International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 12, No. 8, August, 2025, pp. 2845-2854
Journal homepage: http://iieta.org/journals/mmep

Inventory Optimization for Perishables: A Probabilistic and Fuzzy Modeling Approach

Amir Ahmad Dar[1], H. A. Bhavithra[2], A. K. Awasthi[1], Saqib Ul Sabha[3], Javid Gani Dar[4], Aseel Smerat[5,6], S. Sindhuja[2*]

[1] Department of Mathematics, Lovely Professional University, Phagwara 144411, India
[2] Department of Mathematics, Faculty of Science and Humanities, SRM Institute of Science and Technology, Ramapuram Campus, Chennai 600089, India
[3] School of Computer Science and Engineering, Lovely Professional University, Phagwara 144411, India
[4] Department of Applied Sciences, Symbiosis Institute of Technology, Symbiosis International (Deemed University), Pune 412115, India
[5] Faculty of Educational Sciences, Al-Ahliyya Amman University, Amman 19328, Jordan
[6] Department of Biosciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, India

Corresponding Author Email: sindukarti09@gmail.com

https://doi.org/10.18280/mmep.120825

Received: 9 June 2025
Revised: 25 July 2025
Accepted: 1 August 2025
Available online: 31 August 2025

Keywords:
inventory, preservation technology, deterioration, probabilistic demand, defuzzification, Graded Mean Integration Method, Signed Distance Method

ABSTRACT

Inventory is an essential component of the business world. A scarcity of stock could cause frequent interruptions to the production schedule, potentially leading to underutilization of capacity and diminished sales. The objective of this paper is to develop an inventory model for food products that involves the preservation technology with probabilistic demand. This study also aims to explore the potential impacts of minimizing total costs. This study focuses on the key aspects of deterioration, shortages, and preservation technology investment. It also considers a review of existing literature on inventory models and a comparative analysis using fuzzy logic. A numerical example is included in the paper to illustrate the model's viability. The proposed models undergo sensitivity analysis to show how the sensitivity of the output variable changes with the variation of each input parameter.

1. INTRODUCTION

The field of inventory management encompasses a wide array of factors that need careful consideration. This includes achieving a balance between the time needed for restocking, cost of carrying inventory, controlling assets, inventory projections, valuation of inventory, inventory visibility, and forecasting future inventory prices. It also includes conducting physical inventories, making the best use of the physical premises, ensuring compliance with quality standards, controlling replenishment, managing returns and faulty products, and accurately anticipating demand. Achieving a balance among these multifaceted demands is essential for attaining optimal inventory levels. Furthermore, this process is a continuous endeavor as businesses must constantly adapt and respond to the ever-changing external landscape.

Inventory management involves the utilization of mathematical models that consist of an objective function and a set of constraints. These models serve the purpose of minimizing costs while considering various aspects such as the dynamics of supply and demand, inventory completion feasibility, inventory management strategies, and other relevant limiting factors. The nature of mathematical models in inventory management can be highly intricate, which often necessitates simplifications to accurately represent specific management tasks. In a mathematical model, the system is symbolically represented, allowing for modifications through mathematical rules. By developing a mathematical model, numerous inventory systems can be effectively solved, enabling the derivation of optimal decision rules.

Inventory management is a critical aspect of manufacturing operations, with demand and deterioration being key factors to consider. Various elements, including demand rate, shortage cost, and deteriorating items, significantly influence the development of “inventory models. The nature of demand can vary depending on the circumstances and product type. For the past few decades, researchers have explored different demand patterns, such as “constant demand, time-dependent demand, price-dependent demand, and time and price-dependent demand. For example, when newly released fashion products like cosmetics or clothing enter the market, their demand tends to increase over time before stabilizing. This type of demand is commonly referred to as ramp-type demand. Some researchers have also incorporated the consideration of item deterioration into their models. Deterioration can manifest in various ways, such as spoilage, obsolescence, or a decrease in utility or value compared to its original state. Items like alcohol, gasoline, radioactive chemicals, pharmaceuticals, blood, fish, fruits, and vegetables are susceptible to deterioration. Hence, examining the impact of physical item deterioration on inventory management is crucial.

In recent years, there has been significant research on production “inventory models. Khurana et al. [1] proposed an “economic order quantity model” that allows for shortages in the case of decaying products with inconstant demand rates. Another study by Khara et al. [2] developed a “quantity model for economic growth”, considering both perfect and imperfect products. This model incorporates factors such as purchase price, product reliability, and advertising in determining demand. These studies highlight the ongoing efforts to enhance “inventory models, considering factors like demand variations, product deterioration, and other influential variables. By incorporating such elements, researchers aim to improve inventory management practices and optimize decision-making for businesses in various industries.

To develop an “EOQ model, it is essential to consider a determinate planning horizon. In 2018, Palanivel and Uthayakumar [3] introduced a probabilistic function-based “EOQ model” specifically designed for non-immediately decaying products. This model takes into account factors such as inflation and the time value of money within the finite planning horizon. In addition, Saha and Sen [4] proposed an “inventory model” tailored for decaying products, considering both time and price-based demand.

Existing literature suggests that a Weibull distribution is often applied in the case of time-varying demand. However, in real-world scenarios it is not commonly observed that an unchanging adjustment in the item demand rate per unit time is implied by the Weibull distribution. The Weibull distribution” is more suitable for items with a demand rate that increases over time. The location parameter of the three-parameter Weibull distribution is particularly useful in representing the shelf-life of items, which is an important consideration for most degrading items with time-varying demand.

To emphasize this concept, in this paper, an inventory model for deteriorating items with preservation technology with a time-dependent shortage and the probabilistic demand is proposed, as well. Sindhuja et al. [5] discussed a mathematical model based on quadratic demand and also considered time-dependent demand. When this proposed model is compared to the existing model, the total cost is reduced even more.

The remainder of the paper is organized as follows: Section 2 offers a review of the relevant literature. Section 3 outlines the notations and assumptions. The mathematical model is developed in Section 4. Solution procedure represented in Section 5. Section 6 presents a numerical example based on the fuzzy parameters. Comprehensive sensitivity analysis is provided in Section 7, followed by the conclusion and future research directions in Section 8.

2. LITERATURE SURVEY

In the field of modelling techniques, researchers have made significant advancements by refining specific assumptions to better align with real-world scenarios. An extensive body of research has been dedicated to analyzing inventory systems through the development of mathematical models, providing valuable insights for supply chain decision-makers.

Over the past century, numerous books and research articles have inquired into this area of study, focusing on various case scenarios. The concepts presented in this chapter's introduction draw inspiration from the works of many researchers dedicated to inventory issues, which primarily focuses on practical applications rather than theoretical foundations or derivations. This work explores different extensions of the basic lot size concept and highlights their practical utility.

Understanding demand patterns is a crucial aspect of formulating “inventory models. Demand patterns illustrate how consumer demand fluctuates over time. In real-world circumstances, it has been noted that the demand for high-tech innovative products, seasonal commodities, pharmaceuticals, and other goods tends to increase during the growth phase and decrease during the decline phase. Sales throughout a product's life cycle are typically characterized by time-varying demand patterns in the market. Researchers [6-17] have contributed additional research on demand patterns resembling ramps. Their studies have further enriched the understanding of inventory management in the face of such demand dynamics.

Deterioration is a natural process that affects various products such as dairy products, pharmaceuticals, human blood, fruits and vegetables, and other consumer items. It occurs due to factors like expiry, decay, damage, and pilferage, leading to a decrease in the quality and usefulness of the product. Recognizing the significance of managing deteriorating inventory, researchers have shown great interest in developing models for inventory replenishment plans specifically tailored for such products. One notable study by Covert and Philip [18] presented a scenario where deterioration follows a Weibull distribution. This research contributed to the understanding of how deterioration patterns can be modelled and incorporated into inventory control strategies.

The exploration of deteriorating products and their inventory management has attracted a considerable amount of attention from numerous researchers. Among them, researchers [19-32] have made significant contributions to the field through their respective studies. These researchers have inquired into various aspects of deteriorating inventory management, adding valuable insights to the existing body of knowledge.

In the context of carbon management, many findings have been documented in the existing literature. “To begin with, Huang et al. [33] studied inventory models that include green investment and assessed the effects of various carbon emission policies. Rahimi et al. [34] tackled inventory models of the stochastic routing problem by integrating profit, service level, and environmental criteria. Carbon emission considerations in inventory models for perishable items were investigated by Bozorgi [35]. Beccera et al. [36] conducted research on sustainable green inventory model. Mahato and Mahata [37] explored a sustainable partial backordering inventory model linked to order credit policy and all-unit discount with capacity constraints and carbon emissions.

According to the literature, a Weibull time-varying demand signifies a uniform variance in the item demand rate per unit time, which is unusual in the real world. The Weibull distribution is appropriate for things whose rate of demand rises with time, and the location parameter of the three-parameter Weibull distribution is used to show the item shelf-life, which is an essential parameter of most degrading items with time-varying demand.

In this paper, an application of the preservation technology inventory model for deteriorating goods is addressed, as well as a time-dependent shortage with the rate of demand as probabilistic demand is also considered.

3. ASSUMPTIONS

When constructing an EOQ model, while using Weibull demand with preservation technology, several key assumptions are made. These assumptions shape the fundamental framework of the model and provide a basis for analysis. The following are the key assumptions considered:

Single Item: The EOQ model focuses on a single item within the inventory system. This simplifies the analysis by isolating the dynamics and characteristics of a specific product.

Deterministic Demand: The demand pattern in this model is assumed to be deterministic, meaning that the demand for the item is known with certainty. This assumption enables precise calculations and predictions based on the given demand function.

Shortages Allowed: The model accounts for shortages, meaning that stockouts are permitted. This assumption acknowledges the possibility of not meeting the entire demand, resulting in backorders or unfulfilled orders during periods of stock depletion.

Non-Replacement of Deteriorated Items: The model assumes that items that have deteriorated during the cycle period are not replaced. This reflects the real-world scenario where the deteriorated items cannot be rejuvenated or restored to their original state.

Zero Lead Time: The lead time, which refers to the time between placing an order and receiving it, is assumed to be zero. In reality, some lead time always exists, but businesses try to reduce it to improve responsiveness and service levels.

4. MATHEMATICAL MODEL

The suggested model employs a Weibull demand function to better reflect the dynamic and time-dependent characteristics of perishable goods demand. The existing model relies on a quadratic demand structure, which restricts its ability to accurately reflect realistic demand patterns. In contrast, the Weibull distribution offers a flexible framework capable of representing both increasing and decreasing trends.

4.1 Description about an existing model

Sindhuja et al. [5] proposed an inventory model, considering the effect of the quadratic demand and the deterioration rate, the inventory level at any point of time t, $\eta(t)$ can be expressed by the following differential equation:

$$\frac{d\eta(t)}{dt} + \theta\eta(t) = -(\alpha + \beta t + ct^2)\ 0 \le t \le \tau_1$$

With the initial condition $\eta(0) = q_1$ and boundary condition $\eta(\tau_1) = 0$, the inventory level can be expressed as:

$$\frac{d\eta(t)}{dt} = -(\alpha + \beta t + ct^2)\ \tau_1 \le t \le \tau$$

In this model, the demand is considered as a quadratic demand in the existing model. Comparing with proposed model this kind of demand rapid decline in demand as expiry approaches. While the quadratic demand model captures a decline in demand near expiry, the proposed Weibull-based model offers greater flexibility and accuracy in representing the deterioration and time-dependent behavior of perishable products.

5. THE PROPOSED MATHEMATICAL MODEL

Considering the effect of the probabilistic demand based on Weibull demand with preservation technology, the inventory level at any point of time t, $I(t)$ can be expressed by the following differential equation:

$$\frac{dI(t)}{dt} + \theta(1-\xi)I(t) = -\alpha\beta\gamma^{(\beta-1)} \quad 0 \le t \le T_1 \tag{1}$$

With the initial condition $I(0) = q_1$ and boundary condition $I(T_1) = 0$, the inventory level can be expressed as:

$$\frac{dI(t)}{dt} = -\alpha\beta\gamma^{(\beta-1)} \quad T_1 \le t \le T \tag{2}$$

The solution of Eq. (1) with condition $I(T_1)$ can be obtained as

$$I(t) = \frac{-\alpha\beta\gamma^{-1+\beta}(-\alpha+\gamma\beta)}{\theta(-1+\xi)} + (-\alpha+\gamma\beta)^{\frac{1}{\beta}} \tag{3}$$

Using the initial condition $I(0) = q_1$, the value of q_1 is given by

$$q_1 = (-\alpha)^{\frac{1}{\beta}} \tag{4}$$

The solution of Eq. (2) with the condition $I(T_1) = 0$, is given by

$$I(t) = -T_1\alpha\beta(T_1)^{-1+\beta} + t\alpha\beta T \tag{5}$$

Maximum storage quantity q_2 is given by

$$q_2 = a(T - T_1) + \frac{b}{2}(T^2 - T_1^2) \tag{6}$$

Initial order quantity is $Q_i = q_1 + q_2$

$$Q_i = \frac{\alpha[b\alpha + ba(-1+2\beta)]}{(\theta(-1+\xi))} + (-\alpha)^{\frac{1}{\beta}} + a(T - T_1) + \frac{b}{2}(T^2 - T_1^2) \tag{7}$$

Inventory carrying cost in the system during the time intervals $(0, T_1)$ is given by

$$C_{ih} = h\int_0^{T_1} I(t)\,dt$$

On solving the above equation which yields:

$$C_{ih} = h\int_0^{T_1} I(t)\,dt = h\left[-\frac{(\gamma\beta-\alpha)T_1(2b\alpha+(-2+4\beta)+(-1+\beta)T_1)}{(\theta(-1+\xi))} + (\gamma\beta-\alpha)^{\frac{1}{\beta}}T_1\right] \tag{8}$$

The stock out cost between the time intervals (T_1, T) is given by:

$$C_s = s\int_{T_1}^{T} -I(t)\,dt$$

On solving the above equation which yields:

$$C_s = s\begin{bmatrix} (\gamma\beta-\alpha)^{\frac{1}{\beta}}(T-T_1) \\ -\left[\frac{(\gamma\beta-\alpha)(T-T_1)(2\alpha+T(-1+\beta)+a(-2+4\beta)+(-1+\beta)T_1)}{\theta(-1+\xi)}\right]\end{bmatrix} \quad (9)$$

No. of units purchased in the beginning is $Q_i = q_1 + q_2$.

The number of units deteriorated is $Q = T_1\left(\alpha + \beta\frac{T_1}{2}\right)$.

Hence, the deterioration cost is $D_r = d\left\{q_1 - T_1\left(\alpha + \beta\frac{T_1}{2}\right)\right\}$.

On solving the above equation which yields:

$$D_r = d\left\{\frac{\alpha[b\alpha+ba(-1+2\beta)]}{(-1+\beta)(-1+2\beta)} + (-\alpha)^{\frac{1}{\beta}} - T_1\left(\alpha+\beta\frac{T_1}{2}\right)\right\} \quad (10)$$

The total cost per cycle z is defined as $z = \frac{1}{T}[A + C_{ih} + C_s + D_r]$ and substituting the value of Eqs. (8)-(10), it is obtained as below.

6. DEDUCTION OF OPTIMAL COST

$$z = \frac{1}{T}\begin{Bmatrix} A + h\left[-\frac{(\gamma\beta-a)T_1(2b\alpha+(-2+4\beta)+(-1+\beta)T_1)}{\theta(-1+\xi)} + (\gamma\beta-\alpha)^{\frac{1}{\beta}}T_1\right] \\ +s\begin{bmatrix}(\gamma\beta-\alpha)^{\frac{1}{\beta}}(T-T_1) \\ -\left[\frac{(\gamma\beta-\alpha)(T-T_1)(2b\alpha+bT(-1+\beta)+a(-2+4\beta)+b(-1+\beta)T_1)}{\theta(-1+\xi)}\right]\end{bmatrix} \\ +d\left\{\frac{\alpha[b\alpha+ba(-1+2\beta)]}{(\theta(-1+\xi))} + (-\alpha)^{\frac{1}{\beta}} - T_1\left(\alpha+\beta\frac{T_1}{2}\right)\right\}\end{Bmatrix} \quad (11)$$

Taking the first and the second order partial differentiation for Eq. (11) with respect to T_1 and T, the Eqs. (12) and (13) are obtained:

$$\frac{\partial z}{\partial T_1} = -\frac{1}{T}\begin{Bmatrix} h\left[\frac{(3+4\beta(-1)T_1)}{\theta(-1+\xi)} - \frac{6(\alpha-\beta a)+3(\beta+\alpha)T_1+2\beta(-1)T_1^2}{\theta(-1+\xi)}\right] \\ -s(T-T_1)(\alpha+\beta T_1) + d(-\alpha-\beta T_1)\end{Bmatrix} \quad (12)$$

$$\frac{\partial z}{\partial T} = -\frac{1}{T^2}\begin{Bmatrix} A + h\begin{bmatrix}\frac{-T_1(6a(\alpha-\beta a-2\alpha b)+3(\beta a+\beta(-1))T_1+2d(-1)T_1^2)}{\theta(-1+\xi)} \\ -\frac{\alpha T_1}{\beta}+\frac{T_1^2}{2}\end{bmatrix} \\ +\frac{s(T-T_1)^2}{6}\{3\alpha+\beta(T+2T_1\}+ \\ d\left\{\frac{\alpha[d\alpha+dc(-1+2\beta)]}{\theta(-1+\xi)} + (-a)^{\frac{1}{\beta}} - T_1\left(\alpha+\beta\frac{T_1}{2}\right)\right\}\end{Bmatrix} + \frac{1}{T}\left\{\frac{s(T-T_1)}{2}(2\alpha+\beta T-\beta T_1)\right\} \quad (13)$$

$$\frac{\partial^2 TC(Z)}{\partial T^2} = \frac{1}{T^4}\left[T^2\frac{2s(T_1-T)}{6}3a + T^2\frac{2\beta s(T_1-T)}{6}(T+2T_1) - S\frac{(T_1-T)^2}{6}\right] + \frac{1}{T^2}\left[Ts\frac{(2a+\beta T-\beta T_1)}{2} - \frac{s(T-T_1)}{2}\right] \quad (14)$$

$$\frac{\partial^2 TC(Z)}{\partial {T_1}^2} = \frac{1}{T}\left[h\left(\frac{4\beta}{\theta(\xi-1)} + \frac{4T_1\beta-3(\beta+a)}{\theta(\xi-1)}\right) - s(T-1)(\alpha+\beta T_1) + S\big((T_1-T)\beta - d\beta\big)\right] > 0 \quad (15)$$

The optimal values of $T_1 = T_1^*$ and $T = T^*$ are obtained by solving simultaneously for $\frac{\partial z}{\partial T_1} = 0$ and $\frac{\partial z}{\partial T} = 0$.

The optimal (minimum) values of $T_1 = T_1^*$ and $T = T^*$ are obtained.

MATLAB R2013a is used for finding optimal solution of $T_1 = T_1^*, T = T^*, Q_i = Q_i^*, z = z^*$.

7. FUZZY MODEL

Implementation of the proposed model with Pentagon Fuzzy Numbers to confirm the reliability of the suggested Weibull-based inventory model when faced with uncertain parameters (such as deterioration rate, demand, and holding cost), fuzzy logic is utilized, particularly employing Pentagon Fuzzy Numbers.

Steps in the Fuzzy Implementation:

Fuzzification

Transform uncertain parameters in the model (such as demand rate, deterioration rate, and holding cost) into Pentagon Fuzzy Numbers.

Substitute Fuzzy Parameters into the Model

In the inventory equations of Weibull model, substitute crisp values for fuzzy numbers.

Let $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $\tilde{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, $\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$, $\tilde{d} = (d_1, d_2, d_3, d_4, d_5)$ are as pentagon fuzzy number. In a fuzzy sense, the system's total cost per unit of time is

$$\tilde{z} = \frac{1}{T}\left\{A + h\left[\frac{-(\tilde{\gamma}\tilde{\beta}-a)T_1(2b\tilde{\alpha})+(-2+4\tilde{\beta})+(-1+\tilde{\beta})T_1}{\theta(-1+\xi)} + (\tilde{\gamma}\,\tilde{\beta} - \tilde{\alpha})^{\frac{1}{\tilde{\beta}}}T_1\right] + s\left[(\tilde{\gamma}\,\tilde{\beta} - \tilde{\alpha})^{\frac{1}{\tilde{\beta}}}(T - T_1) - \left[\frac{(\tilde{\gamma}\tilde{\beta}-\tilde{\alpha})(T-T_1)(2b\tilde{\alpha}+bT(-1+\tilde{\beta})+a(-2+4\tilde{\beta})+b(-1+\tilde{\beta})T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\tilde{\alpha}[b\tilde{\alpha}+ba(-1+2\tilde{\beta}]}{\theta(-1+\xi)} + (-\tilde{\alpha})^{\frac{1}{\tilde{\beta}}} - T_1\left(\tilde{\alpha}+\tilde{\beta}\frac{T_1}{2}\right)\right)\right]\right\}$$

Apply Defuzzification Techniques

Utilize Graded Mean Integration and the Signed Distance Method to derive crisp values from fuzzy outputs.

(i) *By Graded Mean Integration Method, Total Cost is given by*

$$Z_{GMT} = \frac{1}{6}\left[Z_{GM_1T} + 2\,Z_{GM_2T} + 2Z_{GM_4T} + Z_{GM_5T}\right]$$

$$Z_{GM_1T} = \frac{1}{T}\left\{A + h\left[\frac{-(\gamma_1\beta_1-a)T_1(2b\alpha_1)+(-2+4\beta_1)+(-1+\beta_1)T_1}{\theta(-1+\xi)} + (\gamma_1\beta_1-\alpha_1)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_1\beta_1-\alpha_1)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_1\beta_1-\alpha_1)(T-T_1)(2b\alpha_1+bT(-1+\beta_1)+a(-2+4\beta_1)+b(-1+\beta_1)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_1[b\alpha_1+ba(-1+2\beta_1)]}{\theta(-1+\xi)} + (-\alpha_1)^{\frac{1}{\beta}} - T_1\left(\alpha_1+\beta_1\frac{T_1}{2}\right)\right)\right]\right\}$$

$$Z_{GM_2T} = \frac{1}{T}\left\{A + h\left[\frac{-(\gamma_2\beta_2-a)T_1(2b\alpha_2)+(-2+4\beta_2)+(-1+\beta_2)T_1}{\theta(-1+\xi)} + (\gamma_2\beta_2-\alpha_2)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_2\beta_2-\alpha_2)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_2\beta_2-\alpha_2)(T-T_1)(2b\alpha_2+bT(-1+\beta_2)+a(-2+4\beta_2)+b(-1+\beta_2)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_2[b\alpha_2+ba(-1+2\beta_2]}{\theta(-1+\xi)} + (-\alpha_2)^{\frac{1}{\beta}} - T_1\left(\alpha_2+\beta_2\frac{T_1}{2}\right)\right)\right]\right\}$$

$$Z_{GM_3T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_3\beta_3 - a)T_1(2b\alpha_3)+(-2+4\beta_3)+(-1+\beta_3)T_1}{\theta(-1+\xi)} + (\gamma_3\beta_3 - \alpha_3)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_3\beta_3 - \alpha_3)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_3\beta_3-\alpha_3)(T-T_1)(2b\alpha_3+bT(-1+\beta_3)+ a(-2+4\beta_3)+b(-1+\beta_3)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_3[b\alpha_3+ba(-1+2\beta_3]}{\theta(-1+\xi)} + (-\alpha_3)^{\frac{1}{\beta}} - T_1\left(\alpha_3 + \beta_3\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{GM_4T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_4\beta_4 - a)T_1(2b\alpha_4)+(-2+4\beta_4)+(-1+\beta_4)T_1}{\theta(-1+\xi)} + (\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+ a(-2+4\beta_4)+b(-1+\beta_4)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{GM_5T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_5\beta_5 - a)T_1(2b\alpha_5)+(-2+4\beta_5)+(-1+\beta_5)T_1}{\theta(-1+\xi)} + (\gamma_5\beta_5 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+ a(-2+4\beta_4)+b(-1+\beta_4)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{GMT} = \frac{1}{6}\Bigg[\frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_1\beta_1 - a)T_1(2b\alpha_1)+(-2+4\beta_1)+(-1+\beta_1)T_1}{\theta(-1+\xi)} + (\gamma_1\beta_1 - \alpha_1)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_1\beta_1 - \alpha_1)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_1\beta_1-\alpha_1)(T-T_1)(2b\alpha_1+bT(-1+\beta_1)+a(-2+4\beta_1)+b(-1+\beta_1)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_1[b\alpha_1+ba(-1+2\beta_1]}{\theta(-1+\xi)} + (-\alpha_1)^{\frac{1}{\beta}} - T_1\left(\alpha_1 + \beta_1\frac{T_1}{2}\right)\right)\Bigg]\Bigg\} + 2\frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_2\beta_2 - a)T_1(2b\alpha_2)+(-2+4\beta_2)+(-1+\beta_2)T_1}{\theta(-1+\xi)} + (\gamma_2\beta_2 - \alpha_2)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_2\beta_2 - \alpha_2)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_2\beta_2-\alpha_2)(T-T_1)(2b\alpha_2+bT(-1+\beta_2)+a(-2+4\beta_2)+b(-1+\beta_2)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_2[b\alpha_2+ba(-1+2\beta_2]}{\theta(-1+\xi)} + (-\alpha_2)^{\frac{1}{\beta}} - T_1\left(\alpha_2 + \beta_2\frac{T_1}{2}\right)\right)\Bigg]\Bigg\} + 2\frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_4\beta_4 - a)T_1(2b\alpha_4)+(-2+4\beta_4)+(-1+\beta_4)T_1}{\theta(-1+\xi)} + (\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+a(-2+4\beta_4)+b(-1+\beta_4)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\Bigg]\Bigg\} + \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_5\beta_5 - a)T_1(2b\alpha_5)+(-2+4\beta_5)+(-1+\beta_5)T_1}{\theta(-1+\xi)} + (\gamma_5\beta_5 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+a(-2+4\beta_4)+b(-1+\beta_4)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}\Bigg]$$

The necessary condition for Z_{GMT} to be minimum is

$$\frac{\partial Z_{GMT}}{\partial T} = 0$$

Z_{GMT} is minimum only if $\frac{\partial^2 Z_{GMT}}{\partial T^2} > 0$, for all T > 0.

(ii) *By Signed Distance Method, total cost is given by*

$$Z_{SDT} = \frac{1}{9}\left[Z_{SD_1T} + 2\,Z_{SD_2T} + 3\,Z_{SD_3T} + 2Z_{SD_4T} + Z_{SD_5T}\right]$$

$$Z_{SD_1T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_1\beta_1 - a)T_1(2b\alpha_1)+(-2+4\beta_1)+(-1+\beta_1)T_1}{\theta(-1+\xi)} + (\gamma_1\beta_1 - \alpha_1)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_1\beta_1 - \alpha_1)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_1\beta_1-\alpha_1)(T-T_1)(2b\alpha_1+bT(-1+\beta_1)+ a(-2+4\beta_1)+b(-1+\beta_1)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_1[b\alpha_1+ba(-1+2\beta_1]}{\theta(-1+\xi)} + (-\alpha_1)^{\frac{1}{\beta}} - T_1\left(\alpha_1 + \beta_1\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{SD_2T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_2\beta_2 - a)T_1(2b\alpha_2)+(-2+4\beta_2)+(-1+\beta_2)T_1}{\theta(-1+\xi)} + (\gamma_2\beta_2 - \alpha_2)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_2\beta_2 - \alpha_2)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_2\beta_2-\alpha_2)(T-T_1)(2b\alpha_2+bT(-1+\beta_2)+ a(-2+4\beta_2)+b(-1+\beta_2)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_2[b\alpha_2+ba(-1+2\beta_2]}{\theta(-1+\xi)} + (-\alpha_2)^{\frac{1}{\beta}} - T_1\left(\alpha_2 + \beta_2\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{SD_3T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_3\beta_3 - a)T_1(2b\alpha_3)+(-2+4\beta_3)+(-1+\beta_3)T_1}{\theta(-1+\xi)} + (\gamma_3\beta_3 - \alpha_3)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_3\beta_3 - \alpha_3)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_3\beta_3-\alpha_3)(T-T_1)(2b\alpha_3+bT(-1+\beta_3)+ a(-2+4\beta_3)+b(-1+\beta_3)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_3[b\alpha_3+ba(-1+2\beta_3]}{\theta(-1+\xi)} + (-\alpha_3)^{\frac{1}{\beta}} - T_1\left(\alpha_3 + \beta_3\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{SD_4T} = \frac{1}{T}\Bigg\{A + h\left[\frac{-(\gamma_4\beta_4 - a)T_1(2b\alpha_4)+(-2+4\beta_4)+(-1+\beta_4)T_1}{\theta(-1+\xi)} + (\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\Bigg[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+ a(-2+4\beta_4)+b(-1+\beta_4)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\Bigg]\Bigg\}$$

$$Z_{SD_5T} = \frac{1}{T}\left\{A + h\left[\frac{-(\gamma_5\beta_5 - a)T_1(2b\alpha_5)+(-2+4\beta_5)+(-1+\beta_5)T_1}{\theta(-1+\xi)} + (\gamma_5\beta_5 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+a(-2+4\beta_4)+b(-1+\beta_4)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\right]\right\}$$

$$Z_{SDT} = \frac{1}{9}\left[Z_{SD_1T} + 2\,Z_{SD_2T} + 3\,Z_{SD_3T} + 2Z_{SD_4T} + Z_{SD_5T}\right]$$

$$= \frac{1}{9}\left[\frac{1}{T}\left\{A + h\left[\frac{-(\gamma_1\beta_1 - a)T_1(2b\alpha_1)+(-2+4\beta_1)+(-1+\beta_1)T_1}{\theta(-1+\xi)} + (\gamma_1\beta_1 - \alpha_1)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_1\beta_1 - \alpha_1)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_1\beta_1-\alpha_1)(T-T_1)(2b\alpha_1+bT(-1+\beta_1)+a(-2+4\beta_1)+b(-1+\beta_1)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_1[b\alpha_1+ba(-1+2\beta_1]}{\theta(-1+\xi)} + (-\alpha_1)^{\frac{1}{\beta}} - T_1\left(\alpha_1 + \beta_1\frac{T_1}{2}\right)\right)\right]\right\} + \right.$$

$$2\,\frac{1}{T}\left\{A + h\left[\frac{-(\gamma_2\beta_2 - a)T_1(2b\alpha_2)+(-2+4\beta_2)+(-1+\beta_2)T_1}{\theta(-1+\xi)} + (\gamma_2\beta_2 - \alpha_2)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_2\beta_2 - \alpha_2)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_2\beta_2-\alpha_2)(T-T_1)(2b\alpha_2+bT(-1+\beta_2)+a(-2+4\beta_2)+b(-1+\beta_2)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_2[b\alpha_2+ba(-1+2\beta_2]}{\theta(-1+\xi)} + (-\alpha_2)^{\frac{1}{\beta}} - T_1\left(\alpha_2 + \beta_2\frac{T_1}{2}\right)\right)\right]\right\} +$$

$$3\,\frac{1}{T}\left\{A + h\left[\frac{-(\gamma_3\beta_3 - a)T_1(2b\alpha_3)+(-2+4\beta_3)+(-1+\beta_3)T_1}{\theta(-1+\xi)} + (\gamma_3\beta_3 - \alpha_3)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_3\beta_3 - \alpha_3)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_3\beta_3-\alpha_3)(T-T_1)(2b\alpha_3+bT(-1+\beta_3)+a(-2+4\beta_3)+b(-1+\beta_3)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_3[b\alpha_3+ba(-1+2\beta_3]}{\theta(-1+\xi)} + (-\alpha_3)^{\frac{1}{\beta}} - T_1\left(\alpha_3 + \beta_3\frac{T_1}{2}\right)\right)\right]\right\} +$$

$$2\,\frac{1}{T}\left\{A + h\left[\frac{-(\gamma_4\beta_4 - a)T_1(2b\alpha_4)+(-2+4\beta_4)+(-1+\beta_4)T_1}{\theta(-1+\xi)} + (\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{((\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+a(-2+4\beta_4)+b(-1+\beta_4)T_1)}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\right]\right\} +$$

$$\left.\frac{1}{T}\left\{A + h\left[\frac{-(\gamma_5\beta_5 - a)T_1(2b\alpha_5)+(-2+4\beta_5)+(-1+\beta_5)T_1}{\theta(-1+\xi)} + (\gamma_5\beta_5 - \alpha_4)^{\frac{1}{\beta}}T_1\right] + s\left[(\gamma_4\beta_4 - \alpha_4)^{\frac{1}{\beta}}(T - T_1) - \left[\frac{(\gamma_4\beta_4-\alpha_4)(T-T_1)(2b\alpha_4+bT(-1+\beta_4)+a(-2+4\beta_4)+b(-1+\beta_4)T_1}{\theta(-1+\xi)}\right] + d\left(\frac{\alpha_4[b\alpha_4+ba(-1+2\beta_4]}{\theta(-1+\xi)} + (-\alpha_4)^{\frac{1}{\beta}} - T_1\left(\alpha_4 + \beta_4\frac{T_1}{2}\right)\right)\right]\right\}\right]$$

The necessary condition for Z_{SDT} to be minimum is

$$\frac{\partial Z_{SDT}}{\partial T} = 0$$

Z_{SDT} is minimum only if $\frac{\partial^2 Z_{SDT}}{\partial T^2} > 0$, for all T > 0.

7.1 Solution procedure

In the context of the Economic Order Quantity (EOQ) inventory model, checking the convexity condition is crucial because it ensures that the objective function (typically the total cost function) has a single minimum point, which represents the optimal order quantity. Here's why checking convexity is important:

Single Optimal Solution: The EOQ model seeks to minimize total inventory costs, which include ordering costs and holding costs. If the total cost function is convex, it guarantees that there is only one minimum point. This ensures that there is a unique order quantity that minimizes the total cost. Without convexity, the function could have multiple local minima or points of inflection, making it difficult to determine the true optimal order quantity.

Mathematical Derivatives: Convexity simplifies the mathematical analysis of the EOQ model. For convex functions, if the first derivative (slope of the cost function) is zero at a point, that point is a global minimum. This property allows for straightforward calculation of the EOQ using calculus, ensuring accuracy and reliability in determining the optimal order quantity.

To summarize, checking convexity in the EOQ inventory model ensures that the cost function has a unique minimum point, simplifying both theoretical analysis and practical application of the model in inventory management.

7.2 Lemma

For a given (Q_i, T) the expected total cost per cycle TC (Z) is jointly convex in (T, T_1).

Proof

To prove TC (Z) is jointly convex in (T, T_1) it is enough to prove that Hessian matrix is positive semi definite. That is to prove all principal minors are non-negative.

We have found that

$$\frac{\partial^2 TC(Z)}{\partial T^2} = \frac{1}{T^4}\left[T^2\frac{2s(T_1-T)}{6}3a + T^2\frac{2\beta s(T_1-T)}{6}(T + 2T_1) - S\frac{(T_1-T)^2}{6}\right] + \frac{1}{T^2}\left[Ts\frac{(2a+\beta T-\beta T_1)}{2} + \frac{s(T-T_1)}{2}\right] > 0$$

$$\frac{\partial^2 TC(Z)}{\partial {T_1}^2} = \frac{1}{T}[h(\frac{4\beta}{\theta(\xi-1)} + \frac{4T_1\beta-3(\beta+a)}{\theta(\xi-1)} - s(T-1)(\alpha+\beta T_1)s + S\big((T_1-T)\beta - d\beta\big] > 0$$

$$\frac{\partial^2 TC(Z)}{\partial T\partial T_1} = \frac{\partial^2 TC(Z)}{\partial T_1\partial T} = 0$$

The Hessian matrix is given by $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix}$

$$H = \begin{bmatrix} \frac{\partial^2 TC(Z)}{\partial T^2} & \frac{\partial^2 TC(Z)}{\partial {T_1}^2} \\ \frac{\partial^2 TC(Z)}{\partial T\partial T_1} & \frac{\partial^2 TC(Z)}{\partial T_1\partial T}\end{bmatrix}$$

From the above findings, we found that the first principal minor $H_{11} > 0$ and the second principal minor $H_{22} > 0$.

Thus, we found that the expected total cost per cycle TC (Z) is jointly convex in (T, T_1).

7.3 Theorem

An optimal point of $T_1 = T_1^*$ and $T = T^*$ of the cost function $Z = Z^*$ is obtained from the solving Eqs. (12) and (13) simultaneously, and substituting these positive values in Eq. (11) to calculate the value of total cost.

Proof:

Note that for each fixed value for T >0, in the segment of extreme points (0, T_1) and (T_1, T) the function z is continuous and attains its minimum only in a point $T_1 = T_1^*$ and $T = T^*$ determined by the Eqs. (12) and (13).

Also, on the considered segment, the function z is decreasing for all $T_1 \in (0, T_1)$ and increasing for all $T_1 \in$ (t, T). Moreover, in Eq. (11), T⟶0, it is not possible to attain the minimum value because the total cost value tends to infinity. Furthermore, if T⟶∞, the total cost values tend to infinity. Thus, the function Z attains its global minimum ($T_1 = T_1^*$ and $T = T^*$) in at least a finite interior point of the feasible region.

Therefore, at this point ($T_1 = T_1^*$ and $T = T^*$) the necessary conditions of first and second order to have a local minimum should be satisfied.

Thus, provided that these values of $T_1 = T_1^*$ and $T = T^*$.

$$\frac{\partial^2 TC(Z)}{\partial T^2} = \frac{1}{T^4}\left[T^2 \frac{2s(T_1-T)}{6} 3a + T^2 \frac{2\beta s(T_1-T)}{6}(T + 2T_1) - S\frac{(T_1-T)^2}{6}\right] + \frac{1}{T^2}\left[Ts\frac{(2a+\beta T-\beta T_1)}{2} + \frac{s(T-T_1)}{2}\right] > 0$$

$$\frac{\partial^2 TC(Z)}{\partial {T_1}^2} = \frac{1}{T}\left[h\left(\frac{4\beta}{\theta(\xi-1)} + \frac{4T_1\beta-3(\beta+a)}{\theta(\xi-1)} - s(T-1)(\alpha+\beta T_1) + S\big((T_1-T)\beta - d\beta\right]\right. > 0$$

The pair ($T_1 = T_1^*$ and $T = T^*$) should be satisfy the inventory policy. Having the values of ($T_1 = T_1^*$ and $T = T^*$) from the Eqs. (12) and (13) can calculate the total cost value through the numerical example.

8. NUMERICAL EXAMPLES

Examine a system of inventory where the parametric values are in the appropriate units [38].

Example 8.1 (Graded Mean Integration Method)

Suppose A= 200, $\tilde{\alpha}$ = (50, 75, 125 ,175 , 200) , $\tilde{\beta}$ = (4,7,10,12,20) , $\tilde{\gamma}$ = (8,12,15,18,25) , $\tilde{d}$ = (0.5,1.2,2,3,5) , s=20, h=10, Ɛ from 0.001 to 0.1

The values of $T_1 = T_1^* = 0.53$, $T = T^* = 0.905$, $Q_i = Q_i^* = 89.9327$ units and $z = z^* = 746.1315$ obtain from Mathematica.

Example 8.2 (Signed Distance Method)

Suppose A= 200, $\tilde{\alpha}$ = (50, 75, 125 ,175 , 200) , $\tilde{\beta}$ = (4,7,10,12,20) , $\tilde{\gamma}$ = (8,12,15,18,25) , $\tilde{d}$ = (0.5,1.2,2,3,5) , s=20, h=10, Ɛ from 0.001 to 0.1

The values of $T_1 = T_1^* = 0.52, T = T^* = 0.902$ $Q_i = Q_i^* = 90.9327$ units and $z = z^* = 745.1315$ obtain from Mathematica.

8.1 Sensitivity analysis based on pentagon fuzzy values by Graded Mean Integration Method

The sensitivity analysis based on pentagon fuzzy values by Graded Mean Integration Method as shown in Table 1.

Table 1. Graded mean integration method

Parameter	% of Change	T_1	T	Q_i	Z
	50%	0.53	0.9	109.67	752.45
	25%	0.52	0.89	108.19	750.32
α	0%	0.51	0.88	107.47	749.65
	-25%	0.49	0.86	106.74	748.66
	-50%	0.48	0.85	105.97	747.88
	50%	0.53	0.9	109.78	751.89
	25%	0.52	0.89	108.78	750.77
β	0%	0.51	0.89	107.53	749.88
	-25%	0.49	0.88	106.72	748.77
	-50%	0.48	0.85	105.92	746.66
	50%	0.44	0.71	109.8	748.99
	25%	0.44	0.71	109.6	750.09
γ	0%	0.43	0.7	109.4	752.03
	-25%	0.42	0.69	109.52	753.97
	-50%	0.41	0.68	109.44	755.89
	50%	0.42	0.65	105.64	745.67
	25%	0.41	0.64	104.56	745.56
d	0%	0.4	0.63	103.56	745.46
	-25%	0.4	0.62	102.67	744.67
	-50%	0.4	0.61	101.45	744.56

8.2 Graphical representation of sensitivity analysis based on pentagon fuzzy values by Graded Mean Integration Method

The Graphical representation of sensitivity analysis based on pentagon fuzzy values by Graded Mean Integration Method as shown in Figures 1-4.

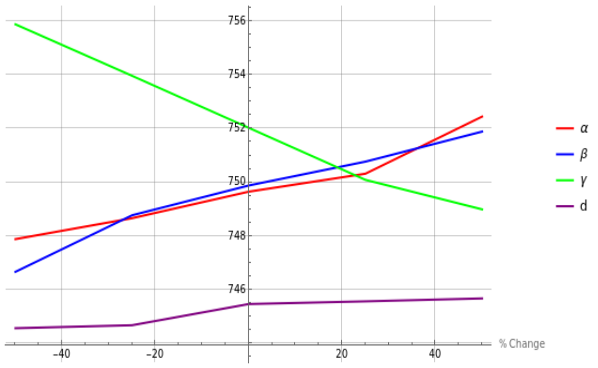

Figure 1. Sensitivity analysis of the total cost value Z with respect to the parameters α, β, γ, d

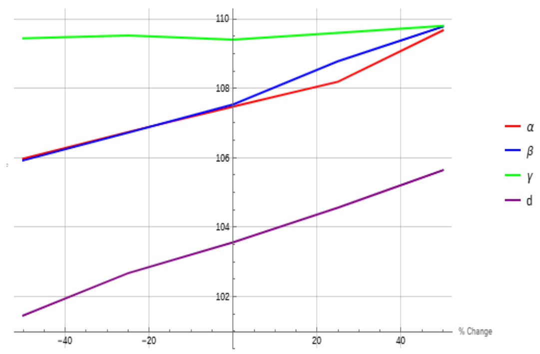

Figure 2. Sensitivity analysis of the initial order quantity Q_i with respect to the parameters α, β, γ, d

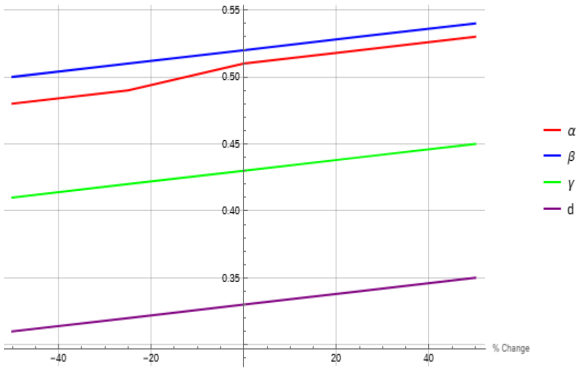

Figure 3. Sensitivity analysis of T_1 with respect to the parameters α, β, γ, d

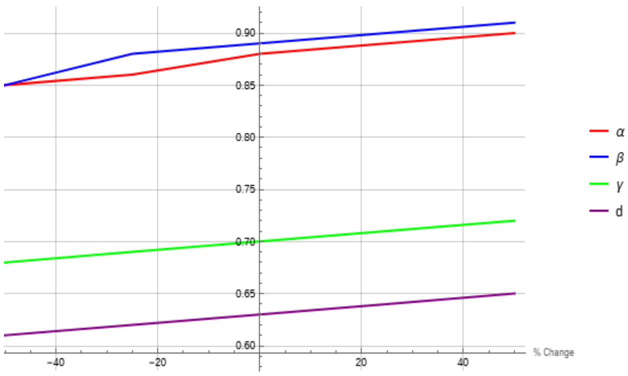

Figure 4. Sensitivity analysis of T with respect to the parameters α, β, γ, d

8.3 Sensitivity analysis based on pentagon fuzzy values by Signed Distance Method

The sensitivity analysis based on pentagon fuzzy values by Signed Distance Method as shown in Table 2.

Table 2. Signed Distance Method

Parameter	% of Change	T1	T	QI	Z
α	50%	0.55	0.9	110.67	754.45
	25%	0.54	0.89	109.56	752.32
	0%	0.53	0.88	108.45	751.88
	-25%	0.52	0.86	107.89	750.77
	-50%	0.51	0.85	106.7	749.66
β	50%	0.53	0.9	110.66	756.77
	25%	0.52	0.89	109.7	755.89
	0%	0.51	0.89	108.5	754.87
	-25%	0.49	0.88	107.6	752.66
	-50%	0.48	0.85	106.5	751.22
γ	50%	0.44	0.71	109.8	746.88
	25%	0.43	0.71	108.7	745.92
	0%	0.42	0.7	107.6	744.52
	-25%	0.41	0.69	106.8	743.41
	-50%	0.4	0.68	105.6	755.89
d	50%	0.43	0.65	107.7	752.87
	25%	0.42	0.64	106.8	751.66
	0%	0.41	0.63	105.6	750.23
	-25%	0.4	0.62	104.2	749.34
	-50%	0.39	0.61	103.3	748.23

8.4 Graphical representation of sensitivity analysis based on pentagon fuzzy values by Graded Mean Integration Method

The graphical representation of sensitivity analysis based on pentagon fuzzy values by Graded Mean Integration Method as shown in Figures 5-8.

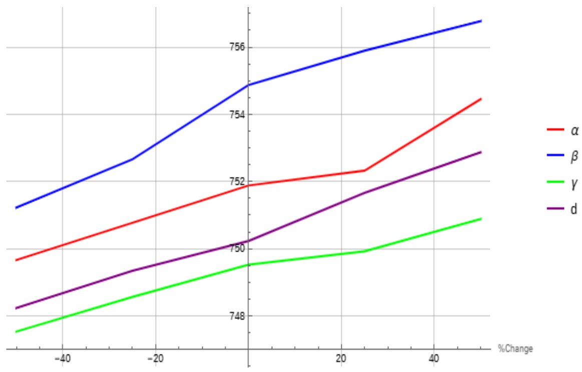

Figure 5. Sensitivity analysis of the total cost value Z with respect to the parameters α, β, γ, d

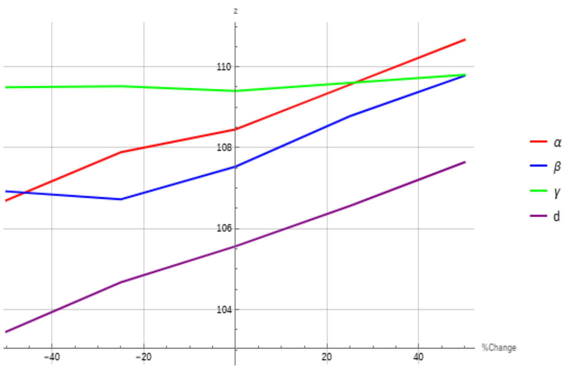

Figure 6. Sensitivity analysis of the initial order quantity Q_i with respect to the parameters α, β, γ, d

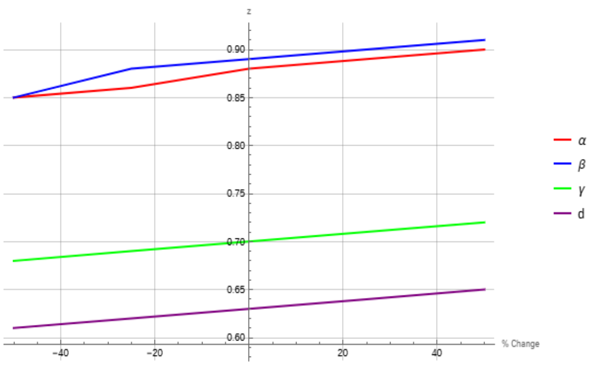

Figure 7. Sensitivity analysis of T with respect to the parameters α, β, γ, d

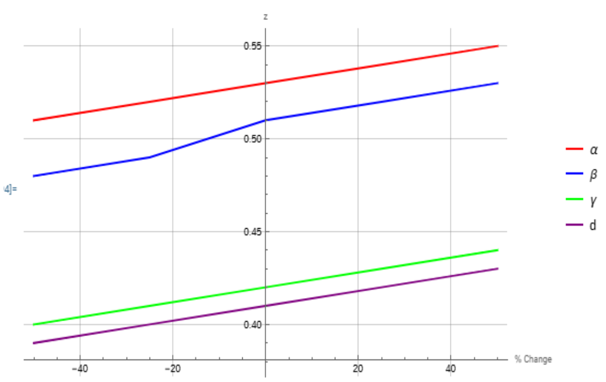

Figure 8. Sensitivity analysis of T_1 with respect to the parameters α, β, γ, d

8.5 Managerial implications of Weibull demand parameters in the proposed model

The Weibull distribution for demand characterization, providing more flexibility than linear or quadratic models. The Weibull demand function is capable of representing demand that increases, decreases, or remains constant over time — an essential aspect for the management of perishable goods.

- α - Assists in estimating the overall anticipated demand over the product's shelf life.
- β - Aids in the timing of preservation measures.
- γ - Grasping time dependence aids in determining the timing for promotions, restocking, and discounts.
- d - Facilitates the alignment of strategy with policies on quality and sustainability

9. CONCLUSION

This research created a preservation-focused inventory model that integrates Weibull demand and deterioration in environments characterized by uncertainty and fuzziness. The analytical and numerical findings underscored the impact of crucial factors, including preservation cost, deterioration rate, and carbon emission charges, on optimal inventory strategies. Through the comparison of crisp and pentagonal fuzzy approaches, the model's robustness in dealing with uncertainty was shown.

The results highlight how vital it is to incorporate preservation technology in order to minimize waste and enhance sustainability in perishable inventory systems. The model supports managers in making decisions that weigh environmental effects against cost efficiency.

To improve the model's relevance for practical situations, future studies could investigate deterioration functions of greater complexity, multi-echelon supply chains, and dynamic pricing in relation to demand uncertainty.

REFERENCES

[1] Khurana, D., Tayal, S., Singh, S.R. (2018). An EPQ model for deteriorating items with variable demand rate and allowable shortages. International Journal of Mathematics in Operational Research, 12(1): 117-128. https://doi.org/10.1504/IJMOR.2018.088582

[2] Khara, B., Dey, J.K., Mondal, S.K. (2019). Effects of product reliability dependent demand in an EPQ model considering partially imperfect production. International Journal of Mathematics in Operational Research, 15(2): 242-264. https://doi.org/10.1504/IJMOR.2019.101621

[3] Palanivel, M., Uthayakumar, R. (2016). Finite horizon EOQ model for non-instantaneous deteriorating items with probabilistic deterioration and partial backlogging under inflation. International Journal of Mathematics in Operational Research, 8(4): 449-476. https://doi.org/10.1504/IJMOR.2016.076784

[4] Saha, S., Sen, N. (2019). An inventory model for deteriorating items with time and price dependent demand and shortages under the effect of inflation. International Journal of Mathematics in Operational Research, 14(3): 377-388. https://doi.org/10.1504/IJMOR.2019.099385

[5] Sindhuja, S., Arathi, P., Mohan, V. (2021). A quadratic demand EOQ model for deteriorating items with time dependent shortage. International Journal of Mathematics in Operational Research, 20(1): 41-59. https://doi.org/10.1504/IJMOR.2021.117636

[6] Mandal, B., Pal, A.K. (1998). Order level inventory system with ramp type demand rate for deteriorating items. Journal of interdisciplinary Mathematics, 1(1): 49-66. https://doi.org/10.1080/09720502.1998.10700243

[7] Manna, P., Manna, S.K., Giri, B.C. (2016). An economic order quantity model with ramp type demand rate, constant deterioration rate and unit production cost. Yugoslav Journal of Operations Research, 26(3): 305-316. https://doi.org/10.2298/YJOR140505020M

[8] Ukil, S.I., Islam, M.E., Uddin, M.S. (2015). A production inventory model of power demand and constant production rate where the products have finite shelf-life. Journal of Service Science and Management, 8(6): 874-885. https://doi.org/10.4236/jssm.2015.86088

[9] Mishra, V.K. (2014). Controllable deterioration rate for time-dependent demand and time-varying holding cost. Yugoslav Journal of Operations Research, 24(1): 87-98. https://doi.org/10.2298/YJOR120823018M

[10] Deng, P.S. (2005). Improved inventory models with ramp type demand and Weibull deterioration. International Journal of Information and Management Sciences, 16(4): 79-86.

[11] Skouri, K., Konstantaras, I., Papachristos, S., Ganas, I. (2009). Inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate. European Journal of Operational Research, 192(1): 79-92. https://doi.org/10.1016/j.ejor.2007.09.003

[12] Sarkar, B., Sarkar, S. (2013). Variable deterioration and demand—An inventory model. Economic Modelling, 31: 548-556. https://doi.org/10.1016/j.econmod.2012.11.045

[13] Chandra, S. (2017). An inventory model with ramp type demand, time varying holding cost and price discount on backorders. Uncertain Supply Chain Management, 5(1): 52-58.

[14] Chang, C.T., Chen, Y.J., Tsai, T.R., Shuo-Jye, W. (2010). Inventory models with stock-and price-dependent demand for deteriorating items based on limited shelf space. Yugoslav Journal of Operations Research, 20(1): 55-69. https://doi.org/10.2298/YJOR1001055C

[15] Park, K.S. (1982). Inventory model with partial backorders. International Journal of Systems Science, 13(12): 1313-1317. https://doi.org/10.1080/00207728208926430

[16] Rai, V., Sharma, B.K. (2016). A deterministic inventory model for non-instantaneous deteriorating items with ramp-type demand rate and shortages under permissible delay in payments. Le Matematiche, 71(2): 129-159.

[17] Giri, B.C., Chaudhuri, K.S. (1998). Deterministic models of perishable inventory with stock-dependent demand rate and nonlinear holding cost. European Journal of Operational Research, 105(3): 467-474. https://doi.org/10.1016/S0377-2217(97)00086-6

[18] Covert, R.P., Philip, G.C. (1973). An EOQ model for items with Weibull distribution deterioration. AIIE Transactions, 5(4): 323-326. https://doi.org/10.1080/05695557308974918

[19] Michalski, G. (2009). Inventory management optimization as part of operational risk management.

Economic Computation and Economic Cybernetics Studies and Research, 213-222.
[20] Padmanabhan, G., Vrat, P. (1995). EOQ models for perishable items under stock dependent selling rate. European Journal of Operational Research, 86(2): 281-292. https://doi.org/10.1016/0377-2217(94)00103-J
[21] Rau, H., Wu, M.Y., Wee, H.M. (2003). Integrated inventory model for deteriorating items under a multi-echelon supply chain environment. International Journal of Production Economics, 86(2): 155-168. https://doi.org/10.1016/S0925-5273(03)00048-3
[22] Chakrabarty, T., Giri, B.C., Chaudhuri, K.S. (1998). An EOQ model for items with Weibull distribution deterioration, shortages and trended demand: An extension of Philip's model. Computers & Operations Research, 25(7-8): 649-657. https://doi.org/10.1016/S0305-0548(97)00081-6
[23] Sarbjit, S., Shivraj, S. (2009). An optimal inventory policy for items having linear demand and variable deterioration rate with trade credit. Journal of Mathematics and Statistics, 5(4): 330-333.
[24] Yang, H.L. (2011). A partial backlogging production-inventory lot-size model for deteriorating items with time-varying production and demand rate over a finite time horizon. International Journal of Systems Science, 42(8): 1397-1407. https://doi.org/10.1080/00207720903576464
[25] Philip, G.C. (1974). A generalized EOQ model for items with Weibull distribution deterioration. AIIE Transactions, 6(2): 159-162. https://doi.org/10.1080/05695557408974948
[26] Yang, P.C., Wee, H.M. (2003). An integrated multi-lot-size production inventory model for deteriorating item. Computers & Operations Research, 30(5): 671-682. https://doi.org/10.1016/S0305-0548(02)00032-1
[27] Bahari-Kashani, H. (1989). Replenishment schedule for deteriorating items with time-proportional demand. Journal of the Operational Research Society, 40(1): 75-81. https://doi.org/10.1057/jors.1989.7
[28] Sarker, B.R., Jamal, A.M.M., Wang, S. (2000). Optimal payment time under permissible delay in payment for products with deterioration. Production Planning & Control, 11(4): 380-390. https://doi.org/10.1080/095372800232117
[29] Lo, S.T., Wee, H.M., Huang, W.C. (2007). An integrated production-inventory model with imperfect production processes and Weibull distribution deterioration under inflation. International Journal of Production Economics, 106(2): 439-505. https://doi.org/10.1016/j.ijpe.2006.07.001
[30] Pang, Z. (2011). Optimal dynamic pricing and inventory control with stock deterioration and partial backordering. Operations Research Letters, 39(5): 375-379. https://doi.org/10.1016/j.orl.2011.06.009
[31] Hou, K.L., Lin, L.C. (2006). An EOQ model for deteriorating items with price-and stock-dependent selling rates under inflation and time value of money. International Journal of Systems Science, 37(15): 1131-1139. https://doi.org/10.1080/00207720601014206
[32] Prasad, K., Mukherjee, B. (2016). Optimal inventory model under stock and time dependent demand for time varying deterioration rate with shortages. Annals of Operations Research, 243: 323-334. https://doi.org/10.1007/s10479-014-1759-3
[33] Huang, Y.S., Fang, C.C., Lin, Y.A. (2020). Inventory management in supply chain with coordination of logistics, green investment, and different carbon emissions policies. Computers and Industrial Engineering, 139: 106207. https://doi.org/10.1016/j.cie.2019.10620
[34] Rahimi, M., Baboli, A., Rekik, Y. (2017). Multi-objective inventory routing problems: A stochastic model to consider profit, service level and green criteria. Transportation Research Part E: Logistics and Transportation Review, 101: 59-83. https://doi.org/10.1016/j.tre.2017.03.001
[35] Bozorgi, A. (2016). Multi-product inventory model for cold items with cost and emission consideration International Journal of Production Economics, 176: 123-142. http://doi.org/10.1016/j.ijpe.2016.03.011
[36] Becerra, P., Mula, J., Sanchis, R. (2021). Green supply chain quantitative models for sustainable inventory management: A review. Journal of Cleaner Production, 328: 129544. https://doi.org/10.1016/j.jclepro.2021.129544
[37] Mahato, C., Mahata, G.C. (2023). Sustainable partial backordering inventory model under linked-to-order credit policy and all-units discount with capacity constraint and carbon emissions. Flexible Services and Manufacturing Journal, 35: 896-944. https://doi.org/10.1007/s10696-022-09456-9
[38] Mahapatra, R.P., Peddoju, S.K., Roy, S., Parwekar, P., Goel, L. (2022). Proceedings of International Conference on Recent Trends in Computing. Springer Singapore. https://doi.org/10.1007/978-981-16-7118-0

NOMENCLATURE

A	Cost of placing an order
h	Cost of holding the stock
d	Purchase cost of one unit
Q_i	Initial stock
T_1	Time of positive stock
T	Cycle time
α, β, γ	Demand parameters
T_1^*	Optimal time of positive inventory
z	Total cost per cycle
ξ	Preservation parameter
D	Demand rate $D = \alpha\beta\gamma^{(\beta-1)}$
D_n	Number of deteriorated units
d_r	Rate of deterioration at any time
q_1	Maximum stock level
q_2	Maximum stock shortage level
z^*	Optimal total cost per cycle