Modified Single Finite Fourier Cosine Integral Transform Method for Finding the Critical Elastic Buckling Loads of First Order Shear Deformable Beams with Fixed Ends

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ABSTRACT

This paper proposes a novel modified single finite Fourier cosine integral transform, which determines the elastic buckling loads of moderately-thick, fixed-end beams made of homogeneous, isotropic, linear elastic materials. First, such a beam was modelled with a fourth-order ordinary differential equation, according to the first-order shear deformation theory. Then, the single finite Fourier cosine integral transform was modified to satisfy all the boundary conditions at the fixed ends, using the apriori knowledge, and provide an exact buckling mode (shape) function for the beam. Through the modified transform, the boundary value problem was converted to an algebraic eigenvalue problem, which canbe described by a set of homogeneous algebraic equations. For nontrivial solutions, the characteristic buckling equation was drived from the vanishing of the determinant of the coefficient matrix. Solving the characteristic buckling equation, the authors obtained the eigenvalues and thus derived the buckling loads. The critical buckling load was found to correspond to the first buckling mode. The proposed modified transform gave exact expressions for the buckling loads and the critical buckling load of the fixed ends, because the integral kernel function satisfies all the boundary conditions, and the transform on the domain governing equation satisfies all the domain equations. The solutions of our method agree well with the previous studies.

1. INTRODUCTION

Several theories have been derived in the literature for beam stability problems, where beams are subjected to axial compressive forces and it is required to determine the loads at which instabilities occur and their modes/shapes [1-5]. The Euler-Bernoulli beam theory (EBBT) derived and developed using the Euler-Bernoulli hypothesis of orthogonality of the beam cross-sections to the middle plane before and after deformation has been extensively used to describe thin beam flexure and stability problems [6, 7]. The implication of the orthogonality assumption of the plane cross-sections to the middle plane is the disregard of transverse shear deformation, hence constraining the scope and use of the theory to thin (slender) beams where elasticity theory prove that transverse shear deformation effects are insignificant [8-13]. The EBBT thus overestimates the buckling load capacities of moderately thick beams since the effects of transverse shear deformation substantially impact the stability behaviour.

Timoshenko derived the Timoshenko beam theory (TBT), a first order shear deformation beam theory by assuming a relaxation of the orthogonality conditions used in formulation of the EBBT [14]. TBT can be formulated using equilibrium methods or variational methods. TBT assumes a constant transverse shear strain across the thickness, thus violating the stress-free boundary conditions on the beam surfaces. It thus requires problem defined shear stress modification factors and is formulated in terms of two unknown displacement functions, rendering the boundary value problem (BVP) complex for analysis and solution.

Further research into buckling analysis have resulted in the derivation and development of higher order shear deformation theories by Krishna Murty, Levinson, Heyliger and Reddy, Ghugal, et al. [13-26].

In this study, the elastic buckling problem of moderately thick beams of prismatic cross-sections, made of homogeneous, isotropic material, modelled as first order shear deformable beam was solved, for the first time, using a modified single (one dimensional) finite Fourier cosine transform method for the case when the beam is fixed at the two ends.

The moderately thick beam studied as shown in Figure 1 for elastic buckling behaviour is defined by $-\frac{l}{2} \leq x \leq \frac{l}{2}$, $-\frac{h}{2} \leq y \leq \frac{h}{2}$ and $-\frac{b}{2} \leq z \leq \frac{b}{2}$. When the cross-section lies in the yz plane and x is the longitudinal axis, l is the beam span, while h is the thickness, b is the width.
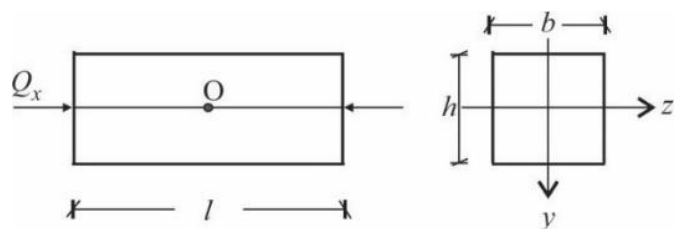

Figure 1. Moderately thick beam studied and the coordinate system

2. THEORY

2.1 Assumptions

The assumptions of the formulation are as follows [8, 9]:
(i) the transverse displacement field component in the y coordinate direction depends only on the position on the longitudinal coordinate axis of the beam.
(ii) the stress strain equations are one-dimensional.
(iii) the displacements are so small compared to the thickness that the resulting strains are infinitesimal.
(iv) the transverse displacement $v_t(x, y)$ is the sum of the bending component $v(x)$ and shear component $v_s(x)$.
(v) the material of the beam is homogeneous, linear elastic and isotropic.

2.2 Displacement components

The displacement components are:

$$u_x(x, y) = -y\theta(x) \tag{1}$$

$$u_y(x, y) = u_y(x) = v(x) + v_s(x) \tag{2}$$

where, $\theta(x)$ is the rotation of the cross-section at the neutral axis, $u_y(x)$ is the displacement field in the y direction, $v(x)$ is the flexural component of the transverse displacement, and $v_s(x)$ is the shear component of the transverse displacement.

$$\theta(x) = -\frac{dv(x)}{dx} \tag{3}$$

y is the transverse coordinate direction.

2.3 Strain field components

The strain field components are from the small deformation elasticity theory, given by:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -y\theta'(x) \tag{4}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \tag{5}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = u'_y(x) - \theta(x) \tag{6}$$

where, ε_{xx}, ε_{yy} are normal strains in the x and y directions respectively, γ_{xy} is the shear strain and the primes represent derivatives with respect to x.

2.4 Stress fields

The stress fields for plane stress are from the one dimensional stress-strain relations, given by:

$$\sigma_{xx} = E\varepsilon_{xx} = -Ey\theta'(x) \tag{7}$$

$$\sigma_{yy} = E\frac{\mu}{1-\mu^2}\varepsilon_{xx} = -\frac{E\mu}{1-\mu^2}y\theta'(x) \tag{8}$$

$$\tau_{xy} = kG\gamma_{xy} = kG\left(u'_y(x) - \theta(x)\right) \tag{9}$$

where, k is the shear correction/modification factor, σ_{xx}, σ_{yy} are the normal stresses, τ_{xy} is the shear stress, μ is the Poisson's ratio, E is the Young's modulus of elasticity, G is the shear modulus or the modulus of rigidity.

2.5 Total potential energy functional, Π

The total potential energy functional is given by:

$$\Pi = \frac{1}{2}\iiint \sigma_{xx}\varepsilon_{xx}dv + \frac{1}{2}\iiint \tau_{xy}\gamma_{xy}dv -$$
$$\frac{1}{2}\int_0^l Q_x\left(\frac{du_y}{dx}\right)^2 dx - \int_0^l q(x)u_y(x)dx \tag{10}$$

Using the stress-strain relations, Π is given as:

$$\Pi = \frac{1}{2}\iiint E\varepsilon_{xx}^2 dv + \frac{1}{2}\iiint kG\gamma_{xy}^2 dv -$$
$$\frac{1}{2}\int_0^l Q_x\left(\frac{du_y}{dx}\right)^2 dx - \int_0^l q(x)u_y(x)dx \tag{11}$$

Using the strain-displacement relations, Π is simplified as:

$$\Pi = \frac{1}{2}\iiint Ey^2\left(\theta'(x)\right)^2 dAdx +$$
$$\frac{1}{2}\iiint kG\left(u'_y(x) - \theta(x)\right)^2 dAdx -$$
$$\frac{1}{2}\int_0^l Q_x\left(\frac{du_y}{dx}\right)^2 dx - \int_0^l q(x)u_y(x)dx \tag{12}$$

Using the Euler-Lagrange condition for the minimization of Π, the governing stability equation is obtained for rectangular cross-sections with $k = 5/6$ and $q(x) = 0$, as:

$$\left(EI - Q_x\frac{h^2}{5}(1+\mu)\right)\frac{d^4v(x)}{dx^4} + Q_x\frac{d^2v(x)}{dx^2} = 0 \tag{13}$$

where, I is the moment of inertia of the beam cross-section, h is the beam thickness.

3. METHOD

The one dimensional (single) finite Fourier cosine transform of a function of one independent variable $f(x)$ defined over the finite integral $0 \le x \le a$ is denoted by $F_c(n)$ and is defined as:

$$F_c(n) = \int_0^a f(x)\cos\frac{n\pi x}{a}dx \tag{14}$$

where, n is a nonnegative integer.
The inverse finite Fourier cosine transform of $F_c(n)$ is given by:

$$f(x) = \frac{1}{a} F_c(0) + \frac{2}{a} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{a} \qquad (15)$$

By integration by parts, the finite Fourier cosine transform of derivatives of $f(x)$ could be determined.

$$\int_0^a \frac{df(x)}{dx} \cos \frac{n\pi x}{a} dx = \left[f(x) \cos \frac{n\pi x}{a} \right]_0^a +$$

$$\frac{n\pi}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \qquad (16)$$

$$F_c\left(f'(x)\right) = (-1)^n f(a) - f(0) + \frac{n\pi}{a} F_s(n) \qquad (17)$$

where, $F_s(n)$ is the single finite Fourier sine transform of $f(x)$. If $f(x)$ vanishes at $x = 0$, and $x = a$, a simpler expression is obtained for $F_c(f'(x))$ as:

$$F_c\left(f'(x)\right) = \int_0^a f'(x) \cos \frac{n\pi x}{a} dx = \frac{n\pi}{a} F_s(n) \qquad (18)$$

The expressions for the single finite Fourier cosine transforms of higher derivatives of $f(x)$ are obtained by the repeated application of the fundamental results for the first derivatives.

$$F_c\left(f''(x)\right) = \int_0^a f''(x) \cos \frac{n\pi x}{a} dx =$$

$$(-1)^n f'(a) - f'(0) - \left(\frac{n\pi}{a}\right)^2 F_c(n) \qquad (19)$$

If $\dfrac{df}{dx} = f'(x)$ vanishes at the end points $x = 0$, and $x = a$, a simpler expression results as follows:

$$F_c\left(f''(x)\right) = -\left(\frac{n\pi}{a}\right)^2 F_c(n) \qquad (20)$$

For the fourth order derivative $f^{iv}(x)$, by induction, it is obtained for $f(x)$ vanishing at $x = 0$, and $x = a$ that:

$$F_c\left(f^{iv}(x)\right) = \int_0^a \frac{d^4}{dx^4} f(x) \cos \frac{n\pi x}{a} dx \qquad (21)$$

$$F_c\left(f^{iv}(x)\right) = -\left(\frac{n\pi}{a}\right)^2 \int_0^a \frac{d^2 f(x)}{dx^2} \cos \frac{n\pi x}{a} dx = \left(\frac{n\pi}{a}\right)^4 F_c(n)$$

$$(22)$$

4. RESULTS

The governing BVP is represented as the fourth order ordinary differential equation (ODE):

$$\frac{d^4 v(x)}{dx^4} + \lambda^2 \frac{d^2 v(x)}{dx^2} = 0 \qquad (23)$$

where,

$$\lambda^2 = \frac{Q_x}{EI - \dfrac{Q_x h^2 (1 + \mu)}{5}} \qquad (24)$$

Following the definition of the single finite Fourier cosine integral transform, we modify the Fourier cosine kernel function to ensure that it satisfies the fixed-fixed boundary conditions as:

$$K(x) = \cos \frac{2n\pi x}{l} \qquad (25)$$

where, $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and the origin is defined at the middle of the beam.

The single finite Fourier cosine integral transform of the governing BVP becomes:

$$\int_0^{l/2} \left(\frac{d^4 v(x)}{dx^4} + \lambda^2 \frac{d^2 v(x)}{dx^2} \right) \cos \frac{2n\pi x}{l} dx = 0 \qquad (26)$$

From the linearity property of the finite Fourier cosine integral transform, we have:

$$\int_0^{l/2} \frac{d^4 v(x)}{dx^4} \cos \frac{2n\pi x}{l} dx + \int_0^{l/2} \lambda^2 \frac{d^2 v(x)}{dx^2} \cos \frac{2n\pi x}{l} dx = 0 \qquad (27)$$

$$\left(\frac{2n\pi}{l}\right)^4 \int_0^{l/2} v(x) \cos \frac{2n\pi x}{l} dx -$$

$$\lambda^2 \left(\frac{2n\pi}{l}\right)^2 \int_0^{l/2} v(x) \cos \frac{2n\pi x}{l} dx = 0 \qquad (28)$$

But,

$$\int_0^{l/2} v(x) \cos \frac{2n\pi x}{l} dx = v_c(n) \qquad (29)$$

where, $v_c(n)$ is the single finite Fourier cosine transform of $v(x)$. Then, we have:

$$\left(\frac{2n\pi}{l}\right)^4 v_c(n) - \lambda^2 \left(\frac{2n\pi}{l}\right)^2 v_c(n) = 0 \qquad (30)$$

Simplification yields:

$$\left(\left(\frac{2n\pi}{l}\right)^4 - \lambda^2 \left(\frac{2n\pi}{l}\right)^2 \right) v_c(n) = 0 \qquad (31)$$

Simplifying, we have the algebraic eigenvalue problem:

$$\left(\left(\frac{2n\pi}{l}\right)^2 - \lambda^2\right)v_c(n) = 0 \qquad (32)$$

For nontrivial solutions, $v_c(n) \neq 0$, and the characteristic buckling equation is found as the system of homogeneous algebraic equations:

$$\left(\frac{2n\pi}{l}\right)^2 - \lambda^2 = 0 \qquad (33)$$

The eigenvalues are obtained as:

$$\lambda^2 = \left(\frac{2n\pi}{l}\right)^2 = 4\left(\frac{n\pi}{l}\right)^2 \qquad (34)$$

The eigenvalues are used to obtain the n buckling loads as follows:

$$\lambda^2 = 4\left(\frac{n\pi}{l}\right)^2 = \frac{Q_x}{EI - \frac{Q_x h^2(1+\mu)}{5}} \qquad (35)$$

Solving,

$$Q_x = \frac{4\left(\frac{n\pi}{l}\right)^2 EI}{1 + \frac{4}{5}h^2(1+\mu)\left(\frac{n\pi}{l}\right)^2} \qquad (36)$$

$$Q_x = \frac{4(n\pi)^2}{1 + 0.8h^2(1+\mu)\left(\frac{n\pi}{l}\right)^2}\frac{EI}{l^2} \qquad (37)$$

For $\mu = 0.25$,

$$Q_x = \frac{4(n\pi)^2}{1 + \frac{h^2}{l^2}(n\pi)^2}\frac{EI}{l^2} \qquad (38)$$

$$Q_{x_{cr}} = \frac{4\pi^2}{1 + \pi^2\frac{h^2}{l^2}}\frac{EI}{l^2} = Q_x(n=1) \qquad (39)$$

For $\mu = 0.30$,

$$Q_x = \frac{4(n\pi)^2}{1 + \frac{26}{25}\frac{h^2}{l^2}(n\pi)^2}\frac{EI}{l^2} \qquad (40)$$

$$Q_{x_{cr}} = Q_x(n=1) = \frac{4\pi^2}{1 + \frac{26}{25}\pi^2\frac{h^2}{l^2}}\frac{EI}{l^2} \qquad (41)$$

Values of the critical buckling load for the first order shear deformable beam for values of Poisson's ratio μ given as: $\mu = $

0.25 and for $\mu = 0.30$ are calculated using Equations (39) and (41) for various values of h/l and presented in Table 1 and 2, respectively.

Table 1. Critical elastic buckling load Q_{xcr} for first order shear deformable beams with Poisson's ratio $\mu = 0.25$ for various values of h/l (fixed ends)

h/l	Q_{xcr} ($\mu = 0.25$) $\left(\times EI/l^2\right)$	Exact solution, Onah *et al.* [17], (for $\mu = 0.25$) $\left(\times EI/l^2\right)$
0.01	39.43949239	39.43949
0.02	39.32317593	39.32317
0.05	38.52778267	38.52778
0.1	35.93206494	35.93206
0.15	32.30465	32.30465
0.2	28.30431997	28.30432
0.25	24.41686668	24.41687
0.3	20.90725096	20.90725
0.35	17.87140938	17.87141
0.4	15.30683408	15.30684
0.45	13.16563892	13.16564
0.5	11.38559297	11.3856

Table 2. Critical elastic buckling load Q_{xcr} for first order shear deformable beams with fixed ends with Poisson's ratio $\mu = 0.30$ for various values of h/l

h/l	Q_{xcr} ($\mu = 0.30$) $\left(\times EI/l^2\right)$	Exact solution for $\mu = 0.30$, Onah *et al.* [17] $\left(\times EI/l^2\right)$
0.01	39.43793697	39.43793
0.02	39.31699165	39.31699
0.05	38.49070863	38.49071
0.10	35.80341588	35.80341
0.15	32.07153655	32.07154
0.20	27.98745363	27.98745
0.25	24.0498529	24.04985
0.30	20.52111487	20.52112
0.35	17.48854188	17.48854
0.40	14.94091706	14.94092
0.45	12.82375231	12.82376
0.50	11.07048294	11.07049

5. DISCUSSION

The elastic buckling problem of first order shear deformable thick and moderately thick beams represented by the fourth order ordinary differential equation, Equation (13) is solved in closed form using the modified single finite Fourier cosine integral transform method for the case of fixed ends. For the elastic buckling problem considered, the integral kernel function for the modified single finite Fourier cosine transform identically satisfies all the geometric and force boundary conditions at the clamped ends.

Another merit of the method is that the modified single finite Fourier cosine transform method can be readily applied to obtain the transform of ordinary derivatives of the function of an independent variable. The modified single finite Fourier cosine transformation is applied to the governing fourth order ODE to obtain Equation (26). The use of the linearity property of the single finite Fourier cosine transform and simplification reduced the problem to an algebraic eigenvalue problem represented by Equation (32). For nontrivial solutions of the

algebraic eigenvalue problem, the determinant of the coefficient matrix is required to vanish resulting in a characteristic buckling equation given by the system of algebraic equations – Equation (33). The eigenvalues (roots) of the characteristic buckling equation are found for all the n buckling modes as Equation (34). The n eigenvalues are used to find the n buckling loads as Equation (37). For $\mu = 0.25$, the n buckling loads are obtained using Equation (38) while for $\mu = 0.30$, the n buckling loads are obtained using Equation (40). The critical buckling load Q_{xcr} which is the minimum value of the buckling loads is found to occur at the first buckling mode when $n = 1$, and found as Equation (39) when $\mu = 0.25$, and Equation (41) for $\mu = 0.30$.

Values of the critical buckling load Q_{xcr} for various values of the ratio h/l and for $\mu = 0.25$, and $\mu = 0.30$ for first order shear deformable beams were calculated and shown in Tables 1 and 2, together with exact results obtained by Onah et al [9] using closed form/analytical solutions of the boundary value problem.

Tables 1 and 2 reveal that the critical elastic buckling load Q_{xcr} obtained by the modified single finite Fourier cosine transform method agreed remarkably well with the exact solutions obtained by Onah et al [9] using analytical/closed form methods of solving the boundary value problem. Tables 1 and 2 further show that for $h/l < 0.02$ (for thin (slender) beams) the critical elastic buckling load is approximately equal to the critical buckling load obtained from the Euler Bernoulli beam buckling theory for fixed ends. Tables 1 and 2 also reveal that for $h/l > 0.02$ the critical elastic buckling load coefficient decreases with increase in h/l, showing that the effect of transverse shear deformation is to significantly reduce the critical elastic buckling load capacity of the beam.

6. CONCLUSION

The conclusions of the study are as follows:

(i) The modified single finite Fourier cosine integral transform method is an effective mathematical tool for solving the elastic buckling problem of moderately thick shear deformable beams with fixed ends for homogeneous isotropic materials.

(ii) The integral kernel (nucleus) function of the modified single finite Fourier cosine integral transform satisfies all the geometric and force boundary conditions at the fixed ends and is an exact buckling modal function for the shear deformable beam with fixed ends for the choice of origin of the coordinate system used.

(iii) The modified single finite Fourier cosine integral transform method transforms the governing fourth order ordinary differential equation for the problem to an algebraic eigenvalue problem given by a system of homogeneous algebraic equations.

(iv) For nontrivial solutions of the system of homogeneous algebraic equations, the determinant of the coefficient matrix vanishes, yielding the characteristic buckling equation.

(v) The n roots (zeros) of the characteristic buckling equation are found as the n eigenvalues of the eigenvalue problem.

(vi) The n buckling loads are found from the n eigenvalues, and the minimum elastic buckling load called the critical buckling load was found to correspond to the first buckling mode for which $n = 1$.

(vii) The modified single finite Fourier cosine integral transform method gave exact expressions for the n buckling loads since the exact buckling mode which identically satisfied all the boundary conditions was used, and the domain governing equation was also satisfied identically.

(viii) For $h/l < 0.02$, i.e. thin (slender) beams, the critical elastic buckling load coefficient is approximately the same as the critical elastic buckling load coefficient of Euler-Bernoulli beam theory.

(ix) For $h/l > 0.02$, the critical elastic buckling load coefficient decreases as h/l increases, and the effect of shear deformation is found to substantially reduce the critical buckling loads.

(x) Shear deformation is an important consideration in the evaluation of the critical buckling load capacities of moderately thick beams for realistic designs.

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