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# Common fixed point theorems on cone metric spaces with c-Distance

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**ABSTRACT**

The purpose of this paper is to prove common fixed point theorems by using the c-Distance in a cone metric space with different types of contractive conditions. Our theorem extends the contractive condition from constant real numbers to some control functions.

## 1. INTRODUCTION

In 2007, Huang and Zhang [11] introduced the cone metric space. Later, many authors proved several fixed and common fixed point results in cone metric spaces (see [4, 6, 7, 8, 9, 10, 12, 17]). Recently, Wang and Guo [14] introduced the concept of c-Distance in a cone metric spaces, which is a cone version of w-Distance of Kada et al [13]. Afterward, large number of fixed point theorems were considered by other authors (see [1, 2, 3, 5, 15, 16, 17]). In this paper, we extend and generalize the results of Kaewkhao et al. [5], Rahimi et al. [7] and Young et al. [17]. Before presenting our theorems, we recall some notations, definitions and examples needed in our subsequent discussions.

**Definition 1.1.** [11] Let $E$ be a real Banach space and $P$ a subset of $E$. Then $P$ is called a cone if and only if

(a) $P$ is closed, non-empty and $P \neq \{\theta\}$;

(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

(c) $if\ x \in P\ and - x \in P, then\ x = \theta$.

For any cone $P \subseteq E$, the partial ordering $\preccurlyeq$ with respect to $P$ is defined by $x \preccurlyeq$ y if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preccurlyeq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in$ int $P$, where int $P$ denotes the interior of $P$. A cone $P$ is called normal if there exists a number $K$ such that for all $x, y \in E$, $\theta \preccurlyeq x \preccurlyeq y$ implies $\parallel x \parallel \leq K \parallel y \parallel$.

The least positive number satisfying the above inequality is called the normal constant of $P$.

**Definition 1.2.** [11] Let $X$ be a non-empty set and $E$ be a real Banach space equipped with

the partial ordering $\preccurlyeq$ with respect to the cone $P \subseteq E$.

Suppose that the mapping $d: X$ x $X \to E$ satisfies the following conditions:

(d1) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

**Definition 1.3. [**11] Let $(X, d)$ be a cone metric space, $\{x_n\}$ a sequence in $X$ and $x \in X$. Then

(1) $\{x_n\}$ converges to $x$ if for every $c \in E$ with $\theta \ll c$ there exists an $n_0 \in \mathbb{N}$ such

that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $lim_{n\to\infty} d(x_n, x) = \theta$.

(2) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists an

$n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$ .We denote this by $lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

(3) If every Cauchy sequence in $X$ is convergent, then $X$ is called a complete cone metric space.

**Lemma 1.4.** [11] Let $(X, d)$ be a cone metric space and $P$ be a normal cone with constant $K$. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ and $x, y \in X$. Then the following hold:

(1) $\{x_n\}$ converges to $x$ if and only if $d(x_n, x) \to \theta\ as\ n \to \infty$.

(2) If $\{x_n\}$ converges to $x$ and $\{x_n\}$ converges to $y$, then $x = y$.

(3) If $\{x_n\}$ converges to $x$, then $\{x_n\}$ is a Cauchy sequence.

(4) If $x_n \to x$ and $y_n \to y$ as $\to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

(5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.

**Lemma 1.5.** [10, 16] Let $E$ be a real Banach space with a cone $P$ in $E$. Then, for all $u, v, w, c \in E$, the following hold:

(1) If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$.

(2) If $\theta \preccurlyeq u \ll c$ for each $c \in int\ P$, then $u = \theta$.

(3) If $u \preccurlyeq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = \theta$.

(4) Let $x_n \to \theta$ in E,$\theta \preccurlyeq x_n$ and $\theta \ll c$. Then there exists positive integer $n_0$ such that $x_n \ll c$ for each $n > n_0$.

(5) If $\theta \preccurlyeq u \preccurlyeq v$ and k is a nonnegative real number, then $\theta \preccurlyeq ku \preccurlyeq kv$.

(6) If $\theta \preccurlyeq u_n \preccurlyeq v_n$ for all $n \in N$ and $u_n \to u, v_n \to v\ as\ n \to \infty$, then $\theta \preccurlyeq u \preccurlyeq v$.

Next, we give the notion of c-Distance on a cone metric space $(X, d)$ of Wang and Guo in [14], which is a generalization of w-Distance of Kada et al. [13] and some properties.

**Definition 1.6.** [14] Let $(X,d)$ be a cone metric space. Then a function $q: X \times X \to E$ is called a c-Distance on $X$ if the following are satisfied:

$(q_1)$ $\theta \preccurlyeq q(x,y)$ for all $x,y \in X$ ;

$(q_2)$ $q(x,z) \preccurlyeq q(x,y) + q(y,z)$ for all $x,y,z \in X$;

$(q_3)$ for each $x \in X$ and $n \geq 1$, if $q(x,y_n) \preccurlyeq u$ for some $u = u_x \in P$, then $q(x,y) \preccurlyeq u$ whenever $\{y_n\}$ is a sequence in $X$ converging to a point $y \in X$;

$(q_4)$ for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll$c.

**Example 1.7.** [14] Let $E = \mathbb{R}$ and $P = \{x \in E: x \geq 0\}$. Let $X = [0,\infty)$ and define a mapping $d: X \times X \to E$ by $d(x,y) = |x-y|$ for all $x,y \in X$. Then $(X,d)$ is a cone metric space. Define a mapping $q: X \times X \to E$ by $q(x,y) = y$ for all $x,y \in X$. Then $q$ is a c-Distance.

**Remark 1.8.** For c-Distance $q$ , $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ and $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

**Lemma 1.9.** [5, 14, 15] Let $(X,d)$ be a cone metric space and let $q$ be a c-Distance on $X$. Also, let $\{x_n\}$ and $\{y_n\}$ be sequence in $X$ and $x,y,z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in $P$ converging to $\theta$. Then the following hold:

$(qp_1)$ If $q(x_n,y) \preccurlyeq u_n$ and $q(x_n,z) \preccurlyeq v_n$ for $n \in \mathbb{N}$, then $y = z$. Specifically, if $q(x,y) = \theta$ and $q(x,z) = \theta$ then $y = z$.

$(qp_2)$ If $q(x_n,y_n) \preccurlyeq u_n$ and $q(x_n,z) \preccurlyeq v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$converges to $z$.

$(qp_3)$ If $q(x_n,x_m) \preccurlyeq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in $X$.

$(qp_4)$ If $q(y,x_n) \preccurlyeq u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in $X$.

## 2. MAIN RESULT

**Theorem 2.1.** Let $(X,d)$ be a cone metric space, $P$ be a normal cone with constant $K$ and $q$ be a c-Distance. Also, let $f,g: X \to X$ be two mappings with $f(X) \subseteq g(X)$ and let $g(X)$ be a complete subspace of $X$. Suppose that there exist mappings $k,l,r: X \to [0,1)$ such that the following conditions hold:

(a) $k(fx) \leq k(gx), l(fx) \leq l(gx),\ r(fx) \leq r(gx)$ for all $x \in X$;

(b) $(k+2l+2r)(x) < 1$ for all $x \in X$;

(c) $q(fx,fy) \preccurlyeq k(gx)q(gx,gy) + l(gx)[q(gx,fy) + q(gy,fx)] + r(gx)[q(gx,fx) + q(gy,fy)]$ for all $x,y \in X$;

(d) $q(fy,fx) \preccurlyeq k(gy)q(gy,gx) + l(gy)[q(fy,gx) + q(fx,gy)] + r(gy)[q(fx,gx) + q(fy,gy)]$ for all $x,y \in X$.

If $f$ and $g$ satisfy $\inf\{\| q(fx,y) \| + \| q(gx,y) \| + \| q(gx,fx) \| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$ or $y \neq gy$, then $f$ and $g$ have a common fixed point in $X$.

**Proof**. Let $x_0$ be an arbitrary point in $X$. Since $f(X) \subseteq g(X)$ there exists a point $x_1 \in X$ such that $fx_0 = gx_1$. By induction we construct the sequence $\{x_n\}$ in $X$ such that

$$fx_n = gx_{n+1} \text{ for } n = 0,1,2,3, \quad (2.1)$$

Now, set $x = x_{n-1}$ and $y = x_n$ in (c). Thus, by $(q_2)$, for $n \geq 1$,

we get $q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n)$

$$q(fx_{n-1},fx_n) \preccurlyeq k(gx_{n-1})q(gx_{n-1},gx_n) + l(gx_{n-1})[q(gx_{n-1},fx_n) + q(gx_n,fx_{n-1})] + r(gx_{n-1})[q(gx_{n-1},fx_{n-1}) + q(fx_n,gx_n)] = k(fx_{n-2})q(gx_{n-1},gx_n) + l(fx_{n-2})[q(gx_{n-1},gx_{n+1}) + q(gx_n,gx_n)] + r(fx_{n-2})[q(gx_{n-1},gx_n) + q(gx_{n+1},gx_n)] \preccurlyeq k(gx_{n-2})q(gx_{n-1},gx_n) + l(gx_{n-2})[q(gx_{n-1},gx_n) + q(gx_n,gx_{n+1})] + r(gx_{n-2})[q(gx_{n-1},gx_n) + q(gx_n,gx_{n+1})] \preccurlyeq k(gx_0)q(gx_{n-1},gx_n) + l(gx_0)[q(gx_{n-1},gx_n) + q(gx_n,gx_{n+1})] + r(gx_0)[q(gx_{n-1},gx_n) + q(gx_n,gx_{n+1})]. \quad (2.2)$$

Similarly, set $x = x_{n-1}$ and $y = x_n$ in (d). Thus by $(q_2)$, for $n \geq 1$, we get

$$q(fx_n,fx_{n-1}) = q(gx_{n+1},gx_n) \preccurlyeq k(gx_n)q(gx_n,gx_{n-1}) + l(gx_n)[q(fx_n,gx_{n-1}) + q(fx_{n-1},gx_n)] + r(gx_n)[q(fx_{n-1},gx_{n-1}) + q(fx_n,gx_n)] = k(fx_{n-1})q(gx_n,gx_{n-1}) + l(fx_{n-1})[q(gx_{n+1},gx_{n-1}) + q(gx_n,gx_n)] + r(fx_{n-1})[q(gx_n,gx_{n-1}) + q(gx_{n+1},gx_n)] \preccurlyeq k(gx_{n-1})q(gx_n,gx_{n-1}) + l(gx_{n-1})[q(gx_{n+1},gx_n) + q(gx_n,gx_{n-1})] + r(gx_{n-1})[q(gx_n,gx_{n-1}) + q(gx_{n+1},gx_n)] \preccurlyeq k(gx_0)q(gx_n,gx_{n-1}) + l(gx_0)[q(gx_{n+1},gx_n) + q(gx_n,gx_{n-1})] + r(gx_0)[q(gx_n,gx_{n-1}) + q(gx_{n+1},gx_n)]. \quad (2.3)$$

Adding up (2.2) and (2.3), we have

$$q(gx_n,gx_{n+1}) + q(gx_{n+1},gx_n) \preccurlyeq (k(gx_0) + l(gx_0) + r(gx_0))[q(gx_{n-1},gx_n) + q(gx_n,gx_{n-1})] + (l(gx_0) + r(gx_0))[q(gx_n,gx_{n+1}) + q(gx_{n+1},gx_n)]. \quad (2.4)$$

Now, set $v_n = q(gx_n,gx_{n+1}) + q(gx_{n+1},gx_n)$ in (2.4), we have $v_n \preccurlyeq \big(k(gx_0) + l(gx_0) + r(gx_0)\big)v_{n-1} + \big(l(gx_0) + r(gx_0)\big)v_n$.

So, $v_n \preccurlyeq \mu v_{n-1}$ for all $n \geq 1$ with $\mu = \frac{k(gx_0)+l(gx_0)+r(gx_0)}{1-l(gx_0)-r(gx_0)} < 1$.

Since $(k+2l+2r)(x) < 1$ for all $x \in X$.

Continuing this process, we get $v_n \preccurlyeq \mu^n v_0$ for $n = 0,1,2$. Thus

$$q(gx_n,gx_{n+1}) \preccurlyeq v_n \preccurlyeq \mu^n(q(gx_0,gx_1) + q(gx_1,gx_0)) \quad (2.5)$$

for all $n = 0,1,2$---. Now, for positive integer $m$ and $n$ with $m > n \geq 1$, it follows from (2.5) and $\mu < 1$, we have

$$q(gx_n,gx_m) \preccurlyeq q(gx_n,gx_{n+1}) + q(gx_{n+1},gx_{n+2}) + -- + q(gx_{m-1},gx_m) \preccurlyeq (\mu^n + \mu^{n+1} + -- - \mu^{m-1})(q(gx_0,gx_1) + q(gx_1,gx_0)) \preccurlyeq \frac{\mu^n}{1-\mu}\big(q(gx_0,gx_1) + q(gx_1,gx_0)\big). \quad (2.6)$$

From Lemma 1.9, we have $\{gx_n\}$ is a Cauchy sequence in $X$. Since $g(X)$ is a complete subspace of $X$, there exists a point $z \in g(X)$ such that $gx_n \to z$ as $n \to \infty$. By (2.6) and $(q_3)$, we have $q(gx_n,z) \preccurlyeq \frac{\mu^n}{1-\mu}(q(gx_0,gx_1) + q(gx_1,gx_0)), n = 0,1,2,\ldots$.

Since $P$ is a normal cone with normal constant $K$, we have

$\| q(gx_n, z) \| \le K \frac{\mu^n}{1-\mu} \| q(gx_0, gx_1) + q(gx_1, gx_0) \| . n = 0, 1, 2,$ (2.7)

And

$$\| q(gx_n, gx_m) \| \le K \frac{\mu^n}{1-\mu} \| q(gx_0, gx_1) + q(gx_1, gx_0) \| \quad (2.8)$$

for all $m > n \ge 1$. If $fz \neq z$ or $gz \neq z$, then by the hypothesis (2.7) and (2.8) with $m = n + 1$, we have
$0 < \inf\{\| q(fx, z) \| + \| q(gx, z) \| + \| q(gx, fx) \| : x \in X\}$
$\le \inf\{\| q(fx_n, z) \| + \| q(gx_n, z) \| + \| q(gx_n, fx_n) \| : n \ge 1\} = \inf\{\| q(gx_{n+1}, z) \| + \| q(gx_n, z) \| + \| q(gx_n, gx_{n+1}) \| : n \ge 1\} \le \inf\{K \frac{\mu^{n+1}}{1-\mu} \| q(gx_0, gx_1) + q(gx_1, gx_0) \| + K \frac{\mu^n}{1-\mu} \| q(gx_0, gx_1) + q(gx_1, gx_0) \|$
$+ K \frac{\mu^n}{1-\mu} \| q(gx_0, gx_1) + q(gx_1, gx_0) \| : n \ge 1\} = 0,$
which is a contradiction. Therefore, we can conclude that $z = fz = gz$. This completes the proof.

The following Corollary is obtained from Theorem 2.1.

**Corollary 2.2.** Let $(X, d)$ be a cone metric space, $P$ be a normal cone with constant $K$ and $q$ be a c-Distance on $X$. Suppose that the mappings $f, g: X \to X$ satisfy the following two contractive conditions:

(i) $q(fx, fy) \preccurlyeq kq(gx, gy) + l[q(gx, fy) + q(gy, fx)] + r[q(gx, fx) + q(gy, fy)]$ for all $x, y \in X$;

(ii) $q(fy, fx) \preccurlyeq kq(gy, gx) + l[q(fy, gx) + q(fx, gy)] + r[q(fx, gx) + q(fy, gy)]$ for all $x, y \in X$;

$k, l, r$ are nonnegative constants such that $k + 2l + 2r < 1$.

If the range of $g$ contains the range of $f$, $g(X)$ is a complete subspace of X, $f$ and $g$ satisfy
$\inf\{\| q(fx, y) \| + \| q(gx, y) \| + \| q(gx, fx) \| : x \in X\} > 0,$
for all $y \in X$ with $y \neq fy$ or $y \neq gy$, then $f$ and $g$ have a common fixed point in $X$.

**Proof**: We can prove this result by applying Theorem 2.1 with $k(x) = k$, $l(x) = l$ and $r(x) = r$.

In Theorem 2.1, if $g = i_X$ is the identity map on $X$, then we get the Theorem 3.3 of Dubey et al. [3] on c-Distance in a cone metric space.

**Theorem 2.3.** Let $(X, d)$ be a complete cone metric space and $P$ be normal cone with constant $K$. Also let $q$ be a c-Distance and $f: X \to X$ be a mapping. Suppose that there exist mappings $k, l, r: X \to [0,1)$ such that the following conditions hold:

(a) $k(fx) \le k(x)$, $l(fx) \le l(x)$, $r(fx) \le r(x)$ for all $x \in X$;

(b) $(k + 2l + 2r)(x) < 1$ for all $x \in X$;

(c) $q(fx, fy) \preccurlyeq k(x)q(x, y) + l(x)[q(x, fy) + q(y, fx)] + r(x)[q(x, fx) + q(y, fy)]$ for all $x, y \in X$;

(d) $q(fy, fx) \preccurlyeq k(y)q(y, x) + l(y)[q(fy, x) + q(fx, y)] + r(y)[q(fx, x) + q(fy, y)]$ for all $x, y \in X$.

If $f$ satisfies $\inf\{\| q(fx, y) \| + \| q(x, y) \| + \| q(x, fx) \| : x \in X\} > 0$, for all $y \in X$ with $y \neq fy$, then $f$ has a fixed point in $X$.

**Corollary 2.4.** Let $(X, d)$ be a complete cone metric space, $P$ be a normal cone with constant $K$ and $q$ be a c-Distance on $X$. Suppose that the mapping $f: X \to X$ satisfies the following two contractive conditions:

(i) $q(fx, fy) \preccurlyeq kq(x, y) + l[q(x, fy) + q(y, fx)] + r[q(x, fx) + q(y, fy)]$ for all $x, y \in X$;

(ii) $q(fy, fx) \preccurlyeq kq(y, x) + l[q(fy, x) + q(fx, y)] + r[q(fx, x) + q(fy, y)]$ for all $x, y \in X$;

where $k, l, r$ are nonnegative constants such that $k + 2l + 2r < 1$.

If $f$ satisfies $\inf\{\| q(fx, y) \| + \| q(x, y) \| + \| q(x, fx) \| : x \in X\} > 0$ for all $y \in X$ with $y \neq fy$ then $f$ has a fixed point in $X$.

**Proof.** We can prove this result by applying Theorem 2.3 with $k(x) = k$, $l(x) = l$ and $r(x) = r$.

**Theorem 2.5.** Let $(X, d)$ be a cone metric space, $P$ be a normal cone with constant $K$ and $q$ be a c-Distance. Also, let $f, g: X \to X$ be two mappings with $f(X) \subseteq g(X)$ and let $g(X)$ be a complete subspace of $X$. Suppose that there exist mappings $k, l, r, t: X \to [0,1)$ such that the following conditions hold:

(a) $k(fx) \le k(gx)$, $l(fx) \le l(gx)$, $r(fx) \le r(gx)$, $t(fx) \le t(gx)$ for all $x \in X$;

(b) $(k + l + r + 2t)(x) < 1$ for all $x \in X$;

(c) $q(fx, fy) \preccurlyeq k(gx)q(gx, gy) + l(gx)q(fy, gy) + r(gx)q(fx, gx) + t(gx)[q(fx, gy) + q(fy, gx)]$ for all $x, y \in X$;

(d) $q(fy, fx) \preccurlyeq k(gy)q(gy, gx) + l(gy)q(gy, fy) + r(gy)q(gx, fx) + t(gy)[q(gy, fx) + q(gx, fy)]$ for all $x, y \in X$.

If $f$ and $g$ satisfy $\inf\{\| q(fx, y) \| + \| q(gx, y) \| + \| q(gx, fx) \| : x \in X\} > 0$ for all $x, y \in X$ with $y \neq fy$ or $y \neq gy$, then $f$ and $g$ have a common fixed point in $X$.

**Proof.** Let $x_0$ be an arbitrary point in $X$. Since $f(X) \subseteq g(X)$, there exists a point $x_1 \in X$ such that $fx_0 = gx_1$. By induction we construct the sequence $\{x_n\}$ in X such that

$$fx_n = gx_{n+1} \text{ for } n = 0, 1, 2, \ldots.. \quad (2.9)$$

Now, set $x = x_{n-1}$ and $y = x_n$ in (c). Thus, by $(q_2)$, for $n \ge 1$, we get

$$q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n)\, q(fx_{n-1}, fx_n) \preccurlyeq k(gx_{n-1})q(gx_{n-1}, gx_n)l(gx_{n-1})q(fx_n, gx_n) + r(gx_{n-1})q(fx_{n-1}, gx_{n-1}) + t(gx_{n-1})[q(fx_{n-1}, gx_n) + q(fx_n, gx_{n-1})] = k(fx_{n-2})q(gx_{n-1}, gx_n) + l(fx_{n-2})q(gx_{n+1}, gx_n) + + r(fx_{n-2})q(gx_n, gx_{n-1}) + t(fx_{n-2})[q(gx_n, gx_n) + q(gx_{n+1}, gx_{n-1})]k(gx_{n-2})q(gx_{n-1}, gx_n) + l(gx_{n-2})q(gx_{n+1}, gx_n) + + r(gx_{n-2})q(gx_n, gx_{n-1}) + t(gx_{n-2})[q(gx_{n+1}, gx_n) + q(gx_n, gx_{n-1})] \preccurlyeq k(gx_0)q(gx_{n-1}, gx_n) + l(gx_0)q(gx_{n+1}, gx_n) + r(gx_0)q(gx_n, gx_{n-1}) + t(gx_0)[q(gx_{n+1}, gx_n) + q(gx_n, gx_{n-1})]. \quad (2.10)$$

Similarly, set $x = x_{n-1}$ and $y = x_n$ in (d). Thus by $(q_2)$, for $n \ge 1$, we get

$$q(gx_{n+1}, gx_n) = q(fx_n, fx_{n-1}) \preccurlyeq k(gx_0)q(gx_n, gx_{n-1}) + l(gx_0)q(gx_n, gx_{n+1}) + r(gx_0)q(gx_{n-1,}, gx_n) + t(gx_0)[q(gx_n, gx_{n+1}) + q(gx_{n-1,}, gx_n)]. \quad (2.11)$$

Adding up (2.10) and (2.11), we have

$$q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)(k(gx_0) + r(gx_0) + t(gx_0))[q(gx_{n-1}, gx_n) + q(gx_n, gx_{n-1})] + (l(gx_0) + t(gx_0))[(q(gx_{n+1}, gx_n) + q(gx_n, gx_{n+1})]. \quad (2.12)$$

Now, set $v_n = q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_n)$ in (2.12), we have

$$v_n \preccurlyeq \left(k(gx_0) + r(gx_0) + t(gx_0)\right)v_{n-1} + (l(gx_0) + t(gx_0))v_n.$$

So $v_n \preccurlyeq \mu v_{n-1}$ for all $n \geq 1$ with $\mu = \frac{k(gx_0)+r(gx_0)+t(gx_0)}{1-l(gx_0)-t(gx_0)} < 1$.

Since $(k + l + r + 2t)(x) < 1$ for all $x \in X$.

Continuing this process, we get $v_n \preccurlyeq \mu^n v_0$ for $n = 0, 1, 2, \ldots\ldots$

Rest of the proof of this theorem is similar as the Theorem 2.1.

**Example 2.6.** Let $E = \mathbb{R}$ and $P = \{x \in E: x \geq 0\}$. Let $X = [0,1]$ and define a mapping $d: X \text{ x } X \rightarrow E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then $(X, d)$ is a cone metric space. Define a mapping $q: X \text{ x } X \rightarrow E$ by $q(x, y) = 2d(x, y)$ for all $x, y \in X$. Then $q$ is a c-Distance. In fact, $(q_1) - (q_3)$ are immediate.

Let $c \in E$ with $0 \ll c$ put $e = \frac{c}{2}$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have $d(x, y) \leq 2d(x, y) = 2|x - y| \leq 2|x - z| + 2|z - y| = q(z, x) + q(z, y) \ll e + e = c$.

This shows that $(q_4)$ holds. Therefore q is a c-Distance.

Let $f, g: X \rightarrow X$ defined by $g(x) = x$ and $f(x) = \frac{x^2}{16}$ for all $x \in X$.

Take mappings $k, l, r, t: X \rightarrow [0,1)$ by $k(x) = \frac{x+1}{16}$, $r(x) = \frac{2x+3}{16}$, $l(x) = \frac{3x+2}{16}$, $t(x) = \frac{x}{16}$ for all $x \in X$. Observe that

(i) $k(fx) = (\frac{x^2}{16} + 1)/16 = \frac{1}{16}\left(\frac{x^2}{16} + 1\right) \leq \frac{1}{16}(x + 1) = k(x) = k(gx)$.

$(ii) r(fx) = (2(\frac{x^2}{16}) + 3)/16 = \frac{1}{16}\left(\frac{2x^2}{16} + 3\right) \leq \frac{1}{16}(2x + 3) = r(x) = r(gx)$.

$(iii) l(fx) = (3(\frac{x^2}{16}) + 2)/16 = \frac{1}{16}\left(\frac{3x^2}{16} + 2\right) \leq \frac{1}{16}(3x + 2) = l(x) = l(gx)$.

(iv) $t(fx) = (\frac{x^2}{16})/16 = \frac{1}{16}\left(\frac{x^2}{16}\right) \leq \frac{1}{16}(x) = t(x) = t(gx)$.

(v) $(k + l + r + 2t)(x) = \left(\frac{x+1}{16}\right) + \left(\frac{3x+2}{16}\right) + \left(\frac{2x+3}{16}\right) + 2(\frac{x}{16}) = \left(\frac{8x+6}{16}\right) < 1$ for all $x \in X$.

(vi) for all $x, y \in X$, we have

$$q(fx, fy) = 2\left|\frac{x^2}{16} - \frac{y^2}{16}\right| \preccurlyeq \frac{2|x+y||x-y|}{16} = \left(\frac{x+y}{16}\right)2|x - y|$$
$$\preccurlyeq k(x)q(x, y) = k(gx)q(gx, gy)$$
$$\preccurlyeq k(gx)q(gx, gy) + l(gx)q(fy, gy) + r(gx)q(fx, gx) + t(gx)[q(fx, gy) + q(fy, gx)].$$

Therefore, all the conditions of Theorem 2.5 are satisfied. Hence $f$ and $g$ have a common fixed point in $X$. This common fixed point is $x = 0$.

## 3. CONCLUSION

In this paper we develop and generalize the common fixed point theorems on c-Distance of Kaewkhao et al. [5], Rahimi et al. [7] and Young et al. [17]. One illustrative example is also furnished to highlight the realized improvements.


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