Advances in Modelling and Analysis B
Vol. 61, No. 3, September, 2018, pp. 127-134
Journal homepage: http://iieta.org/Journals/AMA/AMA_B

Optimal control of four-scroll chaotic system using modal series method

Malihe Roohparvar, Mehdi Neyestani*

Department of Electrical Engineering, Esfarayen university of technology, North khorasan, Iran

Corresponding Author Email: m.neyestani@esfarayen.ac.ir

https://doi.org/10.18280/ama_b.610304

Received: 25 July 2018
Accepted: 1 September 2018

Keywords:
optimal control, chaos, four-scroll chaotic system, pontryagin's minimum principle, modal series method

ABSTRACT

The present paper studies a chaotic system with one cubic nonlinear term and deals with optimal control of this system. The problem analysis technique of this paper, which is a major issue in oscillators, robotics, lasers, etc., has not been proposed in previous studies. Modal Series technique was used to solve the problem of optimal control with infinite time horizon for chaotic system. Nonlinear boundary value obtained in this technique is converted to a sequence of time invariable linear boundary value using Pontryagin's minimum principle. By resolving this sequence, state trajectory and optimal control law are obtained in the form of series with uniform convergence. Moreover, this technique allows for selection of suitable number of answers to reach an appropriate approximation of the main answer. In addition, the number of series terms is not limited. A reverse algorithm for drawing approximate state trajectory and sub-optimal control law. The results of simulations confirmed efficiency and accuracy of the proposed algorithm.

1. INTRODUCTION

Nonlinear dynamics can found in various fields including engineering, physics as well as others. Chaotic systems are nonlinear dynamic systems sensitive to initial conditions. Sensitivity to initial conditions means that slight changes in initial values of a process may result in significant differences in process fate. Thus, synchronization, stabilizing and control of such systems is much difficult. Chaotic systems were first reported by Lorenz in 1963 [1], when he found a three dimensional chaotic system during investigation of climatic patterns. Afterward, many three dimensional chaotic systems were identified including Rössler system [2], Rabinovich system [3], Arneodo system [4], Sprott systems [5], Chen system [6], Lü system [7], Shaw system [8], Cai system [9], Tigan system [10], Colpitt's oscillator [11], Zhou system [12], etc. more chaotic systems have been recently discovered as follows: Li system [13], Sundarapandian system [14], Sundarapandian-Pehlivan system [15], Zhu system [16].

During the past decades, well-known chaotic systems representing n-scroll chaotic attractors were discovered such as: double-scroll attractors (like the Lorenz system [1], Chen system [6], Lü system [7], Tigan system [10]), three-scroll attractors (like Wang system Pan system [17]), and four-scroll chaotic attractors (like Liu system [18]). Chaos theory is widely used in various fields including oscillators [19], lasers [20], robotics and mechanics [21-24], neural networks [25], secure communications [26-27], etc.

Recent studies are oriented toward chaos control including establishment of instable equilibrium points and instable periodic responses [28]. Suitable methods have been developed for suppressing chaos in chaotic systems. These methods include adaptive control, adaptive fuzzy control, sliding mode control, robust control, time-delayed feedback control, double delayed feedback control, bang-bang control, optimal control, intelligent control, etc. [29-40].

During recent decades, optimal control has been an active filed of control theory whose range has been extended to many scientific fields. In this paper, using Pontryagin's minimum principle, nonlinear equation of Four-Scroll chaotic system is converted to nonlinear boundary value problem with infinite time horizon whose solving by analytical approach is too difficult. Various methods have been developed for solving nonlinear boundary value problems. In Successive approximation approach (SAA), for example, instead of direct solving of nonlinear boundary value obtained by Pontryagin's minimum principle, a sequence of variable linear boundary value problems with heterogeneous time is solved in reversible manner [41]. Sensitivity approach is similar to previous approach which only requires solving of a sequence of variable linear boundary value problems with heterogeneous time to propose optimal control law as series form [42]. However, solving of time variable equation is much more difficult than that of time invariable equations.

In this paper, an analytical method called Modal Series Method is proposed for solving the problem of optimal control with infinite time horizon for four-scroll chaotic system. Modal Series Method has been first developed by Dr. Pariz [43]. This method represents a tool for solving nonlinear differential equations such as non-linear optimal control issues [44-47]. Nonlinear boundary value with infinite time horizon in this technique is converted to a sequence of time-independent linear boundary value problems. By solving this sequence, optimal response is obtained as a series with uniform convergence. Moreover, this technique allows for selection of suitable number of answers to reach an appropriate approximation of the main answer. In addition, the number of series terms is not limited.

2. FOUR-SCROLL CHAOTIC SYSTEM

In this section, equilibrium points of Four-scroll chaotic system are evaluated. Nonlinear dynamic equations of Four-scroll chaotic system are presented as follows [48]:

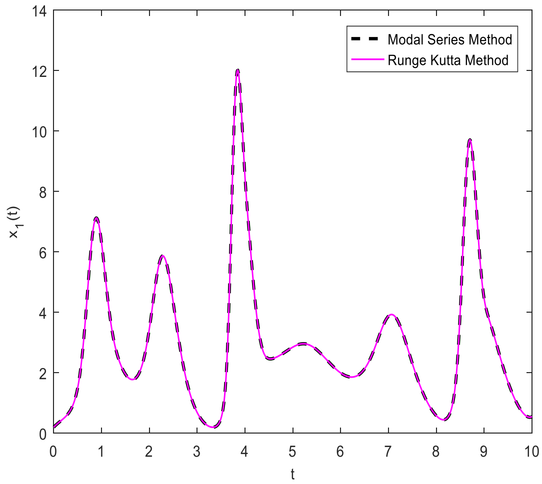
$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + bx_2x_3, \\ \dot{x}_2 = -10x_2^3 - x_2 + 4x_1x_3, \\ \dot{x}_3 = cx_3 - x_1x_2, \end{cases} \tag{1}$$

where, x1, x2 and x3 are state variables and a, b and c represent constant parameters. System Eq. 1 is a polynomial eight term system with three second order nonlinear terms and one cubic nonlinear term. By selecting parameters as presented below, system Eq. 1 describes a four-scroll chaotic attractor.

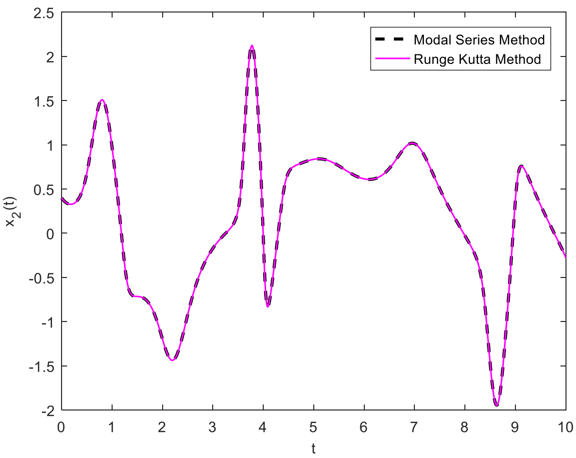
$$a = 3, b = 14, c = 3.9, \tag{2}$$

Time responses according to this initial condition is presented in Fig. 1. In addition, 2D and 3D graphs of the responses are presented in Fig. 2.

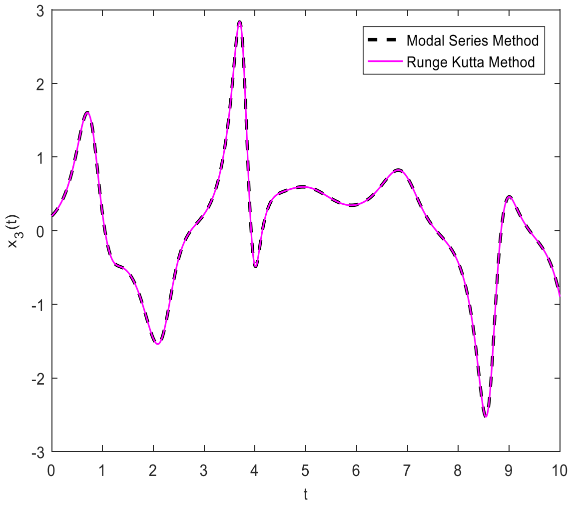
$$\mathrm{x}_0 : \begin{cases} \mathrm{x}_1(0) = 0.2 \\ \mathrm{x}_2(0) = 0.4 \\ \mathrm{x}_3(0) = 0.2 \end{cases} \tag{3}$$



(a)



(b)



(c)

Figure 1. Time response of state variables to Piecewise Modal Series and Runge–Kutta

Simulation was performed using Runge–Kutta method in Matlab and Modal Series analytical method.

3. FOUR-SCROLL CHAOTIC SYSTEM PROPERTIES

In this section, Four-scroll chaotic system and its fundamental properties including dissipativity, symmetry, equilibrium and invariance are discussed as proposed in [48].

3.1. Dissipativity

Right side of system Eq. 1 in vector state can be written as follows:

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} a(x_2 - x_1) + bx_2x_3 \\ -10x_2^3 - x_2 + 4x_1x_3 \\ cx_3 - x_1x_2 \end{bmatrix} \tag{4}$$

Divergence of vector F can be written as follows:

$$\nabla . F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = -a - 1 + c, \tag{5}$$

The necessary and sufficient condition for dissipativity of system Eq. 1 is that vector F divergence be negative. Relation Eq. 5 shows that system Eq. 1 is dissipative if and only if $-a - 1 + c < 0$. According to values of parameters of Eq. 2, this condition is met and thus, system Eq. 1 is dissipative.

3.2. Equilibrium points and stability

Equilibrium points of the chaotic system Eq. 1 is simply obtained by solving the following equation ($a = 3, b = 14, c = 3.9$).

$$\begin{cases} a(x_2 - x_1) + bx_2x_3 = 0, \\ -10x_2^3 - x_2 + 4x_1x_3 = 0, \\ cx_3 - x_1x_2 = 0, \end{cases} \tag{6}$$

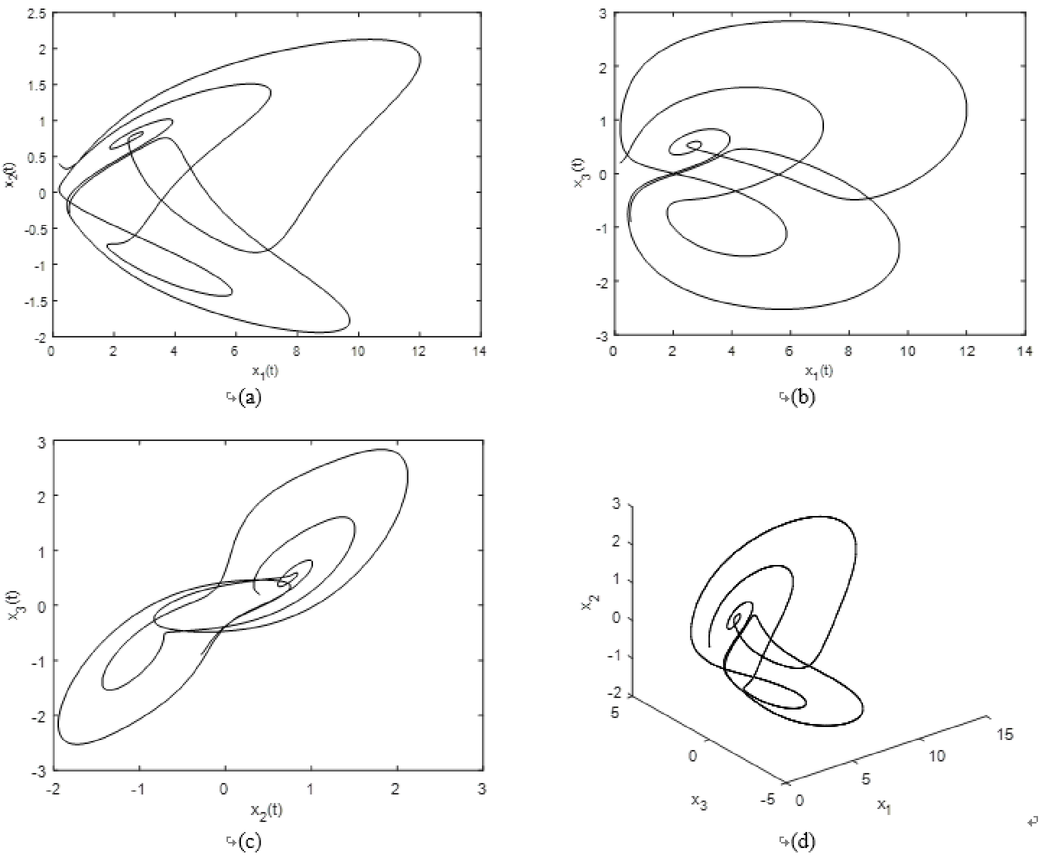


Figure 2. 2D and 3D graph of chaotic system

Equilibrium points are estimated by these calculations:

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, E_1 = \begin{bmatrix} 2.590 \\ 0.7670 \\ 0.5095 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} -2.590 \\ -0.7670 \\ 0.5095 \end{bmatrix}, E_3 = \begin{bmatrix} -3.4089 \\ 1.0449 \\ -0.9134 \end{bmatrix},$$
$$E_4 = \begin{bmatrix} 3.4089 \\ -1.0449 \\ -0.9134 \end{bmatrix}, \tag{7}$$

Jacobian matrix of system Eq. 1 can be presented as follows:

$$J = \begin{bmatrix} -a & bx_3 + a & bx_2 \\ 4x_3 & -30{x_2}^2 - 1 & 4x_1 \\ -x_2 & -x_1 & c \end{bmatrix} = \begin{bmatrix} -3 & 14x_3 + 3 & 14x_2 \\ 4x_3 & -30{x_2}^2 - 1 & 4x_1 \\ -x_2 & -x_1 & 3.9 \end{bmatrix}, \tag{8}$$

Jacobian matrix in equilibrium point E_0 has been estimated as follows:

$$J_0 = J(E_0) = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3.9 \end{bmatrix} \tag{9}$$

Eigenvalue of J_0 is estimated as follows:

$$\lambda_{0,1} = -3, \lambda_{0,2} = -1, \lambda_{0,3} = 3.9, \tag{10}$$

It can be observed that eigenvalue of $\lambda_{0,3}$ is positive and thus equilibrium point E_0 is a saddle point and hence, according to Lyapanov stability theory, equilibrium point E_0 is instable. Now, Jacobian matrix is calculated in equilibrium points of E_1, E_2:

$$J_1 = J(E_1) = \begin{bmatrix} -3.000 & 10.1330 & 10.7380 \\ 2.0380 & -18.6487 & 10.3600 \\ -0.7670 & -2.5900 & 3.9000 \end{bmatrix}, J_2 = J(E_2) = \begin{bmatrix} -3.000 & 10.1330 & -10.7380 \\ 2.0380 & -18.6487 & -10.3600 \\ 0.7670 & 2.5900 & 3.9000 \end{bmatrix} \tag{11}$$

Eigenvalue of J1, J2 is estimated as follows:

$$\lambda_{1,1} = \lambda_{2,1} = -19.1231,$$
$$\lambda_{1,2} = \lambda_{2,2} = 0.6872 + 3.4271i,$$
$$\lambda_{1,3} = \lambda_{2,3} = 0.6872 - 3.4271i, \tag{12}$$

Therefore, equilibrium points of E_1, E_2 are saddle-focus point and hence, these points are instable. Similarly, eigenvalue of equilibrium points of E_3, E_4 are obtained as follows:

$$\lambda_{3,1} = \lambda_{4,1} = -33.9537,$$
$$\lambda_{3,2} = \lambda_{4,2} = 0.5496 + 4.5771i,$$
$$\lambda_{3,3} = \lambda_{4,3} = 0.5496 - 4.5771i, \tag{13}$$

Therefore, equilibrium points of E_3, E_4 are saddle-focus points. Thus all equilibrium points of the chaotic system Eq. 1 are instable. In this paper, only the problem of equilibrium point E0 is investigated.

4. OPTIMAL CONTROL OF FOUR-SCROLL CHAOTIC SYSTEM

In this section, optimal control of Four-scroll chaotic system is investigated. For optimal control, controlled Four-scroll chaotic system is considered:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + bx_2x_3 + u_1, \\ \dot{x}_2 = -10x_2^3 - x_2 + 4x_1x_3 + u_2, \\ \dot{x}_3 = cx_3 - x_1x_2 + u_3, \end{cases} \tag{14}$$

where, u_1, u_2, u_3 stand for control inputs that meet optimality conditions and are achieved by Pontryagin's minimum principle (PMP). The proposed control strategy is designing optimal control inputs of u_1, u_2, u_3 so that state trajectories in limited time interval $[0, t_f]$ are oriented to instable equilibrium point E_0. Thus, boundary condition should be as follows:

$$\begin{aligned} &x_1(0) = x_{10}, \; x_1(t_f) = 0, \\ &x_2(0) = x_{20}, \; x_2(t_f) = 0, \\ &x_3(0) = x_{30}, \; x_3(t_f) = 0, \end{aligned} \tag{15}$$

Objective function that should be minimum is defined as follows:

$$J = \frac{1}{2}\int_0^{t_f} (\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \beta_1 u_1^2 + \beta_2 u_2^2 + \beta_3 u_3^2) dt, \tag{16}$$

where, $\alpha_i (i = 1,2,3)$ and β_i are positive parameters. Now, optimal condition according to Pontryagin's minimum principle is obtained in the form of nonlinear two-point boundary value problems (TPBVPs). The corresponding Hamiltonian function will be as follows:

$$H = -\tfrac{1}{2}[\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \beta_1 u_1^2 + \beta_2 u_2^2 + \beta_3 u_3^2] + \lambda_1[ax_2 - ax_1 + bx_2x_3 + u_1] + \lambda_2[-10x_2^3 - x_2 + 4x_1x_3 + u_2] + \lambda_3[cx_3 - x_1x_2 + u_3] \tag{17}$$

where, $\lambda_i (i = 1,2,3)$ stand for costate variables. According to PMP, Hamiltonian function is obtained as follows:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1}, \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2}, \dot{\lambda}_3 = -\frac{\partial H}{\partial x_3}, \tag{18}$$

By replacing Hamiltonian function H from Eq. 17 in relation Eq. 18, costate equations are achieved as follows:

$$\begin{cases} \dot{\lambda}_1 = a\lambda_1 + x_2\lambda_3 - 4x_3\lambda_2 + \alpha_1 x_1, \\ \dot{\lambda}_2 = \alpha_2 x_2 - a\lambda_1 - b\lambda_1 x_3 + 30\lambda_2 x_2^2 \\ \qquad +\lambda_2 + \lambda_3 x_1, \\ \dot{\lambda}_3 = \alpha_3 x_3 - b\lambda_1 x_2 - 4x_1\lambda_2 - c\lambda_3, \end{cases} \tag{19}$$

Optimal control functions that should be used, are obtained using the condition $\frac{\partial H}{\partial u_i} = 0 \; (i = 1,2,3)$. Thus,

$$u_i^* = \frac{\lambda_i}{\beta_i} \; (i = 1,2,3) \tag{20}$$

By inserting u_i^* from Eq. 20 in Eq. 14, this relation is obtained.

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + bx_2x_3 + \frac{\lambda_1}{\beta_1}, \\ \dot{x}_2 = -10{x_2}^3 - x_2 + 4x_1x_3 + \frac{\lambda_2}{\beta_2}, \\ \dot{x}_3 = cx_3 - x_1x_2 + \frac{\lambda_3}{\beta_3}, \end{cases} \tag{21}$$

Ordinary differential equations (ODEs) Eq. 21 and Eq. 19 are perfect systems for optimal control of Four-scroll chaotic system. Boundary condition of this system is presented in relation Eq. 15. It should be noted that this problem is TPBVP by solving of which, optimal control law and optimal state trajectories are obtained. As can be seen, relations Eq. 19 and Eq. 21 is a nonlinear boundary value problem that is not solved in general manner. Thus, Modal Series Method is used for solving this problem.

5. MODAL SERIES METHOD

In this section, Modal Series approach for solving nonlinear boundary value problem Eq. 21 and Eq. 19 with boundary condition Eq. 15 is used. Thus the following problem equation apparatus is considered:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) + bx_2x_3 + \frac{\lambda_1}{\beta_1} = \\ \qquad g_1(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \\ \dot{x}_2 = -10{x_2}^3 - x_2 + 4x_1x_3 + \frac{\lambda_2}{\beta_2} = \\ \qquad g_2(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \\ \dot{x}_3 = cx_3 - x_1x_2 + \frac{\lambda_3}{\beta_3} = \\ \qquad g_3(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \end{cases} \tag{22}$$

$$\begin{cases} \dot{\lambda}_1 = a\lambda_1 + x_2\lambda_3 - 4x_3\lambda_2 + \alpha_1 x_1 = \\ \qquad g_4(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \\ \dot{\lambda}_2 = \alpha_2 x_2 - a\lambda_1 - b\lambda_1 x_3 + 30\lambda_2 {x_2}^2 + \\ \qquad \lambda_2 + \lambda_3 x_1 = \\ \qquad g_5(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \\ \dot{\lambda}_3 = \alpha_3 x_3 - b\lambda_1 x_2 - 4x_1\lambda_2 - c\lambda_3 = \\ \qquad g_6(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \end{cases} \tag{23}$$

$$x(0) = [x_1(0), x_2(0), x_3(0)] = x_0, \lambda(\infty) = [\lambda_1(\infty), \lambda_2(\infty), \lambda_3(\infty)] = 0 \tag{24}$$

where, $g_i(\cdot)$ ، $i = 1, \cdots, 6$ is analytical function with $g_i(0) = 0$. Taylor series of non-polynomial terms is required in the first step. Since all nonlinear terms in Eq. 21 and Eq. 19 are non-polynomial, there is no need for extension of Taylor series. Obviously, the response is function of initial condition and time; thus response of the abovementioned nonlinear system to initial condition x0 is written as follows:

$$\begin{cases} x(t) = \theta(x_0, t) \\ \lambda(t) = \bar{\theta}(x_0, t) \end{cases} \tag{25}$$

And $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ and $\lambda(t) = \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{bmatrix}$. Moreover, the functions $\theta, \bar{\theta}$ are analytical function in relation to initial condition x_0. It can be shown that $\bar{\theta}(0,t)=\theta(0,t)=0$. Therefore, Maclaurin series of these functions in relation to initial condition x_0 can be written as follows:

$$x(t)=\theta(x_0,t)=\underbrace{\theta(0,t)}_{0}+\underbrace{\left.\frac{\partial\theta(x_0,t)}{\partial x_0}\right|_{x_0=0}x_0}_{g_1(t)}+\underbrace{\frac{1}{2!}\begin{bmatrix} x_0^T\left(\left.\frac{\partial^2\theta_1(x_0,t)}{\partial x_0^2}\right|_{x_0=0}\right)x_0 \\ \vdots \\ x_0^T\left(\left.\frac{\partial^2\theta_i(x_0,t)}{\partial x_0^2}\right|_{x_0=0}\right)x_0 \end{bmatrix}}_{g_2(t)}+\cdots=\sum_{j=1}^{\infty}g_j(t), \quad (26)$$

$$\lambda(t)=\bar{\theta}(x_0,t)=\underbrace{\bar{\theta}(0,t)}_{0}+\underbrace{\left.\frac{\partial\bar{\theta}(x_0,t)}{\partial x_0}\right|_{x_0=0}x_0}_{h_1(t)}+\underbrace{\frac{1}{2!}\begin{bmatrix} x_0^T\left(\left.\frac{\partial^2\bar{\theta}_1(x_0,t)}{\partial x_0^2}\right|_{x_0=0}\right)x_0 \\ \vdots \\ x_0^T\left(\left.\frac{\partial^2\bar{\theta}_i(x_0,t)}{\partial x_0^2}\right|_{x_0=0}\right)x_0 \end{bmatrix}}_{h_2(t)}+\cdots=\sum_{j=1}^{\infty}h_j(t), \quad (27)$$

where, θ_i and $\bar{\theta}_i$ represent the ith entries of vector functions θ and $\bar{\theta}$ .In addition since the functions θ and $\bar{\theta}$ are analytical to x0, presence of Maclaurin in relations Eq. 26 and Eq. 27 and uniformly convergence of these relations are ensured. This theorem is true for every initial condition of x_0. Therefore, if initial condition is εx_0, with ε being an arbitrary scalar parameter, it can be written according to Eq. 26 and Eq. 27 that:

$$x(t)=\theta(\varepsilon x_0,t)=\varepsilon g_1(t)+\varepsilon^2 g_2(t)+\varepsilon^3 g_3(t)+\cdots=\sum_{j=1}^{\infty}\varepsilon^j g_j(t)\rightarrow x_i(t)=\sum_{j=1}^{\infty}\varepsilon^j g_{ij}(t), i=1,2,3 \quad (28)$$

$$\lambda(t)=\bar{\theta}(\varepsilon x_0,t)=\varepsilon h_1(t)+\varepsilon^2 h_2(t)+\varepsilon^3 h_3(t)+\cdots=\sum_{j=1}^{\infty}\varepsilon^j h_j(t)\rightarrow \lambda_i(t)=\sum_{j=1}^{\infty}\varepsilon^j h_{ij}(t), i=1,2,3 \quad (29)$$

Replacing above response in Eq. 22 and Eq. 23 and arrangement according to ε results in:

$$\begin{cases} \varepsilon\dot{g}_{11}(t)+\varepsilon^2\dot{g}_{12}(t)+\cdots= \\ \quad \varepsilon\begin{pmatrix} a\,g_{21}(\mathrm{t})-a\,g_{11}(\mathrm{t})+ \\ \frac{h_{11}(\mathrm{t})}{\beta_1} \end{pmatrix}+ \\ \quad \varepsilon^2\begin{pmatrix} a\,g_{22}(\mathrm{t})-a\,g_{12}(\mathrm{t})+ \\ \mathrm{b}\,g_{21}(\mathrm{t})g_{31}(\mathrm{t})+\frac{h_{12}(\mathrm{t})}{\beta_1} \end{pmatrix}+\cdots, \\ \varepsilon\dot{g}_{21}(t)+\varepsilon^2\dot{g}_{22}(t)+\cdots= \\ \quad \varepsilon\left(-g_{21}(\mathrm{t})+\frac{h_{21}(\mathrm{t})}{\beta_2}\right)+ \\ \quad \varepsilon^2\begin{pmatrix} 4\,g_{11}(\mathrm{t})g_{31}(\mathrm{t})- \\ g_{22}(\mathrm{t})+\frac{h_{22}(\mathrm{t})}{\beta_2} \end{pmatrix}+\cdots, \\ \varepsilon\dot{g}_{31}(t)+\varepsilon^2\dot{g}_{32}(t)+\cdots= \\ \quad \varepsilon\left(\mathrm{c}\,g_{31}(\mathrm{t})+\frac{h_{31}(\mathrm{t})}{\beta_3}\right)+ \\ \quad \varepsilon^2\begin{pmatrix} -g_{11}(\mathrm{t})g_{21}(\mathrm{t})+ \\ c\,g_{32}(\mathrm{t})+\frac{h_{32}(\mathrm{t})}{\beta_3} \end{pmatrix}+\cdots, \end{cases} \quad (30)$$

$$\begin{cases} \varepsilon\dot{h}_{11}(t)+\varepsilon^2\dot{h}_{12}(t)+\cdots= \\ \quad \varepsilon(a\,h_{11}(\mathrm{t})+\alpha_1 g_{11}(\mathrm{t}))+ \\ \quad \varepsilon^2\begin{pmatrix} a\,h_{12}(\mathrm{t})+g_{21}(\mathrm{t})h_{31}(\mathrm{t})- \\ 4g_{31}(\mathrm{t})h_{21}(\mathrm{t})+\alpha_1 g_{12}(\mathrm{t}) \end{pmatrix}+\cdots, \\ \varepsilon\dot{h}_{21}(t)+\varepsilon^2\dot{h}_{22}(t)+\cdots= \\ \quad \varepsilon\big(h_{21}(\mathrm{t})-a\,h_{11}(\mathrm{t})+\alpha_2 g_{21}(\mathrm{t})\big)+ \\ \quad \varepsilon^2\begin{pmatrix} -b\,g_{31}(\mathrm{t})h_{11}(\mathrm{t}) \\ -a\,h_{12}(\mathrm{t})+g_{11}(\mathrm{t})h_{31}(\mathrm{t}) \\ +h_{22}(\mathrm{t})+\alpha_2 g_{22}(\mathrm{t}) \end{pmatrix}+\cdots, \\ \varepsilon\dot{h}_{31}(t)+\varepsilon^2\dot{h}_{32}(t)+\cdots= \\ \quad \varepsilon\big(-c\,h_{31}(\mathrm{t})+\alpha_3 g_{31}(\mathrm{t})\big)+ \\ \quad \varepsilon^2\begin{pmatrix} -b\,g_{21}(\mathrm{t})h_{11}(\mathrm{t})-c\,h_{32}(\mathrm{t})- \\ 4g_{11}(\mathrm{t})h_{21}(\mathrm{t})+\alpha_3 g_{32}(\mathrm{t}) \end{pmatrix}+\cdots, \end{cases} \quad (31)$$

In above relations, by equalizing the terms with identical coefficients of ε powers, a sequence of time invariable nonlinear differential equations is obtained as follows:

$$\varepsilon^2:\begin{cases} \dot{g}_{12}(t)=a\,g_{22}(\mathrm{t})-a\,g_{12}(\mathrm{t})+ \\ \quad \mathrm{b}\,g_{21}(\mathrm{t})g_{31}(\mathrm{t})+\frac{h_{12}(\mathrm{t})}{\beta_1} \\ \dot{g}_{22}(t)=4\,g_{11}(\mathrm{t})g_{31}(\mathrm{t}) \\ \quad -g_{22}(\mathrm{t})+\frac{h_{22}(\mathrm{t})}{\beta_2} \\ \dot{g}_{32}(t)=-g_{11}(\mathrm{t})g_{21}(\mathrm{t}) \\ \quad +c\,g_{32}(\mathrm{t})+\frac{h_{32}(\mathrm{t})}{\beta_3} \\ \dot{h}_{12}(t)=a\,h_{12}(\mathrm{t})+g_{21}(\mathrm{t})h_{31}(\mathrm{t}) \\ \quad -4g_{31}(\mathrm{t})h_{21}(\mathrm{t})+\alpha_1 g_{12}(\mathrm{t}) \\ \dot{h}_{22}(t)=-b\,g_{31}(\mathrm{t})h_{11}(\mathrm{t}) \\ \quad -a\,h_{12}(\mathrm{t})+g_{11}(\mathrm{t})h_{31}(\mathrm{t}) \\ \quad +h_{22}(\mathrm{t})+\alpha_2 g_{22}(\mathrm{t}) \\ \dot{h}_{32}(t)=-b\,g_{21}(\mathrm{t})h_{11}(\mathrm{t}) \\ \quad -c\,h_{32}(\mathrm{t})-4g_{11}(\mathrm{t})h_{21}(\mathrm{t}) \\ \quad +\alpha_3 g_{32}(\mathrm{t}) \end{cases}$$

$$\varepsilon^1:\begin{cases} \dot{g}_{11}(t)=a\,g_{21}(\mathrm{t})-a\,g_{11}(\mathrm{t}) \\ \quad +\frac{h_{11}(\mathrm{t})}{\beta_1} \\ \dot{g}_{21}(t)=-g_{21}(\mathrm{t})+\frac{h_{21}(\mathrm{t})}{\beta_2} \\ \dot{g}_{31}(t)=\mathrm{c}\,g_{31}(\mathrm{t})+\frac{h_{31}(\mathrm{t})}{\beta_3} \\ \dot{h}_{11}(t)=a\,h_{11}(\mathrm{t})+\alpha_1 g_{11}(\mathrm{t}) \\ \dot{h}_{21}(t)=h_{21}(\mathrm{t})-a\,h_{11}(\mathrm{t}) \\ \quad +\alpha_2 g_{21}(\mathrm{t}) \\ \dot{h}_{31}(t)=-c\,h_{31}(\mathrm{t})+\alpha_3 g_{31}(\mathrm{t}) \end{cases} \quad (32)$$

It can be seen that the abovementioned equation system should be solved reversibly. Boundary condition is obviously required for solving this system. For estimation of new boundary condition, $t_0=0$, $t=\infty$ is replaced in Eq. 28 and Eq. 29:

$$x_i(t_0)=\varepsilon x_{i0}=\varepsilon g_{i1}(t_0)+\varepsilon^2 g_{i2}(t_0)+\varepsilon^3 g_{i3}(t_0)+\cdots, \quad (33)$$

$$\lambda_i(\infty)=0=\varepsilon h_{i1}(\infty)+\varepsilon^2 h_{i2}(\infty)+\varepsilon^3 h_{i3}(\infty)+\cdots, \quad (34)$$

In these relations for $i = 1,2,3$, by equalizing the terms with identical coefficients of ε powers, boundary condition is achieved as follows:

$$\begin{cases} g_{i1}(t_0) = x_{i0}, \\ h_{i1}(\infty) = 0, \end{cases} \quad i = 1,2,3$$
$$\begin{cases} g_{ij}(t_0) = 0, \\ h_{ij}(\infty) = 0,. \end{cases} \quad j \geq 2 \tag{35}$$

Finally, the response of nonlinear boundary value problem with infinite time horizon Eq. 19 and Eq. 21 can be written as follows:

$$x_i(t) = \sum_{j=1}^{\infty} g_{ij}(t), \tag{36}$$

$$\lambda_i(t) = \sum_{j=1}^{\infty} h_{ij}(t), \tag{37}$$

where, the j-th order terms of $g_{ij}(t)$ and $h_{ij}(t)$ are obtained only by reversible solving of the sequence of time invariable linear boundary value problems Eq. 32 with boundary condition Eq. 35. Based on aforementioned notes, state variable and optimal control law for nonlinear optimal control problem can be expressed as follows:

$$x_i(t) = \sum_{j=1}^{\infty} g_{ij}(t), \tag{38}$$

$$u_i^* = \frac{\lambda_i(t)}{\beta_i} = \frac{1}{\beta_i} \sum_{j=1}^{\infty} h_{ij}(t), \tag{39}$$

6. PRACTICAL APPLICATION AND DESIGN OF SUBOPTIMAL CONTROL

Since it is impossible to achieve response in series form according to Eq. 38 and Eq. 39 as it contains infinite terms, in practical applications, state variable and M order suboptimal control are obtained by replacing ∞ with a positive integer M as follows:

$$x_i^{(M)}(t) = \sum_{j=1}^{M} g_{ij}(t),$$
$$u_i^{*(M)} = \frac{1}{\beta_i} \sum_{j=1}^{M} h_{ij}(t), \tag{40}$$

where, M is determined according to required accuracy of the problem. Moreover, there is no constraint for selection of M value. State variable and M order suboptimal control law is accurate if the following condition is met for predefined positive constant σ:

$$\left| \frac{J^{(M)} - J^{(M-1)}}{J^{(M)}} \right| < \sigma, \tag{41}$$

$$J^{(M)} = \frac{1}{2} \int_0^{t_f} (\alpha_1 (x_1^M)^2 + \alpha_2 (x_2^M)^2 + \alpha_3 (x_3^M)^2 + \beta_1 (u_1^M)^2 + \beta_2 (u_2^M)^2 + \beta_3 (u_3^M)^2) dt, \tag{42}$$

For achieving suboptimal control law with sufficient accuracy, an algorithm with low calculation has been proposed.

Step 1. The repetition index j is assigned 1.

Step 2. The j -th order terms of $g_{ij}(t)$ and $h_{ij}(t)$ are calculated by solving the sequence of time invariable linear boundary value problems Eq. 32 with boundary condition Eq. 35.

Step3. M is equalized with j and $x_i^{(M)}(t)$ and $u_i^{*(M)}(t)$ are estimated from Eq. 40 and $J^{(M)}$ is calculated from Eq. 42.

Step4. If stop condition Eq. 41 is the case for predefined positive constant σ, go to step 5. Otherwise, add a unit to j and go to step 2.

Step5. The algorithm is stopped and state variable and suboptimal control law and $x_i^{(M)}(t)$ and $u_i^{*(M)}(t)$ are sufficiently accurate.

7. NUMERICAL SOLVING

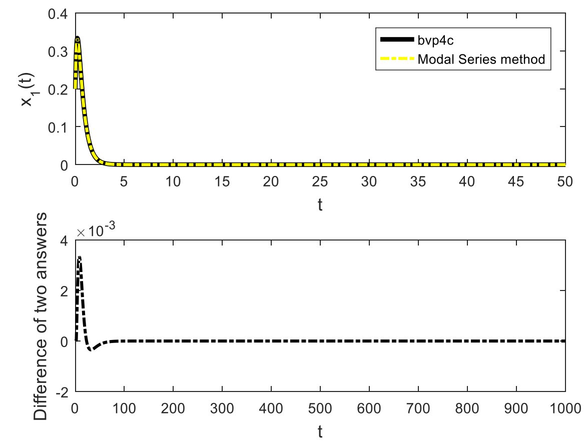


Figure 3. Time response of state variable $x_1(t)$. Difference graph between bvp4e and Modal Series responses with 7 approximate

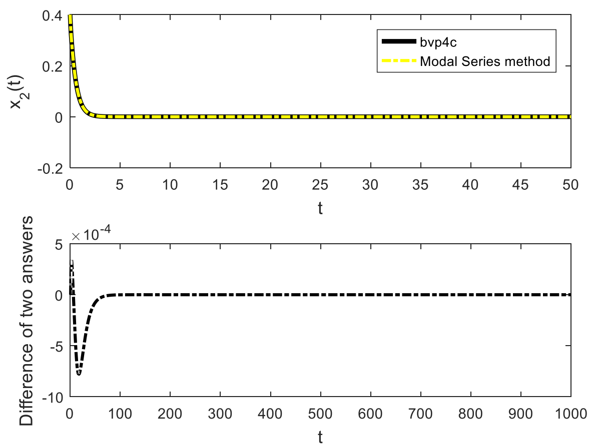


Figure 4. Time response of state variable $x_2(t)$. Difference graph between bvp4e and Modal Series responses with 7 approximate

This section shows efficiency of the proposed method for optimal control of four-scroll chaotic system. In numerical simulations, Modal Series Method and bvp4c of Matlab are used for solving TPBVP. The proposed algorithm for $\sigma = 14 \times 10^{-3}$ and positive constants in the function J with $\alpha_i = 1, \beta_i = 1$ values has been performed to find state variable and suboptimal control law with suitable accuracy. Under this condition, convergence is achieved only after 7 repetitions that is $\left|\frac{J^{(5)}-J^{(4)}}{J^{(5)}}\right| = 13.2 \times 10^{-3} < 15 \times 10^{-3}$. This shows high speed of convergence of the proposed algorithm. The results of simulation of state variable are presented in Fig. 3- Fig. 5. Moreover, simulated graphs have been obtained by direct solving of nonlinear boundary value problem using bvp4c function. The graph of error between real and approximate responses of Modal Series is also presented in Fig. 3- Fig. 5. As seen, the responses achieved by the proposed method accord with those obtained by direct solving. This indicates high accuracy of the proposed method.

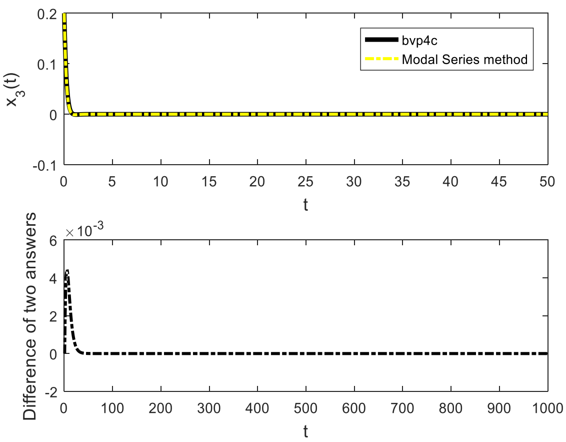


Figure 5. Time response of state variable $x_3(t)$. Difference graph between bvp4e and Modal Series responses with 7 approximate

8. CONCLUSION

In this paper, Modal Series technique was used to solve the problem of optimal control with infinite time horizon for fur-scroll chaotic system. Nonlinear boundary value obtained in this technique is converted to a sequence of time invariable linear boundary value using Pontryagin's minimum principle. By reversible resolving of this sequence, state trajectory and optimal control law are obtained in the form of series with uniform convergence. Then, by considering a limited number of the related series terms, approximate response for state variable and sub-optimal control law is obtained. Moreover, an algorithm was proposed for practical implementation of Modal Series Method that provides close responses with suitable accuracy for state variable and sub-optimal control law.

REFERENCES

[1] Lorenz EN. (1963). Deterministic nonperiodic flow. Journal of the Atmospheric Sciences 20(2): 130-141. https://doi.org/10.1175/15200469(1963)020<0130:DNF>2.0.CO;2

[2] Rössler OE. (1976). An equation for continuous chaos. Physics Letters A 57(5): 397-398. https://doi.org/10.1016/0375-9601(76)90101-8

[3] Rabinovich MI, Fabrikant AL. (1979). Stochastic self-modulation of waves in nonequilibrium media. Journal of Experimental and Theoretical Physics 77: 617-629.

[4] Arneodo A, Coullet P, Tresser C. (1981). Possible new strange attractors with spiral structure. Communications in Mathematical Physics 79(4): 573-579. https://doi.org/10.1007/BF01209312

[5] Sprott JC. (1994). Some simple chaotic flows. Physical Review E 50(2): 647-650. https://doi.org/10.1103/PhysRevE.50.R647

[6] Chen G, Ueta T. (1999). Yet another chaotic attractor. International Journal of Bifurcation and Chaos 9(07): 1465-1466. https://doi.org/10.1142/S0218127499001024

[7] Lü J, Chen G. (2002). A new chaotic attractor coined. International Journal of Bifurcation and Chaos 12(03): 659-661. https://doi.org/10.1142/S0218127402004620

[8] Shaw R. (1981). Strange attractors, chaotic behavior, and information flow. Zeitschrift für Naturforschung A 36(1): 80-112. https://doi.org/10.1515/zna-1981-0115

[9] Cai G, Tan Z. (2007). Chaos synchronization of a new chaotic system via nonlinear control. Journal of Uncertain Systems 1(3): 235-240. https://doi.org/10.1.1.509.4486

[10] Tigan G, Opriş D. (2008). Analysis of a 3D chaotic system. Chaos, Solitons & Fractals 36(5): 1315-1319. https://doi.org/10.1016/j.chaos.2006.07.052

[11] Kennedy MP. (1994). Chaos in the colpitts oscillator. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications 41(11): 771-774. https://doi.org/10.1109/81.331536

[12] Zhou W, Xu Y, Lu H, Pan L. (2008). On dynamics analysis of a new chaotic attractor. Physics Letters A, 372(36): 5773-5777. https://doi.org/10.1016/j.physleta.2008.07.032

[13] Li D. (2008). A three-scroll chaotic attractor. Physics Letters A 372(4): 387-393. https://doi.org/10.1016/j.physleta.2007.07.045

[14] Sundarapandian V. (2013). Analysis and anti-synchronization of a novel chaotic system via active and adaptive controllers. Journal of Engineering Science and Technology Review 6(4): 45-52.

[15] Sundarapandian V, Pehlivan I. (2012). Analysis, control, synchronization, and circuit design of a novel chaotic system. Mathematical and Computer Modelling 55(7-8): 1904-1915. https://doi.org/10.1016/j.mcm.2011.11.048

[16] Zhu CX, Liu YH, Guo Y. (2010). Theoretic and numerical study of a new chaotic system. Intelligent Information Management 2(2): 104. https://doi.org/10.4236/iim.2010.22013

[17] Pan L, Zhou W, Fang JA, Li D. (2010). A new three-scroll unified chaotic system coined. International Journal of Nonlinear Science 10(4): 462-474.

[18] Liu X, Shen XS, Zhang H. (2012). Multi-scroll chaotic and hyperchaotic attractors generated from chen system. International Journal of Bifurcation and Chaos 22(02): 1250033. https://doi.org/10.1142/S0218127412500332

[19] Kengne J, Chedjou JC, Kenne G, Kyamakya K. (2012). Dynamical properties and chaos synchronization of improved colpitts oscillators. Communications in Nonlinear Science and Numerical Simulation 17(7): 2914-2923. https://doi.org/10.1016/j.cnsns.2011.10.038

[20] Li N, Pan W, Yan L, Luo B, Zou X. (2014). Enhanced chaos synchronization and communication in cascade-coupled semiconductor ring lasers. Communications in Nonlinear Science and Numerical Simulation 19(6): 1874-1883. https://doi.org/10.1016/j.cnsns.2013.09.036

[21] Iqbal S, Zang X, Zhu Y, Zhao J. (2014). Bifurcations and chaos in passive dynamic walking: A review. Robotics

and Autonomous Systems 62(6): 889-909. https://doi.org/10.1016/j.robot.2014.01.006
[22] Volos CK, Kyprianidis IM, Stouboulos IN. (2013). Experimental investigation on coverage performance of a chaotic autonomous mobile robot. Robotics and Autonomous Systems 61(12): 1314-1322. https://doi.org/10.1016/j.robot.2013.08.004
[23] Naas TT, Lasbet Y, Benzaoui A, Loubar K. (2016). Characterization of pressure drops and heat transfer of non-newtonian power-law fluid flow flowing in chaotic geometry. International Journal of Heat and Technology 34(2): 251-260. DOI: 10.18280/ijht.340214
[24] Boukhalkhal AL, Lasbet Y, Makhlouf M, Loubar K. (2017). Numerical study of the chaotic flow in three-dimensional open geometry and its effect on the both fluid mixing and heat performances. International Journal of Heat and Technology 35(1). https://doi.org/10.18280/ijht.350101
[25] Kyprianidis IM, Makri AT. (2013). Complex dynamics of FitzHugh-Nagumo type neurons coupled with gap junction under external voltage stimulation. Journal of Engineering Science and Technology Review 6(4): 104-114.
[26] Abdullah A. (2013). Synchronization and secure communication of uncertain chaotic systems based on full-order and reduced-order output-affine observers. Applied Mathematics and Computation 219(19): 10000-10011. https://doi.org/10.1016/j.amc.2013.03.133
[27] Paul S, Dasgupta P, Naskar PK, Chaudhuri A. (2017). Secured image encryption scheme based on DNA encoding and chaotic map. International Information and Engigneering Technology Assocation 1: 0-75. https://doi.org/10.18280/rces.040206
[28] Huber AW. (1989). Adaptive control of chaotic system. Helv Acta 62: 343-346
[29] Li N, Yuan H, Sun H, Zhang Q. (2013). An impulsive multi-delayed feedback control method for stabilizing discrete chaotic systems. Nonlinear Dynamics 73(3): 1187-1199. https://doi.org/10.1016/j.chaos.2003.12.071
[30] Sun K, Liu X, Zhu C, Sprott JC. (2012). Hyperchaos and hyperchaos control of the sinusoidally forced simplified Lorenz system. Nonlinear Dynamics 69(3): 1383-1391. https://doi.org/10.1007/s11071-012-0354-x
[31] Zhang X, Zhu H, Yao H. (2012). Analysis of a new three-dimensional chaotic system. Nonlinear Dynamics 67(1): 335-343. https://doi.org/10.1007/s11071-011-9981-x
[32] Tsai JSH, Fang JS, Yan JJ, Dai MC, Guo SM, Shieh LS. (2018). Hybrid robust discrete sliding mode control for generalized continuous chaotic systems subject to external disturbances. Nonlinear Analysis: Hybrid Systems 29: 74-84. https://doi.org/10.1016/j.nahs.2018.01.001
[33] Zhang R, Zeng D, Park JH, Liu Y, Zhong S. (2018). A new approach to stabilization of chaotic systems with nonfragile fuzzy proportional retarded sampled-data control. IEEE Transactions on Cybernetics. https://doi.org/10.1109/TCYB.2018.2831782
[34] Bouzeriba A, Boulkroune A, Bouden T. (2015, May). Fuzzy adaptive synchronization of a class of fractional-order chaotic systems. In Control, Engineering & Information Technology (CEIT), 2015 3rd International Conference on 1-6. IEEE. https://doi.org/10.1109/CEIT.2015.7233073
[35] Vaidyanathan S. (2017). A new 3-D jerk chaotic system with two cubic nonlinearities and its adaptive backstepping control. Archives of Control Sciences 27(3): 409-439. https://doi.org/10.1515/acsc-2017-0026
[36] Lien CH, Vaidyanathan S, Sambas A, Mamat M, Sanjaya WSM. (2018). A new two-scroll chaotic attractor with three quadratic nonlinearities, its adaptive control and circuit design. In IOP Conference Series: Materials Science and Engineering 332(1): 012010. https://doi.org/10.1088/1757-899X/332/1/012010
[37] Mamat M, Vaidyanathan S, Sambas A, Sanjaya WSM. (2018). A novel double-convection chaotic attractor, its adaptive control and circuit simulation. In IOP Conference Series: Materials Science and Engineering 332(1): https://doi.org/10.1088/1757-899X/332/1/012033
[38] Vaidyanathan S. (2015). Adaptive control design for the anti-synchronization of novel 3-D chemical chaotic reactor systems. Parameters 8: 654–668.
[39] Vaidyanathan S, Volos C. (2015). Analysis and adaptive control of a novel 3-D conservative no-equilibrium chaotic system. Archives of Control Sciences 25(3): 333-353.
[40] Vaidyanathan S. (2015). Dynamics and control of Tokamak system with symmetric and magnetically confined plasma. International Journal of ChemTech Research 8(6): 795-803.
[41] Tang GY. (2005). Suboptimal control for nonlinear systems: a successive approximation approach. Systems & Control Letters 54(5): 429-434. https://doi.org/10.1016/j.sysconle.2004.09.012
[42] Tang GY, Qu HP, Gao YM. (2002). Sensitivity approach of suboptimal control for a class of nonlinear systems. J. Ocean Univ. Qingdao 32(4): 615-620.
[43] Pariz N. (2001). Analysis of nonlinear system behavior: the case of stressed power systems. Ph. D. dissertation, Department of Electrical Engineering, Ferdowsi University, Mashhad, Iran.
[44] Jajarmi A, Pariz N, Effati S, Kamyad AV. (2012). Infinite horizon optimal control for nonlinear interconnected large-scale dynamical systems with an application to optimal attitude control. Asian Journal of Control 14(5): 1239-1250. https://doi.org/10.1002/asjc.452
[45] Jajarmi A, Pariz N, Vahidian Kamyad A, Effati S. (2011). A novel modal series representation approach to solve a class of nonlinear optimal control problems. International Journal of Innovative, Computing, Information and Control 7: 1413–1425.
[46] Jajarmi A, Pariz N, Kamyad AV, Effati S. (2012). A highly computational efficient method to solve nonlinear optimal control problems. Scientia Iranica 19(3): 759-766. https://doi.org/10.1016/j.scient.2011.08.029
[47] Jajarmi A, Pariz N, Effati S, Kamyad AV. (2011). Solving infinite horizon nonlinear optimal control problems using an extended modal series method. Journal of Zhejiang University SCIENCE C 12(8): 667. https://doi.org/10.1631/jzus.C1000325
[48] Sampath S, Vaidyanathan S, Volos CK, Pham VT. (2015). An eight-term novel four-scroll chaotic system with cubic nonlinearity and its circuit simulation. Journal of Engineering Science and Technology Review 8(2): 1-6.