{1}{3}+\frac{5}{2}(2\sigma_1-\sigma_2)+\frac{4\delta}{3}\right\}}{\left\{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}\right\}}\rho^2\right] = \left\{\left[\frac{\mu(1+15(2\sigma_1-\sigma_2))}{3(1-\mu)\left\{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}\right\}}\right](1-\rho)^2\left[1 - 30(2\sigma_1-\sigma_2)\rho + \frac{45}{2}(2\sigma_1-\sigma_2)\rho^2 - \left\{1+\frac{15}{2}(2\sigma_1-\sigma_2)\right\}\rho^3\right]\right\} \quad (16)$$

Using the series as in equation (12), the value of ρ is given as:

$$\rho = \lambda\left[1 - \frac{1}{3}\left\{\frac{1-\frac{43(2\sigma_1-\sigma_2)}{2}+\frac{14\delta}{3}}{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}}\right\}\lambda - \frac{1}{9}\left\{\frac{1+\frac{67(2\sigma_1-\sigma_2)}{2}+\frac{16\delta}{3}}{\left\{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}\right\}^2}\right\}\lambda^2 + \cdots\right] \quad (17)$$

Hence, the solution for L_2 is given as:

$$x = 1 - \mu + \lambda\left[1 - \frac{1}{3}\left\{\frac{1-\frac{43(2\sigma_1-\sigma_2)}{2}+\frac{14\delta}{3}}{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}}\right\}\lambda - \frac{1}{9}\left\{\frac{1+\frac{67(2\sigma_1-\sigma_2)}{2}+\frac{16\delta}{3}}{\left\{1+\frac{7(2\sigma_1-\sigma_2)}{2}+\frac{4\delta}{3}\right\}^2}\right\}\lambda^2 + \cdots\right] \quad (18)$$

3.3 Location of L_3

For the point L_3 substituting $x = x_1 - \rho$ such that $r_1 = \rho$ and $r_2 = 1 + \rho$ in equation (9), we have:

$$\frac{\mu}{1-\mu} = \frac{(n^2\rho^3-q)(1+\rho)^2}{\rho^2\left[1+\frac{3(2\sigma_1-\sigma_2)}{2}(1+\rho)^{-2}-n^2(1+\rho)^3\right]} \quad (19)$$

taking, $\rho = 1 + \alpha$, and using the elementary algorithm for division upto $O[\alpha^4]$, we have:

$$-\frac{\mu}{1-\mu} = \left[\left(-\frac{9}{7}(2\sigma_1-\sigma_2) - \frac{4}{7}\delta\right) + \left(1 - \frac{11}{14}(2\sigma_1-\sigma_2) - \frac{19}{21}\delta\right)\left(-\frac{12}{7}\alpha\right) + \left(1 - \frac{815}{672}(2\sigma_1-\sigma_2) - \frac{989}{1008}\delta\right)\left(-\frac{12}{7}\alpha\right)^2 + \left(\frac{1567}{1728} - \frac{10627}{8064}(2\sigma_1-\sigma_2) - \frac{5465}{6048}\delta\right)\left(-\frac{12}{7}\alpha\right)^3 + \cdots.\right] \quad (20)$$

Now, using the method of successive approximations and Lagrange inversion formula [24], and retaining only linear terms in $\delta, \sigma_1, \sigma_2$ we get:

$$\rho = \left[\left(1 - \frac{3}{4}(2\sigma_1-\sigma_2) - \frac{1}{3}\delta\right) - \frac{7}{12}\left(1 - \frac{25}{14}(2\sigma_1-\sigma_2) - \frac{5}{21}\delta\right)\left(\frac{\mu}{1-\mu}\right) + \frac{7}{12}\left(1 - \frac{4075}{1344}(2\sigma_1-\sigma_2) - \frac{71}{504}\delta\right)\left(\frac{\mu}{1-\mu}\right)^2 + \left(\frac{-13223}{20736} + \frac{22307}{41472}(2\sigma_1-\sigma_2) - \frac{30481}{31104}\delta\right)\left(\frac{\mu}{1-\mu}\right)^3 + O\left[\frac{\mu}{1-\mu}\right]^4\right] \quad (21)$$

Hence, the solution for L_3 is given as:

$$x = -\mu - \left[\left(1 - \frac{3}{4}(2\sigma_1-\sigma_2) - \frac{1}{3}\delta\right) - \frac{7}{12}\left(1 - \frac{25}{14}(2\sigma_1-\sigma_2) - \frac{5}{21}\delta\right)\left(\frac{\mu}{1-\mu}\right) + \frac{7}{12}\left(1 - \frac{4075}{1344}(2\sigma_1-\sigma_2) - \frac{71}{504}\delta\right)\left(\frac{\mu}{1-\mu}\right)^2 + \left(\frac{-13223}{20736} + \frac{22307}{41472}(2\sigma_1-\sigma_2) - \frac{30481}{31104}\delta\right)\left(\frac{\mu}{1-\mu}\right)^3 + O\left[\frac{\mu}{1-\mu}\right]^4\right] \quad (22)$$

4. LINEAR STABILITY OF COLLINEAR POINTS

The stability of motion of the collinear points is decided by the following lemma whose formulation is taken from [22], with some modifications in notations being done for adapting to the present problem.

Lemma*:* At the collinear points:

$$k = \frac{1}{n^2}\left[\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu(4\sigma_1-3\sigma_2)}{2r_2^5} - \frac{15\mu(\sigma_1-\sigma_2)y^2}{2r_2^7}\right] > 1 \quad (23)$$

Proof: For an equilibrium point, we have the condition:

$$x - \frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)q}{r_1^3} + \frac{\mu(x-1+\mu)}{r_2^3} + \frac{3\mu(x-1+\mu)(2\sigma_1-\sigma_2)}{2r_2^5} - \frac{15\mu(x-1+\mu)(\sigma_1-\sigma_2)y^2}{2r_2^7}\right\} = 0 \quad (24)$$

The condition for a collinear equilibrium point is y=0, so the equation (24) gets simplified as:

$$x - \frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)q}{r_1^3} + \frac{\mu(x-1+\mu)}{r_2^3} + \frac{3\mu(x-1+\mu)(2\sigma_1-\sigma_2)}{2r_2^5} - \frac{15\mu(x-1+\mu)(\sigma_1-\sigma_2)y^2}{2r_2^7}\right\} = 0 \quad (25)$$

Rearranging the terms, the equation (25) can be written as:

$$\frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)(r_1-r_1^{-2}q)}{r_1} + \frac{\mu(x-1+\mu)(r_2-r_2^{-2})}{r_2} + \frac{3\mu(x-1+\mu)(2\sigma_1-\sigma_2)(r_2-r_2^{-4})}{2r_2} + \frac{3\mu(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)}{2}\right\} = 0 \quad (26)$$

Next, to prove equation (23) we analyze each collinear equilibrium point separately.

4.1 Stability at collinear point L_1

Now, at the point $L_1, r_1 + r_2 = 1$, where, $r_1 = x + \mu$, and $r_2 = 1 - x - \mu$. Substituting the values in equation (26) and simplifying using equation (3), we have:

$$\left[\frac{1}{n^2}\left[\left(1 - k + \frac{3\mu(2-\mu)(2\sigma_1 - \sigma_2)}{2}\right) r_1 + \frac{3}{2}\mu(2\sigma_1 - \sigma_2) - \mu\left(1 - \frac{1}{r_2^3} - \frac{3(2\sigma_1 - \sigma_2)}{2r_2^5}\right)\right]\right] = 0$$

Since, $\frac{1}{n^2} \neq 0$ and $r_2 < 1$, we have:

$$k = 1 + \left[\frac{\mu}{r_1}\left(\frac{1}{r_2^3} + \frac{3(2\sigma_1 - \sigma_2)}{2r_2^5} - \frac{3}{2}(2\sigma_1 - \sigma_2)\right) - 1 + \frac{3\mu(2-\mu)(2\sigma_1 - \sigma_2)}{2r_1}\right] \tag{27}$$

Hence, $k > 1$ for collinear point L_1.

4.2 Stability at Collinear point L_2

At L_2, $r_1 - r_2 = 1$, $r_1 = x + \mu$ and $r_2 = x + \mu - 1$. Inserting, the values in equation (26), and using equation (3), and proceeding in the same way as for L_1, also for collinear point L_2, we have,

$$k = 1 + \left[\frac{\mu}{r_1}\left(\frac{1}{r_2^3} + \frac{3(2\sigma_1 - \sigma_2)}{2r_2^5} - \frac{3}{2}(2\sigma_1 - \sigma_2)\right) - 1 + \frac{3\mu(2-\mu)(2\sigma_1 - \sigma_2)}{2r_1}\right] \tag{28}$$

Hence, for collinear point L_2, we have , $k > 1$.

4.3 Stability at collinear point L_3

At L_3, $r_2 - r_1 = 1$, $r_1 = -x - \mu$, $r_2 = -x - \mu + 1$. Proceeding in the same manner as in L_1 and L_2, substituting values in equation (26) and using equation (3) we have:

$$k = 1 + \left[\frac{\mu}{r_1}\left\{1 - \left(\frac{1}{r_2^3} + \frac{3(2\sigma_1 - \sigma_2)}{2r_2^5} - \frac{3}{2}(2\sigma_1 - \sigma_2)\right)\right\} + \frac{3\mu(2-\mu)(2\sigma_1 - \sigma_2)}{2r_1}\right] \tag{29}$$

Hence, $k > 1$, for collinear point L_3 also.

Thus, for all collinear points L_1, L_2 *and* L_3, we have $k > 1$. This completes the proof of lemma.

To analyses the stability of motion around the primaries near the collinear points, investigating the roots of the characteristic equations. For this, assuming, that the particle receives a small displacement from the equilibrium position. Then finding the variational equations of motion by substituting the coordinates of displaced point in the equation of motion equation (1) and expanding by Taylor's series about the collinear points and taking only the linear terms, we get the equation following [20] as:

$$\ddot{\xi} - 2\dot{\eta} = \phi[\Omega_{xx}^0 \xi + \Omega_{xy}^0 \eta];\ \ddot{\eta} + 2\dot{\xi} = \phi[\Omega_{yx}^0 \xi + \Omega_{yy}^0 \eta]. \tag{30}$$

where $\phi = \frac{1}{(1+e\cos v)}$ and (x_0, y_0) are the coordinates of the collinear points respectively. The subscript of Ω denotes the second order partial derivatives of Ω with respect to *x* and *y*, as it appears, respectively.

Since, all the collinear points lie on the *x*-axis, hence y=0, resulting, $\Omega_{xy} = 0$. Introducing new variables given by,

$$x_1 = \xi, x_2 = \eta, x_3 = \frac{d\xi}{dv}, x_4 = \frac{d\eta}{dv}$$

Substituting these values in equation (30), the system of equations can be written as:

$$\frac{dx_i}{dv} = P_{i1}x_1 + P_{i2}x_2 + P_{i3}x_3 + P_{i4}x_4;\ i = 1,2,3,4 \tag{31}$$

where,

$$P_{11} = P_{12} = P_{14} = P_{21} = P_{22} = P_{23} = P_{33} = P_{44} = 0;\ P_{13} = P_{24} = 1;\ P_{34} = 2;\ P_{43} = -2\ ;\ P_{31} = \phi\Omega_{xx}^0;\ P_{42} = \phi\Omega_{yy}^0$$

where, superscript *'0'* indicates the values evaluated at respective collinear points.

The coefficients of equation (31) are 2π periodic functions of *v*. Hence, taking the averaged system, given by:

$$\frac{dx_i^{(0)}}{dv} = P_{i1}^{(0)}x_1^{(0)} + P_{i2}^{(0)}x_2^{(0)} + P_{i3}^{(0)}x_3^{(0)} + P_{i4}^{(0)}x_4^{(0)};\ i = 1,2,3,4 \tag{32}$$

where,

$$P_{is}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{is}(v)dv\,;\ i,s = 1,2,3,4$$

So, we get:

$$P_{31}^{(0)} = \frac{1}{\sqrt{1-e^2}}\Omega_{xx}^0;\ P_{42}^{(0)} = \frac{1}{\sqrt{1-e^2}}\Omega_{yy}^0$$

where, superscript *'0'* indicates the values evaluated at respective collinear points L_1, L_2, L_3 .

Thus, the characteristic equation for the system of equations (32) can be given as:

$$\lambda^4 + Q\lambda^2 + R = 0 \tag{33}$$

where,

$$Q = -(4 - P_{31}^0 - P_{42}^0);\ R = P_{31}^0 . P_{42}^0 \tag{34}$$

The motion of the infinitesimal particle will be stable near the collinear point when given a small displacement and small velocity, the particle oscillates for a considerable time about the points [9]. That is, the system will be stable if the roots of the characteristic equation are purely imaginary. Hence, the condition for stable roots can be given as:

$$Q < 0; R > 0.$$

Taking the latter inequality, the condition of stability can be written as:

$$-\frac{1}{2}-\frac{3}{4}(2\sigma_1-\sigma_2)-\frac{3\mu(\sigma_1-\sigma_2)}{r_2^5}<k$$
$$<1+\frac{3}{4}(2\sigma_1-\sigma_2)-\frac{3\mu(\sigma_1-\sigma_2)}{r_2^5}$$

If the values of σ_1,σ_2 are negligibly small. Then, the condition of stability can be written as:

$$-\frac{1}{2}<k<1 \tag{35}$$

But, from the above subsections, it is clear that for all the collinear equilibrium points, L_1, L_2, L_3 we have k>1. Thus, the collinear points are unstable, using the above condition given by equation (35).

The roots of the characteristic equation (36) are given by:

$$\lambda_{1,2}^2=\frac{\left[k-2(1-e^2)-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right]\pm\left[\left(9k^2-8k-\frac{18\mu k(2\sigma_1-\sigma_2)}{r_2^5}\right)+8e^2\left(1+k-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right)\right]^{1/2}}{2\sqrt{1-e^2}} \tag{36}$$

Let, $\lambda_i^2=s_i\,, i=1{,}2$. Hence, the above equation (36) can be written as:

$$s_1=\frac{\left[k-2(1-e^2)-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right]+\left[\left(9k^2-8k-\frac{18\mu k(2\sigma_1-\sigma_2)}{r_2^5}\right)+8e^2\left(1+k-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right)\right]^{1/2}}{2\sqrt{1-e^2}}$$
$$s_2=\frac{\left[k-2(1-e^2)-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right]-\left[\left(9k^2-8k-\frac{18\mu k(2\sigma_1-\sigma_2)}{r_2^5}\right)+8e^2\left(1+k-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right)\right]^{1/2}}{2\sqrt{1-e^2}} \tag{37}$$

As, $k>1$ for collinear points, we have:

$$\left[\left(9k^2-8k-\frac{18\mu k(2\sigma_1-\sigma_2)}{r_2^5}\right)+8e^2\left(1+k-\frac{3\mu(2\sigma_1-\sigma_2)}{r_2^5}\right)\right]^{1/2}>1 \tag{38}$$

for the values of $\sigma_1,\sigma_2<1, e<1$. Thus, from the above equations (37) and (38), we have $s_1>0, s_2<0$. As $\lambda^2=s$, $s_1>0$ gives two real roots of opposite signs, and $s_2<0$, results into two imaginary roots. Hence, the solution of equation (36) can be written in the form:

$$\lambda_i=C_{i1}\epsilon^{p_1 v}+C_{i2}\epsilon^{p_2 v}+C_{i3}\cos(p_3 v-C_{i4})\,, i=1{,}2. \tag{39}$$

where, p_1, p_2, p_3 are the roots of equation (33). The first and second terms of equation (39) cause an exponential growth in the values of the roots λ_i and dominates the third term. Thus, from the values of the two characteristic roots, it is also clear that the motion is unstable near a collinear equilibrium point.

4. CONCLUSION

The formulas derived in the paper can be applied to the Sun and the Earth as primaries and the particle as space craft}.

Hence, for the system, the mass parameter $\mu=\frac{m_2}{m_1+m_2}=3.00317\times10^{-6}$, the eccentricity of the elliptic orbit of the primaries, e=0.0167.

The location of collinear points for the Sun-Earth system are given in table: [1], for different values of δ,σ_1,σ_2 .

The nature of motion around the collinear points can be analysed as:

(i) the motion around the point L_1 is unstable for all the values of δ,σ_1 and σ_2 as $k>1$ and $\lambda_1^2>0, \lambda_2^2<0$. This can be seen from table: [2] and is also evident from graph (figure:1)

(ii) the point L_2 also exhibits the instability of motion in its vicinity as $\lambda_1^2>0$ and $\lambda_2^2<0$ and also k>1 for all values of δ,σ_1 and σ_2 .This is evident from table: [3]. The instability can also be seen from the graph(figure:2)

(iii) For L_3 , the motion appears to be stable for some values of δ,σ_1 and σ_2 because the values of $k<1$ as well as $\lambda_{1,2}^2<0$. As, the value of σ_1 and σ_2 , increases the system becomes unstable, for a fixed value of δ, the roots of the system $\lambda_{1,2}^2<0$. That is, the roots will be imaginary implying the stability of the system which is evident from the table: [4] and also from the graph(figure:3)

When σ_1 and σ_2 are negligibly small, the locations and stability of our results are in confirmation with [16] and [17]. The collinear point L_3 was found to be stable for the value of radiation pressure $\beta\geq0.115$ in [16] and [17]. This result is also in confirmation with [16] and [17] as the collinear point L_3 is found to be stable in our result also, clearly evident from table: [4]. The collinear points L_1 and L_2 are found unstable, as is given in [16-17] and [18].

The existence and stability of collinear equilibrium points of the elliptic restricted three body problem with radiating primary and triaxial secondary has been analysed by the method given by [16, 22]. The rotating coordinate system and dimensionless pulsating variables are used to make the system independent of time. The numerical calculations are performed using the software *Mathematica 10.2* and the graphs are plotted using *Matlab* software.

The location of the collinear points L_1, L_2 and L_3 are given by equations (14), (18) and (22), respectively. The increment in the values of δ,σ_1 and σ_2 affects the locations of the collinear points. The stability of the system can be analysed by the equations (33), (37) and (38) using the condition of stability given by equation (35). The points L_1, L_2 exhibit the unstable motion in their vicinity, which is evident from graphs (figure:1, figure:2). But, the motion around the point L_3 is stable for a particular value of δ,σ_1 and σ_2 (hypothetical values), which can be seen from the figure 3. Also, the stability has been numerically analysed for the Sun-Earth system.

Thus, it can be concluded that the stability of the collinear point L_3 is dependent on the parameters, radiation pressure and the triaxiality parameters, respectively. Increase in the value of the radiation parameter makes motion around the collinear point L_3 stable. On the other hand, increase in the value of triaxiality parameters causes instability of motion around the collinear point L_3 .

Table 1. Location of collinear equilibrium points for Sun-Earth system for different values of δ,σ_1 and σ_2

Collinear Point →	L_1	L_2	L_3
For $\sigma_1=0.00003, \sigma_2=0.00001$			

$\delta = 0.01$	(0.989979,0)	(1.00999,0)	(-0.996615,0)
$\delta = 0.001$	(0.990019,0)	(1.01003,0)	(-0.999615,0)
$\delta = 0.2$	(0.988948,0)	(1.00929,0)	(-0.933282,0)
$\delta = 0.3$	(0.988195,0)	(1.00899,0)	(-0.899949,0)
For $\delta = 0.001$			
For $\sigma_1 = 0.03, \sigma_2 = 0.01$	(0.987962,0)	(1.01216,0)	(-0.962168,0)
For $\sigma_1 = 0.003, \sigma_2 = 0.001$	(0.989777,0)	(1.01029,0)	(-0.995915,0)
For $\sigma_1 = 0.0003, \sigma_2 = 0.0001$	(0.989997,0)	(1.01006,0)	(-0.993144,0)

Table 2. Values of $\lambda_{1,2}^2$, k for different values of $\delta, \sigma_1, \sigma_2$ and e for L1

k, $\lambda_{1,2}^2 \rightarrow$ $\delta\downarrow$	k	λ_1^2	λ_2^2
For $\sigma_1 = 0, \sigma_2 = 0, e = 0$			
$\delta = 0$	4.06052	6.41385	-4.3503
$\delta = 0.001$	4.01058	6.4304	-4.34795
$\delta = 0.01$	4.055579	6.31279	4.30221
$\delta = 0.1$	3.55847	5.40248	-3.84399
$\delta = 0.2$	3.05618	4.338855	-3.33237
$\delta = 0.4$	2.052	2.34335	-2.29135
For $\delta = 0.001$, e=0			
For $\sigma_1 = 0.03, \sigma_2 = 0.01$	1496.23	869.936	-1159.8
For $\sigma_1 = 0.003, \sigma_2 = 0.001$	364.719	257.812	-299.108
For $\sigma_1 = 0.0003, \sigma_2 = 0.0001$	44.5513	35.7023	-38.2083
For $\sigma_1 = 0,\ \sigma_2 = 0, \delta = 0.001$			
e=0.02	4.05579	6.4058	-4.34798
e=0.04	4.05579	6.41198	-4.35134
e=0.06	4.05579	6.4223	-4.35558
e=0.1	4.05579	6.4555	-4.36925
For $\delta = 0.001$			
$\sigma_1 = 0.03, \sigma_2 = 0.04, e = 0.04$	1496.23	870.634	-1160.73
$\sigma_1 = 0.003, \sigma_2 = 0.004, e = 0.02$	364.719	257.864	-299.165
$\sigma_1 = 0.0003, \sigma_2 = 0.0004, e = 0.04$	44.5513	35.7325	-38.2373

Table 3. Values of $\lambda_{1,2}^2$, k for different values of $\delta, \sigma_1, \sigma_2$ and e for L2

k, $\lambda_{1,2}^2 \rightarrow$ $\delta\downarrow$	k	λ_1^2	λ_2^2
For $\sigma_1 = 0, \sigma_2 = 0, e = 0$			
$\delta = 0$	3.94076	6.17231	-4.23155
$\delta = 0.001$	3.9357	6.16213	-4.22643
$\delta = 0.01$	3.89023	6.07061	-4.18039
$\delta = 0.1$	3.43725	5.15805	-3.7208
$\delta = 0.2$	2.93709	4.14759	-3.2105
$\delta = 0.4$	1.9447	2.12041	-2.17634
For $\delta = 0.001$, e=0			
For $\sigma_1 = 0.03, \sigma_2 = 0.01$	1421.36	826.304	-1101.75
For $\sigma_1 = 0.003, \sigma_2 = 0.001$	352.253	248.94	-288.877
For $\sigma_1 = 0.0003, \sigma_2 = 0.0001$	43.3909	34.4808	-36.9571
For $\delta = 0.001$			
$\sigma_1 = 0.03, \sigma_2 = 0.04, e = 0.04$	352.253	248.97	-288.934
$\sigma_1 = 0.003, \sigma_2 = 0.004, e = 0.02$	43.0939	34.5101	-36.9851

Table 4. Values of $\lambda_{1,2}^2$, k for different values of $\delta, \sigma_1, \sigma_2$ and e for L3

k, $\lambda_{1,2}^2 \rightarrow$ $\delta\downarrow$	k	λ_1^2	λ_2^2
For $\sigma_1 = 0, \sigma_2 = 0, e = 0$			
$\delta = 0.001$	1.0000022	0.00000688	-1.00000458
$\delta = 0.01$	0.999969	-0.00009299	-0.999938
$\delta = 0.1$	0.996354	-.0109937	-0.992653
$\delta = 0.2$	0.983968	-0.492158	-0.966816
$\delta = 0.4$	0.921714	-0.278231	-0.80005
For e=0.01			
For $\sigma_1 = 0.003, \sigma_2 = 0.001,\ \delta = 0.1$	1.00054	0.00359971	-1.00246
For $\sigma_1 = 0.0003, \sigma_2 = 0.0001,\ \delta = 0.1$	0.996766	-0.00923806	-0.993846

For $\sigma_1 = 0.0003, \sigma_2 = 0.0001, \delta = 0.11$	0.995963	-0.0116716	-0.992216
$\sigma_1 = 0.003, \sigma_2 = 0.001, \delta = 0.11$	0.999763	-0.000204296	-0.99988
$\sigma_1 = 0.00003, \sigma_2 = 0.00001, \delta = 0.2$	0.984013	-0.0485452	-0.967293
$\sigma_1 = 0.0003, \sigma_2 = 0.0001, \delta = 0.2$	0.984416	-0.0472766	-0.968158
$\sigma_1 = 0.003, \sigma_2 = 0.001, \delta = 0.2$	0.98851	-0.0345095	-0.976831

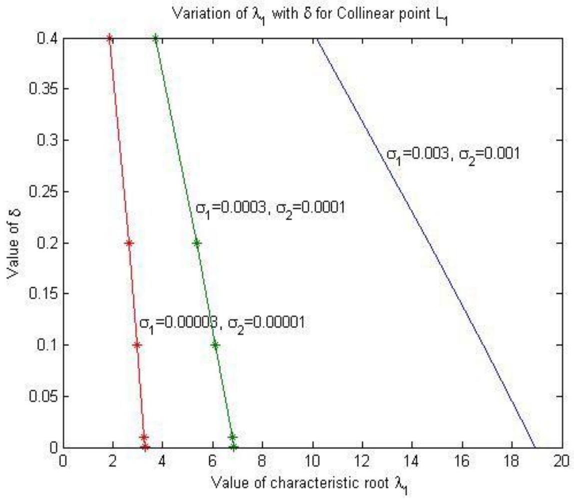

Figure 1. Correlation of characteristic root λ_1 and δ for L_1

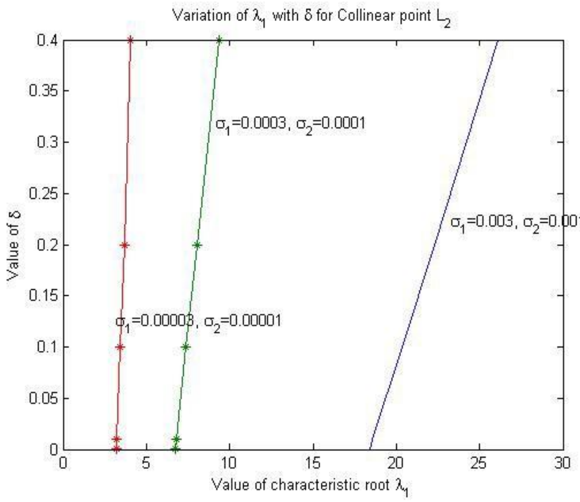

Figure 2. Correlation of characteristic root λ_1 and δ for L_2

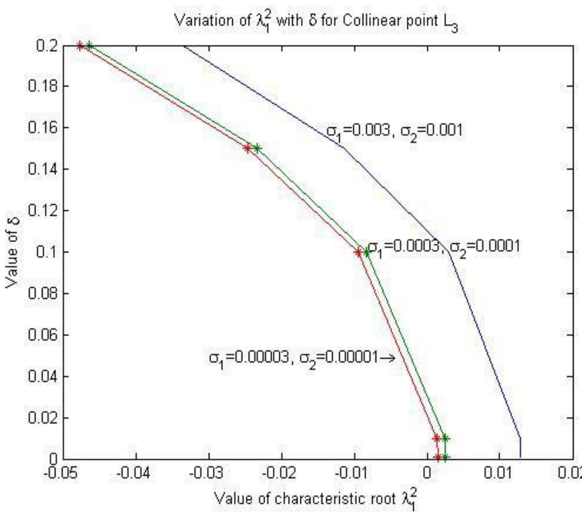

Figure 3. Correlation of characteristic root λ_2 and δ for L_3

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