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FIXED POINT PROPERTIESAND ORBIT SPACES

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ABSTRACT

The aim of this note is to show that if $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an linearly ordered system of compact spaces such that each X_α has fixed point property for continuous multi-valued functions and each projection map is surjective, then the orbit space also has fixed point property for continuous multi-valued functions.

Keywords: Linearly ordered system, Orbit space, Fixed point.

1. INTRODUCTION

A topological space X is said to have the $f.p.p$ (fixed point property) if for every continuous function $f: X \to X$, there exists some $x \in X$ such that $x = f(x)$. Hamilton [1] has proved that the chainable metric continua have the $f.p.p$. A topological space X is said to have the $F.p.p.$ (fixed point property for multi-valued functions) if every continuous multi-valued functions $F: X \to X$ has fixed point; that is, there exist some point $x \in X$ such that $x \in F(x)$. Clearly, a space has $F.p.p.$, then it is has $f.p.p$. But the converse need not be true [2].

Recently, the inverse limits has been widely used in Dynamical Systems,a series of good results were obtained (cf. [3]-[8]). Particularly, Yeng and Lin[9] investigated an inverse system and obtained some fixed point properties in inverse limit spaces. But the applications have been encountered many difficulties because the direction of the chain maps of inverse limits is exactly the opposite to the tracks of the relative dynamical system. In 2011, the authors [10] introduce the concepts of the orbit spaces by reversing the direction of chain maps of inverse limit space, and considered the preserving property of continuums in their orbit space. Naturally, it is posed whether fixed point property can be preserved in orbit space as well as inverse limit space.

The aim of this paper is to give an affirmative answer to this problems. And we show that if $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an linearly ordered system of compact spaces such that each X_α has the $F.p.p.$, then the orbit space of the linearly ordered system $\mathcal{O}\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ also has the $F.p.p.$.

2. PRELIMINARIES

In all that follows, all spaces are assumed to be Hausdorff spaces. A multifunction, $F: X \to Y$, from a space X to a space Y is a point-to-set correspondence such that, for each $x \in X$, $F(x)$ is a subset of Y. For any $x \in Y$, $F^{-1}(x) = \{x \in X : y \in F(x)\}$. Let $A \subset X$ And $B \subset Y$. $F(A) = \bigcup\{F(A) : x \in A\}$, and $F^{-1}(B) = \bigcup\{F^{-1}(y) : y \in B\}$.

Definition 2.1. [9] A multifunction, $F: X \to Y$ is said to be continuous if and only if

(i) $F(x)$ is closed for each $x \in X$.

(ii) $F^{-1}(B)$ is closed for each $B \in 2^Y$.

(iii) $F^{-1}(V)$ is open for each open set V in Y.

The following Lemma is due to [9].

Lemma 2.1 If $F: X \to Y$ is a continuous multifunction and if A is a compact subset of X such that $F(a)$ is compact for each $a \in A$, then $F(A)$ is compact.

Let $\{X_a\}_{\lambda \in \Lambda}$ be a family of topological spaces, where Λ is linearly ordered set. Denote the product space of $\{X_a\}_{\lambda \in \Lambda}$ by $\Pi_{\lambda \in \Lambda} X_a$. Assume that $\pi_\alpha^\beta : X_\alpha \to X_\beta$ is a continuous mapping whenever $\alpha \le \beta$ for each $\alpha, \beta \in \Lambda$. The triples $\{X_\alpha, \pi_\alpha^\beta, \Lambda\}$ is said to be a linearly ordered system if the following two conditions are satisfied:

(a) $\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma$, whenever $\alpha \le \beta \le \gamma$ for each $\alpha, \beta, \gamma \in \Lambda$;

(b) $\pi_\alpha^\alpha = idX_\alpha$ for each $\alpha \in \Lambda$, where π_α^α is an identity mapping from X_α to X_α.

Usually, the subspace

$\{x=(x_\alpha)_{\alpha\in\Lambda}\in\Pi_{\alpha\in\Lambda}X_\alpha:\pi_\alpha^\beta(x_\alpha)=x_\beta,\alpha\le\beta\}$ of the product space $\Pi_{\lambda\in\Lambda}X_a$ is called to be the orbit space of the linearly ordered system $\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$ and is denoted by $\mathcal{O}\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$, where each point

$x\in\mathcal{O}\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$ is said to be an orbit of $\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$, each mapping π_α^β is called to be a link mapping of $\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$.

Assume that $f:X\to X$ is a continuous mapping where X is a topological space. In Dynamical Systems, we call that (X,f) is a discrete dynamical system, and for each $x_0\in X$, the sequence $\{x_0,f(x_0),\cdots\}$ is called to be an orbit of (X,f) and is denoted by $\mathcal{O}_f^+(x_0)$. The set of all orbits of (X,f) is a subspace of the product space $\Pi_{n=0}^\infty X_n$ where each $X_n=X$, we call it an orbit space of (X,f) and denote it by $\mathcal{O}_f^+(x)$.

As a special case of a linearly ordered system, if we take $\Lambda=Z^+$ (the set of all non-negative integers). Each $X_n=X$ and $\pi_n^{n+1}=f$ for any $n\in Z^+$, it is easy to see that the linearly ordered system $\{X,f^{m-n},Z^+\}$ is the discrete system (X,f) and

$\{X,f^{m-n},Z^+\}=O_f^+(X).$

The following two results are due to [10].

Lemma 2.3 Assume that X is the orbit space of a linearly ordered system $\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$. Then

(i) The collections $\{\pi_\alpha^{-1}(U_\alpha):U_\alpha$?} is an open subset of X_α,?$\alpha\in\Lambda$ is a base of X;

(ii) If X is Hausdorff, then X is a closed subset of $\Pi_{\alpha\in\Lambda}$;

(iii) If each $X_\alpha,\alpha\in\Lambda$ is compact, then so is X;

(iv) If each link map π_α^β is onto map, then X is a continuum if and only if each X_α is a continuum.

Lemma 2.4 Let $X=\mathcal{O}\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$, $A\subset X$ and $A_\alpha=\pi_\alpha(A)$ for each $\alpha\in\Lambda$. If $\overline{\pi}_\alpha^\beta=\pi_\alpha^\beta|_{\overline{A_\alpha}}$ whenever $\alpha\le\beta$ for any $\alpha,\beta\in\Lambda$, then $\left\{\overline{A_\alpha},\overline{\pi}_\alpha^\beta,\Lambda\right\}$ is a linearly ordered system and $\mathcal{O}\left\{\overline{A_\alpha},\overline{\pi}_\alpha^\beta,\Lambda\right\}=\overline{A}$.

3. MAIN THEOREM

Now, we state our main result in this paper.

Main Theorem. Let $\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$ be an linearly ordered system of compact spaces such that each X_α has the

$F.p.p.$, then the orbit space of the linearly ordered system $\mathcal{O}\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$ also has the $F.p.p.$

We divide the proof of this theorem into the following steps. In Lemma 3.1, Lemma 3.2 and Lemma 3.3, X will be the orbit space of the linearly ordered system $\mathcal{O}\{X_\alpha,\pi_\alpha^\beta,\Lambda\}$ of compact spaces.

Lemma 3.1 If $F:X\to X$ is a continuous multifunction, define $F:X_\alpha\to X_\alpha$ by $F_\alpha=\pi_\alpha F\pi_\alpha^{-1}$

For $\alpha\in\Lambda$, then F_α is a continuous multifunction.

Proof. This proof is modified by [9, Lemma 4]. By Lemma 2.1 and the definition of the continuous multifunction, this result is directly.

Lemma 3.2 Let $F:X\to X$ is a continuous multifunction, and $F:X_\alpha\to X_\alpha$ be defined as in Lemma 3.1. Then, for each $x\in X$.

$\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}$ and $\{\pi_\alpha F_\alpha(x),\pi_\alpha^\beta,\Lambda\}$ are linearly ordered systems of compact spaces,

$$\mathcal{O}\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}=\mathcal{O}\{\pi_\alpha F_\alpha(x),\pi_\alpha^\beta,\Lambda\}\quad(2)$$

$$F(x)=\mathcal{O}\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}.\quad(3)$$

Proof. (1) It is obvious that each $F_\alpha\pi_\alpha(x)$ is compact. To show that $\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}$ forms an linearly ordered system, it suffices to show $\pi_\alpha^\beta F_\alpha\pi_\alpha(x)\subset F_\beta\pi_\beta(x)$ when ever $\alpha<\beta$. Since

$\pi_\alpha^\beta\pi_\alpha=\pi_\beta,\pi_\alpha(x)\in(\pi_\alpha^\beta)^{-1}\pi_\alpha^\beta\pi_\alpha(x)=(\pi_\alpha^\beta)^{-1}\pi_\beta(x).$

Furthermore, $\pi_\alpha^\beta F_\alpha\pi_\alpha(x)\subset$

$\pi_\alpha^\beta F_\alpha(\pi_\alpha^\beta)^{-1}(x)\subset F_\beta\pi_\beta(x)$. Then $\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}$ is an linearly ordered system.

For each $\alpha\in\Lambda,x\in X,$ we have that

$\pi_\alpha F(x)\subset\pi_\alpha F\pi_\alpha^{-1}\pi_\alpha(x)=(\pi_\alpha F\pi_\alpha^{-1})\pi_\alpha(x)=F_\alpha\pi_\alpha(x),$

Hence, $\mathcal{O}\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}\subset\mathcal{O}\{\pi_\alpha F_\alpha(x),\pi_\alpha^\beta,\Lambda\}$.

Next we shall show that

$X-\mathcal{O}\{\pi_\alpha F_\alpha(x),\pi_\alpha^\beta,\Lambda\}\subset X-\mathcal{O}\{F_\alpha\pi_\alpha(x),\pi_\alpha^\beta,\Lambda\}.$

Indeed, for $y\in X-\mathcal{O}\{\pi_\alpha F_\alpha(x),\pi_\alpha^\beta,\Lambda\}$. By Lemma 3.1, there exists $\beta\in\Lambda,$ such that $\pi_\beta(y)\notin\pi_\beta F(x).$

Let U_β,V_β be two open subsets of X_β with $U_\beta\cap V_\beta=\varnothing$ satisfying $\pi_\beta(y)\in U_\beta,$ $\pi_\beta F(x)\in V_\beta.$

Then $F(x)\subset\pi_\beta^{-1}(V_\beta).$ By Lemma 2.1, there exists $\gamma\in\Lambda$ and some open subset U_γ of X_γ, such that $x\in\pi_\alpha^{-1}(U_\gamma)$ and $F(\pi_\gamma^{-1}(U_\gamma))\subset\pi_\beta^{-1}(V_\beta).$

Let $\alpha_0=\min\{\beta,\gamma\}$. And let $U_{\alpha_0}=\pi_{\alpha_0}^\gamma(U_\gamma).$

Since $(\pi_{\alpha_0})^{-1}(\pi_{\alpha_0}^\gamma)^{-1}=\pi_\gamma^{-1},$

$F(\pi_{\alpha_0}^\gamma(U_{\alpha_0}))\subset\pi_\beta^{-1}(V_\beta).$

Thus,

$F_{\alpha_0}(U_{\alpha_0})=\pi_{\alpha_0}F\pi_{\alpha_0}^{-1}(U_{\alpha_0}))\subset\pi_{\alpha_0}\pi_\beta^{-1}(V_\beta)$

$=\pi_{\alpha_0}(\pi_{\alpha_0}^\beta\pi_{\alpha_0})^{-1}(V_\beta)=(\pi_{\alpha_0}^\beta)^{-1}(V_\beta).$

Particularly, $F_{\alpha_0}\pi_{\alpha_0}(x)\in(\pi_{\alpha_0}^\beta)^{-1}(V_\beta).$

By the similar way, it is easy to check that $\pi_{\alpha_0}(y)\in(\pi_{\alpha_0}^\beta)^{-1}(U_\beta).$

Since $(\pi_{\alpha_0}^\beta)^{-1}(U_\beta)\cap(\pi_{\alpha_0}^\beta)^{-1}(V_\beta)=\varnothing,$

$\pi_{\alpha_0}(y)\notin F_{\alpha_0}\pi_{\alpha_0}(x).$

Thus, $y\in X-\mathcal{O}\{\pi_\alpha F_\alpha(x),\pi_\alpha^\beta,\Lambda\}.$

Moreover, $\mathcal{O}\{\pi_\alpha F_\alpha(x), \pi_\alpha^\beta, \Lambda\} \subset \mathcal{O}\{F_\alpha \pi_\alpha(x), \pi_\alpha^\beta, \Lambda\}$.

So, (2) is true.

(3) It is directly by Lemma 3.1 and (2).

The following Lemma is directly.

Lemma 3.3 Let $F: X \to X$ is a continuous multifunction, and $F_\alpha : X_\alpha \to X_\alpha$ be defined as in Lemma 3.1. Let $P_\alpha = \{p_\alpha : X_\alpha \cap F_\alpha(p_\alpha)\}$.

Then $\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$ forms an inverse system.

Proof. It suffices to prove $\pi_\alpha^\beta(P_\alpha) \subset P_\beta$, whenever $\alpha < \beta$, which follows in a routine Way.

4. PROOF OF MAIN THEOREM

Since each X_α has $F.p.p.$ and F_α is continuous, P_α is nonempty closed subset. By Lemma 3.3, $\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$ is an linearly ordered systems of compact spaces, so it has a orbit space $\mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$. We assert that

$\forall x \in \mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$, $x \in F(x)$.

Let $\forall x \in \mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$, Then $\forall \alpha \in \Lambda$,

$\pi_\alpha(x) \in P_\alpha$. That is to say $\pi_\alpha(x) \in F_\alpha \pi_\alpha(x)$. So, by Lemma 2.4 and Lemma 3.2,

$$x = \mathcal{O}\{\pi_\alpha(x), \pi_\alpha^\beta, \Lambda\} \in \mathcal{O}\{F_\alpha \pi_\alpha(x), \pi_\alpha^\beta, \Lambda\} = F(x).$$

In fact, with the assumption of the main theorem and the notation of Lemma 3.3 together with the notation $P = \{x : x \in F(x)\}$, we have the following sharper assertion.

Theorem 3.4 $P = \mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$.

Proof. By the main theorem, we have

$P \supset \mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$. It remains to be proved that $P \subset \mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$.

Let $\forall x \in P$. Then $x \in F(x)$ and $\forall \alpha \in \Lambda$,

$$\pi_\alpha(x) \in \pi_\alpha F(x) \subset \pi_\alpha F(\pi_\alpha^{-1}\pi_\alpha)(x) = F_\alpha(\pi_\alpha(x)).$$

That is, $\pi_\alpha(x) \in P_\alpha$; It follows from Lemma 3.3 that $P \subset \mathcal{O}\{P_\alpha, \pi_\alpha^\beta, \Lambda\}$.

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