FEM-BEM iterative solution of electrostatic problems with floating potential conductors

Giovanni Aiello, Salvatore Alfonzetti, Nunzio Salerno

Dipartimento di Ingegneria Elettrica, Elettronica e Informatica (DIEEI)
Università di Catania
Viale A. Doria, 6 , 95125 Catania, Italy
alfo@dieei.unict.it

ABSTRACT. *This paper describes two iterative procedures to solve efficiently the global algebraic systems of equations obtained by applying the hybrid FEM-BEM method to the solution of open-boundary electrostatic problems in the presence of floating potential conductors. In both methods, non-standard boundary elements are used. In the first procedure the conjugate gradient solver is used to solve the FEM equations, whereas the BEM equations are solved by the direct LU solver. In the second method, the GMRES solver is applied to a reduced system virtually available, in which the unknowns are the values of the normal derivatives of the electric potential on the truncation boundary. The proposed methods are also applicable to other kind of electromagnetic problems such as magnetostatic and static current density problems.*

RÉSUMÉ. *Cet article décrit deux procédures itératives pour résoudre efficacement les systèmes globaux d'équations algébriques obtenus en appliquant la méthode FEM-BEM à la solution de problèmes électrostatiques en domaines illimités en présence de conducteurs avec potentiels flottant. Des éléments de frontière non-standard sont utilisés. Dans la première procédure, le solveur du gradient conjugué est utilisé pour résoudre les équations aux éléments finis, tandis que les équations des éléments de frontière sont résolues par le solveur direct avec la décomposition LU. Dans la seconde procédure, le solveur GMRES est appliqué à un système réduit virtuellement disponible, dans lequel les inconnues sont les valeurs nodales de la dérivée normale du potentiel électrique sur la frontière de troncature. Les deux méthodes proposées sont également applicables à d'autres types de problèmes électromagnétiques, tels que des problèmes magnétostatiques et de champ de courant statique.*

KEYWORDS: *finite element method, Boundary element method, hybrid methods, iterative solutions, GMRES, floating potentials.*

MOTS-CLÉS : *méthode des éléments finis, méthode des éléments de frontière, méthodes hybrides, solutions itératives, GMRES, potentiels flottants.*

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1. Introduction

In computational electromagnetics (Ferrari *et al.*, 1996; Jin, 1993), very often, open-boundary field problems are faced with by means of hybrid methods, such as FEM-BEM (Finite Element Method - Boundary Element Method) (Brebbia *et al.*, 1984; Zienkiewicz *et al.*, 1977; Salon *et al.*, 1988) and FEM-DBCI (Dirichlet Boundary Condition Iteration) (Aiello *et al.*, 1994, 1996).

Both methods introduce a fictitious truncation boundary, say Γ_T, and couple a differential Equation for the interior problem with an integral one, which expresses the unknown boundary condition on Γ_T by involving the free-space Green function.

The differences are the following. FEM-BEM assumes an unknown Neumann boundary condition on Γ_T, whereas FEM-DBCI a Dirichlet one. The FEM-BEM truncation boundary is the support of the integral equation, whereas in the FEM-DBCI this support is another surface, strictly enclosed by the truncation boundary Γ_T, so that singularities are avoided (Aiello *et al.*, 1994).

The FEM-DBCI global system is partly sparse and partly dense and is solved efficiently in an iterative way. Assuming an initial guess for the Dirichlet condition on the truncation boundary, the sparse FEM equations are solved by means of the conjugate gradient (CG) solver; then, the dense DBCI equations are used to improve the Dirichlet condition (Aiello *et al.*, 1994), this procedure is iterated until convergence is reached. This solution strategy is efficient because the CG is applied to the sparse equations only, and the dense equations are used only a few times.

The authors have shown that a similar procedure can also be used for the solution of the FEM-BEM algebraic systems for electrostatic problems (Aiello *et al.*, 2007). This solution is obtained by making use of the CG solver for the FEM equations and the direct LU solver for the BEM equations. Moreover, the BEM equations are written in a non-conventional way, by making the nodes of the potential non-coinciding with those of its normal derivative (Alfonzetti *et al.*, 2009).

A more robust approach to the solution of the hybrid FEM-BEM and FEM-DBCI systems is obtained by means of the use of the GMRES (Generalized Minimal Residual) solver (Saad *et al.*, 1986), virtually applied to a suitable reduced system (Aiello *et al.*, 1997; 2008; 2013).

This paper deals with the case in which some conductors with prescribed total charges are present in the open-boundary electrostatic system. In literature, very often these problems are treated by substituting to the true conductors dielectric objects with very high permittivity (Konrad *et al.*, 1996; 1997). This approach is very simple but suffers from some drawbacks: 1) it is inaccurate (Amman *et al.*, 2014); 2) if the objects are large, the FEM mesh is large too, because the interior of such objects need to be meshed; 3) it cannot take into account nonzero total charges, which may be prescribed to lie on some floating conductors.

In the following we show that these shortcomings are completely overcome by the two methods proposed, in which the basic iterative procedure and the GMRES-based one to solve the FEM-BEM algebraic systems are extended to the case in which some floating potential conductors are present in the electrostatic system. The methods incorporate in the solving iterations those relative to the floating potentials.

The paper is structured as follows. In the Section 2 the mathematical and numerical formulations of the relevant equations are derived for a generic electrostatic system. In Section 3 two solving strategies are described for the solution of the global FEM-BEM system. In Section 4 two examples are given. Finally, the authors' conclusions follow in Section 5.

2. FEM-BEM formulation

Consider the electrostatic system depicted in Figure 1, constituted by dielectric objects, charge distributions and conductors embedded in an unbounded dielectric medium (free space). Some conductors are voltaged at given potential values V_k, k=1,.., N_C, with respect to infinity, where the potential is assumed to be zero. The other conductors have assigned total charges Q_h, h=1,…N_F (floating potential conductors).

In order to compute numerically the electric potential v(x,y,z), a fictitious truncation boundary Γ_T is introduced. This boundary must include all the conductors and non-homogeneities, but it may leave out some (lumped or distributed) charges. Note that this boundary may be constituted by several disjoint closed surfaces, so that the whole domain D is decomposed in detached subdomains. In the bounded domain D so obtained, the Poisson equation holds:

$$-\varepsilon_0 \nabla \cdot (\varepsilon_r \nabla v) = \rho \tag{1}$$

where ε_0 is the vacuum electric permittivity, ε_r is the relative permittivity and ρ is the volume charge density. Unknown Neumann and Dirichlet conditions are assumed to hold, respectively, on the truncation boundary Γ_T and on the surfaces Γ_h, h=1,…N_F, of the floating potential conductors.

Discretizing the bounded domain D by means of simplex nodal finite elements of a given order, the following FEM algebraic system is derived (Alfonzetti *et al.*, 2009):

$$\begin{bmatrix} \mathbf{A} & \mathbf{A}_T \\ \mathbf{A}_T^t & \mathbf{A}_{TT} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_T \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A}_F \\ \mathbf{0} \end{bmatrix} \mathbf{v}_F - \begin{bmatrix} \mathbf{0} \\ \mathbf{C} \end{bmatrix} \mathbf{q}_T \tag{2}$$

where: $\mathbf{v}$ and $\mathbf{v}_T$ are the vectors of the unknown values of the potential v at the nodes inside the domain and on the truncation boundary Γ_T, respectively; $\mathbf{v}_F$ is the vector of the unknown values of the potential of the floating conductors; $\mathbf{A}$, $\mathbf{A}_T$, $\mathbf{A}_{TT}$, $\mathbf{A}_F$ and $\mathbf{C}$ are sparse matrices of geometrical coefficients; $\mathbf{b}_0$ is the known term array

due to the voltaged conductors and internal sources; $\mathbf{q}_T$ is the vector of the unknown values of the inward normal derivative $q=\varepsilon_r \partial v/\partial n$ of the potential evaluated at suitable nodes on the truncation boundary. For simplicity, in (2) it is assumed that there are no finite elements that have nodes on the truncation boundary and on the floating conductors simultaneously.

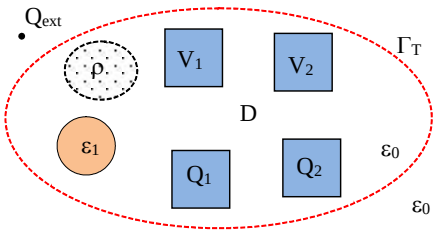

Figure 1. An open-boundary electrostatic system constituted by voltaged and floating conductors, non-homogeneous dielectric objects and distributed charges, enclosed by a fictitious truncation boundary Γ_T

In the boundary elements (segments for triangular and quadrangular finite elements in 2D, triangles for tetrahedral elements in 3D, quadrangles for bricks in 3D) lying on the truncation boundary Γ_T, the nodes P_n of the field variable v are placed in the canonical positions (Silvester *et al.*, 1996), whereas the nodes Q_m of its normal derivative q are placed in between them and internally to the boundary element (see Figure 2), in such a way that the fact that one variable is the derivative of the other is fully exploited (Alfonzetti *et al.*, 2009).

The BEM integral equations are written by using the nodes Q_m of the q variable as poles of the Green function:

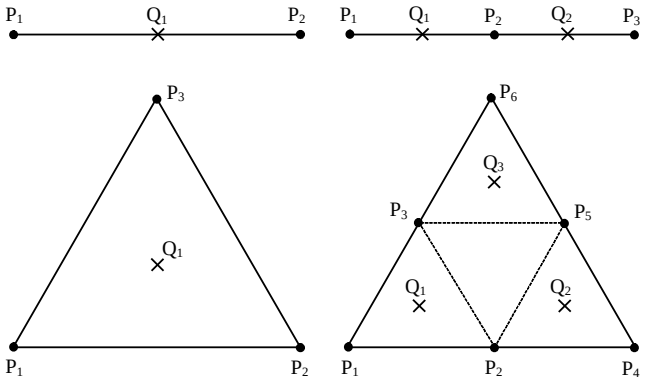

Figure 2. Boundary elements lying on the fictitious boundary. The symbol • denotes the nodes P_n of the potential v, whereas the symbol × denotes the nodes Q_m of its normal derivative q

$$\frac{1}{2}v(Q_m)+\int_{\Gamma_T} v(P)\frac{\partial G(P,Q_m)}{\partial n_P}d\Gamma = v_{ext}(Q_m)+\int_{\Gamma_T} q(P)\,G(P,Q_m)\,d\Gamma \qquad (3)$$

where G is the free-space Green function, given by:

$$G(P,Q)=\frac{1}{2\pi}\ln\frac{1}{r} \qquad \text{or} \qquad G(P,Q)=\frac{1}{4\pi r} \qquad (4)$$

for 2- or 3-dimensional problems, respectively, and where r is the distance between points P and Q.

In matrix form the BEM equations are rewritten as (Alfonzetti *et al.*, 2009):

$$\mathbf{H}\,\mathbf{v}_T = \mathbf{v}_{ext} + \mathbf{G}\,\mathbf{q}_T \qquad (5)$$

where **H** and **G** are dense matrices. Note that matrix **G** is square by construction, whereas **H** may be rectangular.

In order to solve the electrostatic problem, it is necessary to derive other N_F equations for the unknown floating potentials. This can be accomplished in two ways. The most obvious one is by means of the following equations:

$$-\varepsilon_0\int_{\Gamma_h}\varepsilon_r\frac{\partial v}{\partial n}dS = Q_h \qquad h=1,...,N_F \qquad (6)$$

where Γ_h is the surface of the h-th floating conductor and n is the outward versor normal to the surface Γ_h. In the numerical approximation these equations read:

$$\mathbf{F}\,\mathbf{v}_F = \mathbf{q} - \mathbf{E}\,\mathbf{v} \qquad (7)$$

where $\mathbf{q}=[Q_1\ Q_2\ ...Q_{N_F}]_t$ is the column vector of the assigned total charges in the floating conductors, **F** is a diagonal matrix and **E** is a dense rectangular matrix in which non-null entries appear only for the nodes of the elements adjacent to the surfaces Γ_h, and having a face (side) lying on it.

The other way is to write the following equations:

$$\varepsilon_0\int_D \varepsilon_r \nabla v\cdot\nabla w_h\,dD = Q_h \qquad h=1,...,N_F \qquad (8)$$

where w_h is an arbitrary function defined in the whole domain D and such that w_h=1 on the surface Γ_h of the h-th floating conductor and w_h=0 on Γ_T and on the surfaces of all the other conductors. A convenient choice for w_h is to set its nodal values to 1 on the nodes lying on Γ_h and to 0 in all the other nodes. In this way, the function w_h coincides with the sum of the shape functions α_n of all the nodes lying on the surface of the h-th floating conductor:

$$w_h = \sum_{n \in \Gamma_h} \alpha_n \tag{9}$$

and for the numerical approximation of (8), only the Dirichlet FE matrices of the elements adjacent to Γ_h have to be computed (Dular *et al.*, 1998). This equation has a form similar to (7), but the entries of the two matrices are different, so that it is convenient to rewrite it as follows:

$$\mathbf{A}_{FF}\,\mathbf{v}_F = \mathbf{q} - \mathbf{A}_F^t\,\mathbf{v} \tag{10}$$

where $\mathbf{A}_{FF}$ is a diagonal matrix and the matrix which multiplies $\mathbf{v}$ is the transposed of the matrix $\mathbf{A}_F$, already defined in (2). Then the matrix equation (10) can be added to (2) to obtain the global algebraic FE system:

$$\begin{bmatrix} \mathbf{A} & \mathbf{A}_T & \mathbf{A}_F \\ \mathbf{A}_T^t & \mathbf{A}_{TT} & \mathbf{0} \\ \mathbf{A}_F^t & \mathbf{0} & \mathbf{A}_{FF} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_T \\ \mathbf{v}_F \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{0} \\ \mathbf{q} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{C} \\ \mathbf{0} \end{bmatrix} \mathbf{q}_T \tag{11}$$

in which the global matrix in the first side is symmetric and positive definite, and can be solved for $\mathbf{v}$, $\mathbf{v}_T$ and $\mathbf{v}_F$, once the array $\mathbf{q}_T$ is known.

3. Solution of the global system

The global algebraic system is constituted by Equations (2), (5) and (7) or (10). Equation (10) can be added to (2) to obtain (11). This system can be solved by means of several strategies, which will be discussed in the following.

A simple approach to solve the global system

$$\begin{bmatrix} \mathbf{A} & \mathbf{A}_T & \mathbf{A}_F & \mathbf{0} \\ \mathbf{A}_T^t & \mathbf{A}_{TT} & \mathbf{0} & \mathbf{C} \\ \mathbf{A}_F^t & \mathbf{0} & \mathbf{A}_{FF} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} & \mathbf{0} & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v}_T \\ \mathbf{v}_F \\ \mathbf{q}_T \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{0} \\ \mathbf{q} \\ \mathbf{v}_{ext} \end{bmatrix} \tag{12}$$

is by means of an iterative CG-like solver for non-symmetric matrices (Golub *et al.*, 1996). A similar approach can also be used in which the array $\mathbf{q}_T$ is derived from the BEM equations and substituted in the FEM ones, to obtain a reduced system. Both these approaches suffer from the fact that in each iterative step the matrix-vector multiplication is very costly, due to the presence of dense parts in the system matrix. To alleviate the heaviness of the integral equations, whose complexity is N^2, various FMM (Fast Multipole Method) techniques have been used to reach an NlogN complexity (Jin, 1993; Sabariego *et al.*, 2004). However, the integral equation still remains the most time-consuming part of the solution algorithm. Irrespective of whether FMM is used for the BEM equations, it is of great importance to minimize

the number of times that these equations are solved in the context of an iterative solution.

The very different nature of Equations (2) and (5) calls for a special technique to solve them. The sparse Equation (2) could be efficiently solved by means of the standard CG solver for symmetrical matrices, whereas Equation (5) could be efficiently solved by means of the LU factorization. To meet these two opposing requirements, an iterative solution is proposed, similar to that in described in (Aiello *et al.*, 2007):

1') assume initial guesses for the Neumann boundary condition $\mathbf{q}_T$ on Γ_T (e.g. $\mathbf{q}_T=\mathbf{0}$) and for the Dirichlet condition $\mathbf{v}_F$ on the floating potentials;

2') solve the interior FEM problem (2), by means of the CG solver;

3') first decompose the square matrix **G** into **L** and **U** matrices, and then solve equation (5) for $\mathbf{q}_T$: formally:

$$\mathbf{q}_T = \mathbf{G}^{-1}\left\{\mathbf{H}\ \mathbf{v}_T - \mathbf{v}_{ext}\right\} \tag{13}$$

4') solve Equation (10) (or (7)) for $\mathbf{v}_F$:

$$\mathbf{v}_F = \mathbf{A}_{FF}^{-1}\left\{\mathbf{q} - \mathbf{A}_F^t\ \mathbf{v}\right\} \tag{14}$$

5') obtain new guesses for $\mathbf{q}_T$ and $\mathbf{v}_F$ by means of relaxation schemes

$$\mathbf{q}_T^{new} = \gamma_1\,\mathbf{q}_F + (1-\gamma_1)\,\mathbf{q}_T^{old} \tag{15}$$

$$\mathbf{v}_F^{new} = \gamma_2\,\mathbf{v}_F + (1-\gamma_2)\,\mathbf{v}_F^{old} \tag{16}$$

where γ_1 and γ_2 are suitable relaxation coefficients;

6') repeat steps 2'-5' until convergence is reached.

This simple iterative scheme exhibits some advantageous characteristics: i) since the first guess for the CG solver in each step is the solution obtained in the previous iteration step, the various solutions of system (2) get faster as the iteration proceeds; ii) the LU decomposition is performed only once at the beginning of the iterative procedure; iii) the whole iterative procedure is rapidly convergent if appropriate relaxation coefficients γ_1 and γ_1 are selected; iv) consequently, the integral Equation (5) is used only a few times, if compared to its use in an iterative CG-like solver for the whole non-symmetric system (12).

A more robust approach is based on the solution of the reduced system:

$$\mathbf{M}\,\mathbf{q}_T = \mathbf{k} \tag{17}$$

where matrix **M** and vector **k** are formally defined as

$$\mathbf{M} = \mathbf{G} + \mathbf{H}\,\mathbf{A}_Z^{-1}\mathbf{C} \tag{18}$$

$$\mathbf{k} = -\mathbf{v}_{\text{ext}} - \mathbf{H}\,\mathbf{A}_{\text{Z}}^{-1}\mathbf{A}_{\text{T}}^{\text{t}}\mathbf{A}_{\text{S}}^{-1}(\mathbf{b}_0 - \mathbf{A}_{\text{F}}\mathbf{A}_{\text{FF}}^{-1}\mathbf{q}) \tag{19}$$

with: $\mathbf{A}_{\text{S}} = \mathbf{A} - \mathbf{A}_{\text{F}}\mathbf{A}_{\text{FF}}^{-1}\mathbf{A}_{\text{F}}^{\text{t}}$ (20)

$$\mathbf{A}_{\text{Z}} = \mathbf{A}_{\text{TT}} - \mathbf{A}_{\text{T}}^{\text{t}}\mathbf{A}_{\text{S}}^{-1}\mathbf{A}_{\text{T}} \tag{21}$$

Matrix **M** and vector **k** are not directly available. However, vector **k** is simply built as follows:

1) assume a zero initial guess $\mathbf{q}_{\text{T}}^{(0)} = \mathbf{0}$;

2) solve (11) by means of CG to obtain $\mathbf{v}^{(0)}$, $\mathbf{v}_{\text{T}}^{(0)}$ and $\mathbf{v}_{\text{F}}^{(0)}$;

3) compute $\mathbf{k} = -\mathbf{v}_{\text{ext}} + \mathbf{H}\mathbf{v}_{\text{T}}^{(0)}$, which coincides with the initial residual vector.

Matrix **M** can be used to perform matrix-by-vector multiplication $\mathbf{M}\mathbf{q}_{\text{T}}^{(\text{n})}$, as follows:

1) given the vector $\mathbf{q}_{\text{T}}^{(\text{n})}$;

2) solve (11) with $\mathbf{b}_0 = \mathbf{0}$ and $\mathbf{q} = \mathbf{0}$ to obtain $\mathbf{v}^{(\text{n})}$, $\mathbf{v}_{\text{T}}^{(\text{n})}$ and $\mathbf{v}_{\text{F}}^{(\text{n})}$;

3) compute $\mathbf{M}\mathbf{q}_{\text{T}}^{(\text{n})} = \mathbf{G}\mathbf{q}_{\text{T}}^{(\text{n})} - \mathbf{H}\mathbf{v}_{\text{T}}^{(\text{n})}$.

Then, due to this virtual availability of the algebraic system (17), various non-stationary iterative CG-like solvers for non symmetric matrices (Golub *et al.*, 1996) may be used, such as BiCG (BiConjugate Gradient), QMR (Quasi Minimal Residual), CGS (Conjugate Gradient Squared), BiCGstab (BiCG stabilized) and GMRES.

GMRES should be preferred because the matrix-by-vector multiplications in this context are much more expensive than in a system where the coefficient matrix is directly available. In fact, GMRES performs a true minimization of the residual and is thus the optimal method for accelerating the iterative solution of (17) as it minimizes the number of matrix-by-vector multiplications. The residual are computed by using the orthonormal basis of the Krylov subspace, as explained in (Saad *et al.*, 1986). In general, the greater drawbacks of GMRES are the computing time and memory required to compute and store the orthonormal basis, which increases linearly with the number of iterations. In this case, however, the computing time and memory required for the orthonormal basis are only a small fraction of the total, because GMRES works on a reduced system, the number of unknowns being the values of the normal derivative of the potential on the nodes of the truncation boundary.

Applying the method described in this paper, a fast convergence has always been observed: typically from 10 to 20 GMRES steps are needed. For this reason, no preconditioning of the reduced system (17) has been implemented.

Comparing the GMRES-based solving procedure with the simple iterative one, it is to be noted that the GMRES solution does not require the LU decomposition of the square matrix **G**. This is a great advantage, especially for problems with large number of unknowns. Conversely, the various solutions of the FEM equations by means of the CG solver are not related to each other, so the number of CG steps does not decrease as the solution proceeds.

4. Numerical examples

The first example concerns an electrostatic system constituted by a charged square capacitor (thickness t=1cm, edge length l=10t, relative permittivity of the dielectric ε_r=9, Q_C=5nC) in the presence of an external lumped charge Q_{ext}=nC, placed at a distance d=5 t on the top of the center of the upper positive armature of the capacitor, as shown in Figure 3.

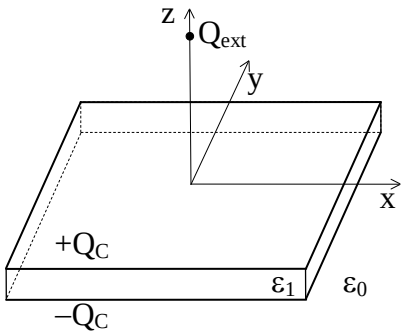

Figure 3. A charged capacitor in the presence of a lumped charge Q_{ext}

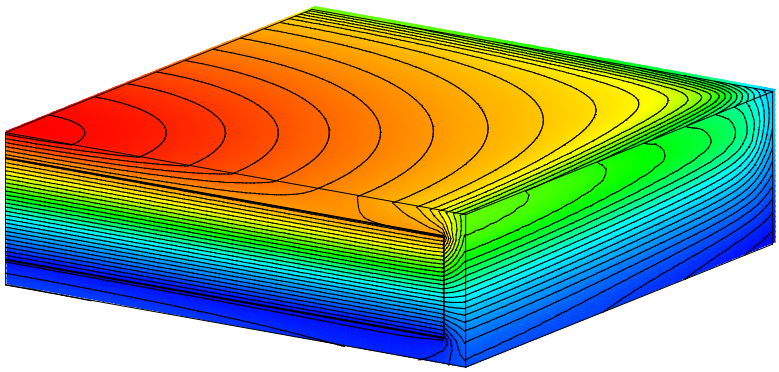

Figure 4. Contours of the potential around the capacitor

Due to symmetry reasons, the analysis is restricted to a quarter of the system, by imposing homogeneous Neumann conditions on the xz and yz planes. The

truncation boundary Γ_T is selected as a parallelepiped, placed at a distance $d_T = t/4$ from the capacitor surface.

The domain is regularly discretized by means of 15876 second-order tetrahedra and 24037 nodes, of which N_T=4633 lie on the truncation boundary and 3362 on the two floating armatures. Having set an end-iteration tolerance of 10^{-4} per cent for the CG solver and 10^{-2} for the GMRES, the procedure converges in 11 iterations. The CPU time is 3.71 s on an AMD Turion 64 X2 Mobile Tech. TL-64, 2.2GHz, 2GB RAM, GNU/Linux 64bit. Figure 4 shows the contours of the electrical potential in the xz symmetry plane and on Γ_T. The potentials of the two armatures are evaluated as V_+=127.5V and V_-=94.32V. The same problem has been solved by means of the basic iterative procedure based on the LU decomposition of matrix **G**. By suitably setting the relaxation parameters to $\gamma_1=\gamma_2$=0.40, convergence is reached in 15 iterations, with a CPU time decrease to about 70% of the GMRES one. The two solutions are in very good agreement: the relative error is less than 10^{-2} per cent. Note that the computing time has decreased even if the number of iterations has increased; this is not strange, because, as said before, in the GMRES-based procedure, the various CG solutions of the FEM equations are not related to each other, whereas in the basic procedure the these solutions are getting faster with the advance of the same (Aiello *et al.*, 2006). However, this good performance of the basic iterative procedure has been obtained with a particular setting of the relaxations parameters; by using other values, the computing time may increase or even convergence may disappear. Since the optimal values of the relaxation parameters are not known *a priori*, it appears more suitable to adopt the GMRES-based procedure, which finds the solution in all cases.

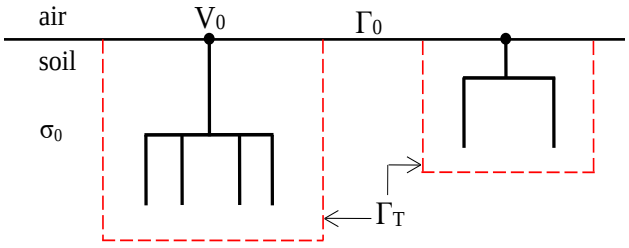

Figure 5. A voltaged grounding system in the proximity of another passive system

The second example deals with the computation of transferred potentials from a grounding system to another one, as shown in Figure 5. The grounding system on the left is voltaged at V_0, whereas the system on the right is floating. The unbounded soil is modeled as a uniform conductive medium.

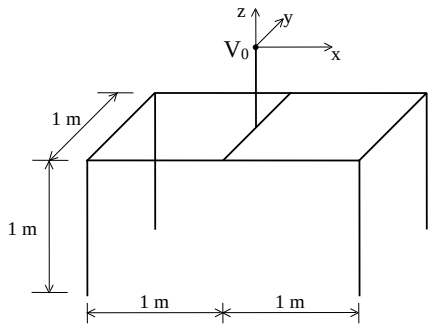

Figure 6. 3D geometry of the voltaged grounding system

In Figure 6 the 3D geometry of the voltaged ground grid is shown. The grid is buried at 0.6 m from the soil surface z=0. The passive grid is a simple bar of length 2 m, buried at 0.6m and aligned along the y-axis, far away 16m from the center of the first one on the x direction. The soil is modeled as a homogeneous material of conductivity σ_0=0.05S/m.

In order to compute the electrical potential v and the current density J around the two grounding systems, a fictitious truncation boundary Γ_T is introduced, enclosing all the conductive objects. For symmetry reasons, the domain analysis is restricted to half the system, by imposing a homogeneous Neumann condition on the xz plane. Due to the great distance between the two systems, this boundary is constituted by two detached parts (see the dashed lines in Figure 5).

In the bounded domain D, delimited by Γ_T, by the xz plane and by the soil surface Γ_0, the Laplace equation holds:

$$\nabla \cdot (\sigma \nabla v) = 0 \tag{22}$$

subject to homogeneous Neumann conditions on Γ_0, Dirichlet conditions on the PEC surfaces, and unknown Neumann conditions on Γ_T, expressed through the integral Equation (3).

The conductors of the grounding systems are modeled as edges of zero thickness on which Dirichlet conditions hold. This condition is equal to V_0=100 V for the conductors of the voltaged system, whereas is a floating value for the other. The two detached subdomains are regularly discretized by means of 36400 and 5200 tetraedra of the second-order, respectively. Having set an end-iteration tolerance of 10^{-4} per cent for the CG solver and 10^{-2} per cent for the GMRES, the procedure converges in 9 iterations. Figure 7 shows the contours of the electrical potential in the xz symmetry plane and on Γ_T around the voltaged grounding system. The floating potential of the other system is evaluated as V_F=3.94V.

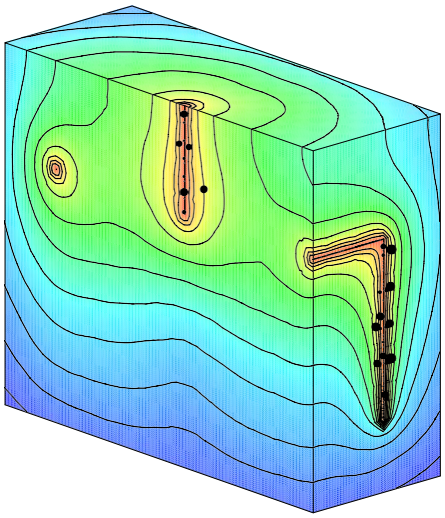

Figure 7. Contours of the potential around the voltaged grounding system

The third example deals with the axisymmetric insulator shown in Figure 8. The upper electrode is voltaged at V_0=100 V with frequency f=50Hz, whereas the lower electrode is grounded to the soil by means of a wire of height h=10m (symbolically represented by the dashed line in Figure 8). The permittivity of the dielectric is $\varepsilon=5\varepsilon_0$. The upper surfaces of the two sheds are covered by a thin film of water, that at this frequency behaves as a good conductor. Then, these surfaces can be modelled as two floating conductors (of zero thickness), whose potentials V_{F1} and V_{F2} are unknown. In order to obtain an enlarged picture of the electric field, the truncation boundary Γ_T (dashed line in Figure 8) is placed sufficiently far from the insulator. The mesh is formed by 2523 triangular finite elements of the second order and 5274 nodes (see Figure 9) (Geuzaine *et al.*, 2009). Note that the boundary condition v=0 on the soil surface (which is outside the analysis domain) is taken into account by means of an appropriate Green's function. The open boundary problem is solved by means of the iterative procedure: convergence is reached in 17 steps, starting from the initial guesses $\mathbf{q}_T=\mathbf{0}$, V_{F1}=52.11V, V_{F2}=80.05V. Figure 10 shows the contours of the electrical potential around the insulator. The same problem has been solved by means of the Coordinate Transformation (CT) method (Lowther *et al.* 1989; Nicolet *et al.* 1994). The inner domain has been selected as the quarter of circle, delimited by the r- and z-axes and by the circumference of radius R=11 m centred at the origin. The inner and outer domains are meshed by means of 9783 and 2048, respectively, second-order triangular finite elements, in such a way that the mesh around the insulator is the same as in the previous analysis. Due to this higher number of finite elements, the CT solutions exhibits a CPU time which is approximately double of that of the FEM-BEM. The two solutions are in good agreement in the common subdomain around the insulator: a mean relative difference of $2.8\cdot10^{-2}$ was observed.

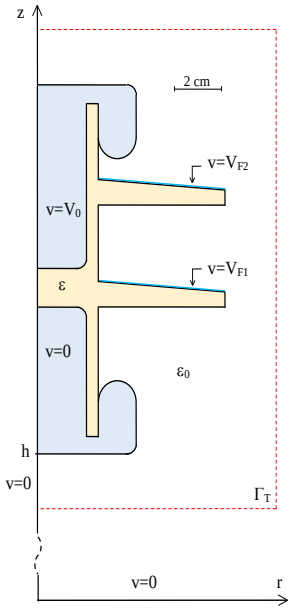

Figure 8. Axisymmetric electric insulator with wet surfaces

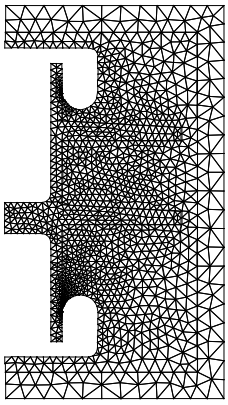

Figure 9. Finite element mesh of the insulator

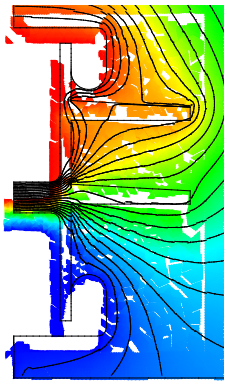

Figure 10. Contours of the electrical potential around the insulator

4. Conclusions

Two iterative methods have been described for the FEM-BEM solution of electrostatic field problems in presence of floating conductors. In both methods the conjugate gradient solver is used to solve the FEM equations, whereas the BEM equations are used only a few times. Non-standard boundary elements are successfully used. The proposed methods are also applicable to other kind of static electromagnetic problems such as current density and magnetic field problems. The main advantage of these methods lies in the simplicity of implementation, accuracy and the reduced computational effort.

From a theoretical point of view we could think that the proposed method is a sort of Domain Decomposition Method (Jin, 2014), where the whole unbounded domain is partitioned into two detached sub domains: a bounded one, which is treated by FEM, and a homogeneous unbounded one, which is treated by BEM. However, this approach, even if correct, does not give any practical implications, so we prefer to describe the proposed method simply as an hybrid method, in which a particular iterative solving strategy is applied to the solution of the global algebraic FEM/BEM system.

The computations were performed by means of ELFIN, a large FEM code developed by the authors for electromagnetic CAD research (Aiello *et al.*, 1999).

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