Solving Fuzzy Volterra Fractional Integro-Differential Equations with a New Least Squares Technique

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ABSTRACT

This work introduces a new Least Squares Technique (LST) aimed at solving both linear and nonlinear fuzzy fractional Volterra integro-differential equations (FFVIDEs). The new method combines a special type of Legendre polynomial with a system for adjusting control points to improve accuracy in finding solutions. Unlike other methods, the fuzzy Volterra Least Squares Technique (FVLST) includes the control point parameters directly in the solution, allowing for real-time adjustments based on the fractional order α and fuzzy membership functions. The method utilizes Caputo's derivative in fuzzy settings and applies convergence analysis to ensure solution stability. Numerical results show that FVLST achieves absolute errors below 0.5% for both linear and nonlinear FFVIDEs at $\alpha=0.9$ and demonstrates convergence rates approximately three times faster than conventional methods. Overall, the proposed FVLST greatly improves the accuracy of solutions and the speed of calculations, making it a robust option for dealing with FFVIDEs.

1. INTRODUCTION

Fractional integro-differential equations (FIDEs) are important for modeling complicated systems in engineering and applied sciences because they can include memory effects and behaviors that are not limited to local interactions. Despite their relevance, these equations rarely admit closed-form solutions, necessitating the development of numerical and semi-analytical methods [1, 2]. Various approaches have been proposed to address such systems, particularly in signal processing, fluid dynamics, and other nonlocal applications [3-9]. Among these methods, fractional differential equations (FDEs) have become a strong tool for understanding long-term relationships in systems that classical calculus can't describe well, and they are used in areas like fluid mechanics, electronics, biological systems, and robotics. The foundational definitions by references [10-14] offer mathematical flexibility yet face challenges related to computational complexity, convergence stability, and sensitivity to uncertain inputs. Models using Riemann–Liouville often have difficulty accurately simulating real-world data, while Caputo's derivative works better for physical problems but doesn't automatically handle uncertainty from unclear starting or boundary conditions. To address this, fuzzy set theory was introduced into fractional modeling, most notably through the work of the reference [15], which incorporated fuzziness using Hukuhara differences. These findings led to the development of fuzzy fractional differential equations (FFDEs), which have since been explored in terms of existence, uniqueness, and approximate solutions [16-21]. However, current fuzzy approximation methods often struggle with controlling how well they work and dealing with uncertainty, especially when it comes to fuzzy fractional Volterra integro-differential equations (FFVIDEs). Even though the Least Squares Technique (LST), created by Legendre in 1805, is a good method that avoids breaking down variables and reduces numerical errors, it doesn't have ways to deal with fuzziness or adjust convergence on the fly. This limitation highlights a clear research gap: existing methods fall short in simultaneously managing fractional dynamics, fuzzy uncertainty, and solution convergence. To tackle these issues, this paper introduces a new approach called the Fuzzy Volterra Least Squares Technique (FVLST), which uses a special type of Legendre polynomial and optimizes control points to improve how well it can adapt and accurately approximate solutions. Unlike conventional methods, FVLST dynamically responds to fuzzy membership levels and fractional order variations, enabling it to solve both linear and nonlinear FFVIDEs effectively in the Caputo sense. The remainder of the paper is structured as follows: Section 2 explains the basic concepts of fractional calculus and fuzzy analysis; Section 3 describes the proposed FVLST; Section 4 shows how to use

the method on specific FFVIDEs; Section 5 discusses how the method converges; Section 6 gives numerical examples to support the method; and Section 7 wraps up the main points and suggests future research directions.

2. PRELIMINARIES

This part provides an overview of the essential definitions and notations of fractional calculus that are necessary to understand the content in the next sections. Definitions 2.1 and 2.2, extracted from references [14, 22], provide precise explanations of the fractional derivative and integral. Additionally, the property that the Caputo derivative of a constant is equal to zero makes them commonly used, especially in the representation of physical and engineering problems. In addition, the Caputo fractional derivative enables the use of beginning or boundary conditions in the problem formulation. Also, this section will provide precise definitions for important terminology in fuzzy set theory, including fuzzy functions and fuzzy integers. The range of numbers is from references [23, 24].

Definition 2.1 [14, 24] (Riemann-Liouville): The R-L fractional integral of order $\alpha > 0$ is define as:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(t) d\tau, t > 0, \qquad \alpha \in \mathbb{R}^+$$

where, Γ represents the Gamma function.

Definition 2.2 [11, 24] (Caputo fractional order derivative): The Caputo fractional order derivative of order $\alpha > 0$ is define as follows:

$$^c\mathfrak{D}^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases}$$

For $\alpha > 0$, the Caputo fractional derivative has the following properties:

(1) $^c\mathfrak{D}^\alpha\big(I_t^\alpha f(t)\big) = f(t)$

(2) $I_t^\alpha\big(^c\mathfrak{D}^\alpha f(t)\big) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}$

(3) $^c\mathfrak{D}^\alpha(C) = 0, C \in \mathbb{R}$

(4) $I_t^\alpha \ t^\mathfrak{B} = \frac{\Gamma(\mathfrak{B}+1)}{\Gamma(\alpha+\mathfrak{B}+1)} \ t^{+\alpha}$

(5) $^c\mathfrak{D}^\alpha(t^k) = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, k \in \{1,2,3,\dots\}, k \ge [\alpha] \\ 0, k \in \{1,2,3,\dots\}, k < [\alpha] \end{cases}$

where, $[\alpha]$ is the floor function of α.

Definition 2.3 [25] Let X be any of elements, a fuzzy set $\widetilde{\mathfrak{B}}$ is characterized by a membership function $\mu_{\widetilde{\mathfrak{B}}}(x): X \to [0,1]$, and may be written as the set of points:

$$\widetilde{\mathfrak{B}} = \{(x, \mu_{\widetilde{\mathfrak{B}}}(x)) x \in X, 0 \le \mu_{\widetilde{\mathfrak{B}}}(x) \le 1\}.$$

Definition 2.4 [26]. The crisp set of components that make up the set $\widetilde{\mathfrak{B}}$ at least to the degree α is called the weak α-level set (or weak α-cut), and is defined by $\mathfrak{B}_b = \{x \in X: \mu_{\widetilde{\mathfrak{B}}}(x) \ge \alpha\}$.

While the strong α-level set (or strong α-cut) is defined by: $\mathfrak{B}'_\alpha = \{x \in X: \mu_{\widetilde{\mathfrak{B}}}(x) > \alpha\}$.

Definition 2.5 [25, 27] (Triangular Fuzzy Number)
A fuzzy number with the following function of membership:

$$\mu(x; c, b, a) = \begin{cases} 0, & x < c \\ \frac{x-c}{b-c}, & c \le x \le b \\ \frac{a-x}{a-b}, & b \le x \le a \\ 0, & x > a \end{cases}$$

which is indicated by the fuzzy number of triangular kinds, where a, b and $c \in \mathbb{R}$ with $a > b > c$.

In Figure 1, we can observe that the graph illustrates the symmetry of the membership function $\mu(x)$ takes the triangular shape with the base over the interval $[c, b]$ and vertex at $x = b$.
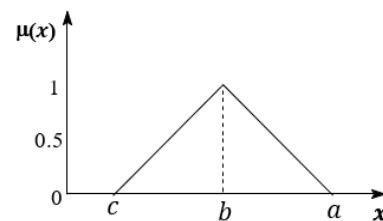

Figure 1. Triangular fuzzy number

The related α-cut is $[\tilde{\mu}]_\alpha = [c + \alpha(b-c), a - \alpha(a-b)], \alpha \in [0,1]$.

Definition 2.6 [28, 29]: Given two fuzzy numbers (two fuzzy sets) $\tilde{a}, \tilde{b} \in \tilde{U}$, then the Hausdorff distance between $\tilde{a}$ and $\tilde{b}$ symbolized by $D_H\big([\tilde{a},\tilde{b}]_\alpha\big)$ is such that $D([\tilde{a},\tilde{b}]) = \sup\{D_H\big([\tilde{a},\tilde{b}]_\alpha\big)|\alpha \in [0,1]\}$, where $(\tilde{U}, D)$ indicates the Cauchy space (complete metric space).

Suppose that $\tilde{U}$ is the set of all semi-continuous upper normal convex fuzzy numbers with bounded α-level sets. The α-cuts of fuzzy numbers are always bounded and closed such that the intervals are $[\tilde{\mu}(x)]_\alpha = \big[\underline{\mu}(x), \overline{\mu}(x)\big]_\alpha$, $x \in \mathbb{R}, \forall \alpha \in [0,1]$.

Let $[\tilde{b}(\alpha)] = [\underline{b}(\alpha), \overline{b}(\alpha)], [\tilde{a}(\alpha)] = [\underline{a}(\alpha), \overline{a}(\alpha)]$ represent two fuzzy numbers [Definition 2.5], for $\gamma \ge 0$. We can define the binary operations on fuzzy numbers $\tilde{b}$ and $\tilde{a}$ of addition and scalar multiplication described as follows:

1. $\underline{(b+a)}(\alpha) = \big(\underline{b}(\alpha) + \underline{a}(\alpha)\big)$
2. $\overline{(b+a)}(\alpha) = \big(\overline{b}(\alpha) + \overline{a}(\alpha)\big)$
3. $\big(\underline{\gamma b}\big)(\alpha) = \gamma.\overline{b}(\alpha)$

Definition 2.7 [29]: Let $f \in C(J; E) \backslash L^1(J; E)$ exhibit characteristics of a fuzzy set-valued function. Function f is considered Caputo's fuzzy differentiable at point x when

$$^{c}\mathfrak{D}^{\alpha}\mathfrak{f}(t) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha}}d\tau, 0 < \alpha \leq m$$

Definition 2.8 [30]: Let $\tilde{f}(x)$ be a fuzzy-valued approximate solution to a fuzzy differential equation, and let $\tilde{R}(\tilde{a}_i, x, \alpha)$ expressed using α-cuts, is defined as:

$$\begin{cases} \underline{E}([\underline{a}_i]^w) = \left(\sum_{i=0}^{n}\underline{a}_i\,\mathcal{P}_{l_{i+1}}(t;w) - I\left(\underline{\mathfrak{f}}(x;w) + \int_0^t k(x,t)\underline{F}\left(\sum_{i=0}^n\underline{a}_i\,\mathcal{P}_{l_{i+1}}(t;w)\right)d\,t\right)\right)^2 \\ \overline{E}([\overline{a}_i]^w) = \left(\sum_{i=0}^{n}\overline{a}_i\,\bar{\mathcal{P}}_{l_{i+1}}(t;w) - I\left(\bar{\mathfrak{f}}(x;w) + \int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^n\overline{a}_i\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)d\,t\right)\right)^2 \end{cases}$$

The recurrence formulae:

$$\tilde{\mathcal{P}}_{l_{i+1}}(t;w) = \frac{2i+1}{i+1}t\tilde{\mathcal{P}}_{l_i}(t;w) - \frac{i}{i+1}\tilde{\mathcal{P}}_{l_{i-1}}(t;w),$$
$$i = 0,1,2,\dots,$$
$$\tilde{\mathcal{P}}_{l_0}(t;w) = 1,$$
$$\tilde{\mathcal{P}}_{l_0}(t;w) = t$$

Introducing $t = 2\mathcal{T} - 1$, the fuzzy Legendre polynomials are defined on the interval (0,1), will be refer to F-SLPs and is denoted by $\tilde{\mathcal{P}}^{*}{}_{l_{i+1}}(\mathcal{T};w)$ are generated using the following recurrence formulae.

$$\tilde{\mathcal{P}}^{*}{}_{l_{i+1}}(\mathcal{T};w) = \frac{2i+1}{i+1}\left(\frac{2i}{l_i}+1\right)\tilde{\mathcal{P}}^{*}{}_{l_i}(\mathcal{T};w)$$
$$- \frac{2i+1}{i+1}\tilde{\mathcal{P}}^{*}{}_{l_{i-1}}(\mathcal{T};w)$$
$$\tilde{\mathcal{P}}^{*}{}_{l_0}(\mathcal{T};w) = 1, \tilde{\mathcal{P}}^{*}{}_{l_1}(\mathcal{T};w) = 2\tilde{\mathcal{T}} - 1$$

4. NEW GENERAL FORM OF FV-FLST

In this section, we will present a new form of standard LST in order to solve FV-FIDEs, consider the following F-FIDE of order $\alpha \in [0,1]$ for $w \in [0,1]$:

$$^{c}\tilde{\mathfrak{D}}^{\alpha}\tilde{y}(x;w) = \tilde{\mathfrak{f}}(x;w)$$
$$+ \int_{0}^{t} k(x,t)F(\tilde{y}(t;w))d\,t \qquad (1)$$
$$0 \leq x, t \leq 1, 0 < \alpha \leq 1$$

Under fuzzy initial conditions:

$$\tilde{y}(0;w) = \tilde{y}_0(w) \qquad (2)$$

where, Eq. (1) can be restated with a broader generalization by applying the concepts of defuzzification, where $\tilde{\mathfrak{f}}(x;w) = [\bar{\mathfrak{f}}, \underline{\mathfrak{f}}]^{w}$ is a fuzzy function hence the approximate solution will be a fuzzy function which can be written as $\tilde{y}(t;w) = [\bar{y}, \underline{y}]^{w}$, where, $^{c}\tilde{\mathfrak{D}}^{\alpha}$ is the fuzzy Caputo fractional derivative of order α is a fuzzy function of the crisp variable x, and $\tilde{y}(x;w)$ is a fuzzy function. Thus, Eq. (1) can be written in terms level sets $w \in [0,1]$ as:

$$^{c}\underline{\mathfrak{D}}^{\alpha}\underline{y}(x;w) = \underline{\mathfrak{f}}(x;w) + \int_{0}^{t} k(x,t)\,\underline{F}\left(\underline{y}(t;w)\right)dt, \qquad (3)$$
$$0 \leq x, t \leq 1, 0 < \alpha \leq 1$$

$$[\tilde{R}(\alpha)] = [[\underline{b}(t,\alpha)] - [f(\alpha)]]_{\alpha} = [\underline{R}(t,\alpha), \overline{R}(t,\alpha)]$$

3. SHIFTED FUZZY LEGENDRE POLYNOMIALS

Based on the information provided [1], the Legendre polynomials of degree i are indicated for each fuzzy level sets $\omega \in [0,1]$ by $\tilde{\mathcal{P}}_{l_i}(t;\omega)$ that defined on the interval (-1,1), by

$$^{c}\overline{\mathfrak{D}}^{\alpha}\overline{y}(x;w) = \bar{\mathfrak{f}}(v;w) + \int_{0}^{t} k(x,t)\overline{F}(\overline{y}(t;w))dt,$$
$$0 \leq x, t \leq 1, 0 < \alpha \leq 1$$
$$\overline{y}(0;w) = \overline{y}_0(w) \text{ and } \underline{y}(0;w) = \underline{y}_0(w)$$

From Section 3 application, the approximate solution of solve $\tilde{y}(x;w)$ in terms of α and ω in the following LST polynomial approximation solution of LST such that

$$\tilde{y}(x;w) = \sum_{i=0}^{n}[\tilde{a}_i]^{\omega}\tilde{\mathcal{P}}_{l_{i+1}}(t;w)$$

then substitute the values of $[\tilde{a}_i]^{\omega} = [\overline{a}_i, \underline{a}_i]^{\omega}$ into Eq. (3) yields:

$$\sum_{i=0}^{n}\underline{a}_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w) = I\left(\underline{f}(x;w) +\right.$$
$$\left.\int_0^t k(x,t)\underline{F}\left(\sum_{i=0}^n\underline{a}_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\right)$$
$$\sum_{i=0}^{n}\overline{a}_i\,\overline{\mathcal{P}}_{l_{i+1}}(t;w) = I\left(\overline{f}(x;w) +\right. \qquad (4)$$
$$\left.\int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^n\overline{a}_i\,\overline{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\right)$$

Next, Eq. (4) can be expanded into the following forms

$$\sum_{i=0}^{n}\underline{a}_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w) - I\left(\underline{f}(x;w) +\right.$$
$$\left.\int_0^t k(x,t)\underline{F}\left(\sum_{i=0}^n\underline{a}_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\right) = 0$$
$$\sum_{i=0}^{n}\overline{a}_i\,\overline{\mathcal{P}}_{l_{i+1}}(t;w) - I\left(\overline{f}(x;w) +\right. \qquad (5)$$
$$\left.\int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^n\overline{a}_i\,\overline{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\right) = 0$$

Then, by minimizing Eq. (5) such that

$$\underline{E}([\underline{a}_i]^w) = \left(\begin{array}{c}\sum_{i=0}^n\underline{a}_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w) - I(\underline{\mathfrak{f}}(x;w) + \\ \int_0^t k(x,t)\underline{F}\left(\sum_{i=0}^n\underline{a}_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\end{array}\right)^2$$
$$\overline{E}([\overline{a}_i]^w) = \left(\begin{array}{c}\sum_{i=0}^n\overline{a}_i\,\overline{\mathcal{P}}_{l_{i+1}}(t;w) - I(\bar{\mathfrak{f}}(x;w) + \\ \int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^n\overline{a}_i\,\overline{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\end{array}\right)^2 \qquad (6)$$

where, $\tilde{E}([\tilde{a}_i]^{w})$ is minimizing requires finding the values of $[\tilde{a}_i]^{w}$, to determining the best approximate solution of Eq. (1). To get the minimum possible value of $\tilde{E}([\tilde{a}_i]^{w})$, then the following partial derivatives with respect to $^{c}\tilde{D}^{\alpha}\tilde{y}(x;w)$ as follows:

$$\frac{\partial \underline{E}([\underline{a}_i]^w)}{\partial[\underline{a}_i]^w} = 0, i = 0, 1, \dots, n$$
$$\frac{\partial \overline{E}([\overline{a}_i]^w)}{\partial[\overline{a}_i]^w} = 0, i = 0, 1, \dots, n \qquad (7)$$

Such that:

$$\frac{\partial E([\underline{a_i}]^w)}{\partial[\underline{a_i}]^w} = 2\left(\sum_{i=0}^{n}\underline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w) - I\left(\bar{f}(x;w) + \int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^{n}\underline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)d\,t\right)\right)$$

$$\frac{\partial \overline{E}([\overline{a_i}]^w)}{\partial[\overline{a_i}]^w} = 2\left(\sum_{i=0}^{n}\overline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w) - I(\bar{f}(x;w) + \int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^{n}\overline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)d\,t)\right)$$

$$(8)$$

$\frac{\partial E([\underline{a_i}]^w)}{\partial[\underline{a_i}]^w}$ and $\frac{\partial \overline{E}([\overline{a_i}]^w)}{\partial[\overline{a_i}]^w}$ for all i, causing $\tilde{E}([\tilde{a_i}]^w)$ to be reduced with each i. By rearranging Eq. (8), we can derive the following normal equations, to find coefficients $[\tilde{a_i}]^w$ such that

$$\frac{\partial E([\underline{a_i}]^w)}{\partial[\underline{a_i}]^w} = \sum_{i=0}^{n}\underline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w) - I\left(\bar{f}(x;w) + \int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^{n}\underline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)dt\right)$$

$$\frac{\partial \overline{E}([\overline{a_i}]^w)}{\partial[\overline{a_i}]^w} = \sum_{i=0}^{n}\overline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w) - I(\bar{f}(x;w) + \int_0^t k(x,t)\overline{F}\left(\sum_{i=0}^{n}\overline{a_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)d\,t)$$

$$(9)$$

Thus, the derivative of the sum represented by Eq. (9) with respect to each of the coefficients $[\tilde{a_i}]^w$. The F-FLST polynomial approximation of Eq. (9) is defined as $\tilde{X}[\tilde{a_i}]^w = f(x;w)$.

5. CONVERGENCE ANALYSIS OF FVLST

While previous studies have addressed the application of the Laplace–Adomian Sumudu Transform (LST) for solving various forms of fuzzy integro-differential equations (FIDEs) [31-33], the convergence discussion remains limited without referencing formal convergence theorems or establishing error bounds. To ensure mathematical rigor, it is crucial to establish a formal rationale for convergence based on existing results. According to references [33, 34], the convergence of Adomian-type decomposition methods and their variants, such as LST depends on the nature of the nonlinear operator and the boundedness of the Adomian polynomials employed in creating the series solution. The key idea is that the solution converges if the successive approximations are uniformly limited and the nonlinear terms meet Lipschitz criteria over the domain of interest. Moreover, the convergence control parameter $^cD^\alpha\tilde{y}(x)$ has a vital role in establishing the region in which the LST solution converges. As the LST gives a family of approximate expressions depending on this parameter, picking an appropriate value of α directly influences the correctness of the solution across all α-cuts. The optimum option of α minimizes the residual error and maximizes faithfulness to the original FFIDE. To generate the residual form for fuzzy fractional integro-differential equations (FFIDEs), substitution of the approximate LST solution from Eq. (1) into the original problem Eq. (6) provides a method for error estimates and iterative refinement. This permits tweaking of the convergence parameter by empirical or analytical methods such as minimizing residual norms or applying fuzzy stability criteria [34].

$$\widetilde{EE}(a_0, a_1, a_2, \ldots, a_n) = {}^cD^\alpha\tilde{y}(x;w) - \bar{f}(x;w) - \int_0^t k(x,t)(\tilde{y}(t;w))dt$$

$$(10)$$

Operating I^α on both sides of Eq. (10) and plugging

property (4) of defined (2), and the residual formula in the following manner:

$$\widetilde{EE}(a_0, a_1, a_2, \ldots, a_n) = {}^cD^\alpha\tilde{y}(x;w) - \bar{f}(x;w) - \int_0^t k(x,t)(\tilde{y}(t;w))d\,t$$

$$(11)$$

Below, for all $\alpha \in [0,1]$, the partial derivatives with respect to (α) can be produced by assigning the values of fractional order $0 < \alpha \leq 1$.

$$\frac{\partial\widetilde{EE}}{\partial a_i} = 0 \qquad (12)$$

For each fuzzy level set $\omega \in [0,1]$, we may find the best value for the convergence control parameter $^cD^\alpha\tilde{y}(x)$ by numerically solving the system of nonlinear equations in Eq. (12) with respect to $^cD^\alpha\tilde{y}(x)$. The goal is to find the best value between the best values of $^cD^\alpha\tilde{y}(x)$ so that the LST series solution $y_0(x)$ is as accurate as possible for any $\alpha \in (0,1]$. To determine the ideal curve or best-fitting line for a given set of data points, one must minimize the sum of the squared differences between the points and the curve. If a curve yields the minimum sum of squared deviations when applied to a specific dataset, it is considered the optimal fit for its category. In a fuzzy environment, determine the contract curves and calculate the optimal value $^cD^\alpha\tilde{y}(x)$ for each $\alpha \in (0,1]$. We must select the optimal value of $^cD^\alpha\tilde{y}(x)$ that yields the most precise solution for FV-FIDEs, along with its corresponding fuzzy level set. Subsequently, we apply α to each level set $\tilde{\omega} = [\underline{w}, \overline{w}]$ to obtain the most accurate lower and upper approximation solution.

The fuzzy coefficients a_i in the series solution function as control points that influence the characteristics of the approximate fuzzy solution throughout all α-cuts. Enhancing these coefficients is crucial for minimizing the fuzzy error at each level (α), thereby enabling the LST-based approximation to converge more closely to the accurate solution. These factors impact how accurate and stable the solution is; just like control points shape the smoothness and shape of a Bézier curve, the coefficients determine how closely the fuzzy solution matches the original. Furthermore, the above analysis in Section 5 can be summarized in the following algorithm:

Step 1: Define the fuzzy FV-FIDEs with fuzzy initial conditions.

Step 2: Given the principles of defuzzification.

Step 3: Follow LST polynomial approximation solution using Eqs. (1)-(3).

Step 4: Minimizing Eq. (3) requires finding the values of $[\tilde{a_i}]^\omega$.

Step 5: Find control points $[\tilde{a_i}]^\omega$.

Step 6: The final control points will be saved once the learning procedure has finished.

6. NUMERICAL IMPLEMENTATION

We will use the methodology from the previous section to test the effectiveness of the FV-FIDEs. Each example showcases the efficacy of the proposed method by contrasting the estimated results with the precise solutions.

Example 6.1 [35]: Given the linear V-FFIDEs

$$\mathfrak{D}^{\alpha}\tilde{y}(x;w) = [w-1, 1-w]$$
$$+ \int_0^x \tilde{y}(t;w)d\,t, \ 0 < w \le 1, 0 \le x, t \le 1 \quad (13)$$

with fuzzy initial condition $\tilde{y}(0,w) = 0$.

$^c\mathfrak{D}^{\alpha}$ represents the α-th Caputo fractional derivative of y(x) and $0 < \alpha \le 1$, $[\omega-1, 1-\omega]$ are fuzzy triangular numbers, $y(x_0)$ represents the fuzzy initial condition, and y (x) represent a crisp function of non-fuzzy independent variable x. The exact solution of Eq. (13) when $\alpha = 1$:

$$[y(x)]^w = [w-1, 1-\omega]\ sinh\,(x) \quad (14)$$

By utilizing the FVLST algorithm as presented previously, the kth order deformation equation of Eq. (11) and operating I^{α} on the both sides of Eq. (13) and plugging property (4) of defined (2).

$$\underline{y}(x;w) = I^{\alpha}((w-1) + \int_0^x \underline{y}(t;w)d\,t),$$
$$0 < \alpha \le 1, 0 \le x, t \le 1$$
$$\overline{y}(x;w) = I^{\alpha}((1-w) + \int_0^x \overline{y}(t;w)d\,t), \quad (15)$$
$$0 < \alpha \le 1, 0 \le x, t \le 1$$

By applying the Shifted Legendre Polynomials from section (3), yields.

$$\sum_{i=0}^2 \underline{a_i}\,\mathcal{P}_{l_{i+1}}(x;w) - I^{\alpha}((w-1) +$$
$$\int_0^x \sum_{i=0}^2 \underline{a_i}\mathcal{P}_{l_{i+1}}(t;w)d\,t) = 0$$
$$\sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w) - I^{\alpha}((1-w) + \quad (16)$$
$$\int_0^x \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)d\,t) = 0$$

Thus, we minimized Eq. (16) as

$$E1(\underline{a_0},\underline{a_1},\underline{a_2})^w = \sum_{i=0}^2 \underline{a_i}\,\mathcal{P}_{l_{i+1}}(x;w) -$$
$$I^{\alpha}((w-1) + \int_0^x \sum_{i=0}^2 \underline{a_i}\mathcal{P}_{l_{i+1}}(t;w)d\,t)$$
$$E2(\overline{b_0},\overline{b_1},\overline{b_2})^w = \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w) - I^{\alpha}((1- \quad (17)$$
$$w) + \int_0^x \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)d\,t)$$

Therefore, minimizing E required for finding the values of a_i, as well as to determining the best estimate for the solution of the fuzzy fractional integro differential Eq. (17). To get the minimum possible value of E, then the following partial derivatives with respect to $^cD^{\alpha}y(x)$ and $0 < \alpha \le 1$ for $i = 0, 1, 2$.

$$\begin{cases} (\frac{\partial E}{\partial \underline{a_i}})^w = 0 \\ (\frac{\partial E}{\partial \overline{b_i}})^w = 0 \end{cases} \quad (18)$$

$$E1(\underline{a_0},\underline{a_1},\underline{a_2})^w =$$
$$\left(\begin{matrix} \sum_{i=0}^2 \underline{a_i}\,\mathcal{P}_{l_{i+1}}(x;w) \\ -I^{\alpha}\left((w-1) + \int_0^x \sum_{i=0}^2 \underline{a_i}\mathcal{P}_{l_{i+1}}(t;w)dt\right) \end{matrix} \right)^2$$
$$E2(\overline{b_0},\overline{b_1},\overline{b_2})^w = \quad (19)$$
$$\left(\begin{matrix} \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w) \\ -I^{\alpha}\left((1-w) + \int_0^x \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)dt\right) \end{matrix} \right)^2$$

Applying Eq. (18) in Eq. (19) we get:

$$\left(\frac{\partial E1}{\partial(\underline{a_0},\underline{a_1},\underline{a_2})}\right)^w = 2\left[\sum_{i=0}^2 \underline{a_i}\,\mathcal{P}_{l_{i+1}}(x;w) - I^{\alpha}\left((w-1) + \int_0^x \sum_{i=0}^2 \underline{a_i}\mathcal{P}_{l_{i+1}}(t;w)dt\right)\right]$$
$$\left(\frac{\partial E2}{\partial(\overline{b_0},\overline{b_1},\overline{b_2})}\right)^w = 2\left[\sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w) - I^{\alpha}((1- \quad (20)$$
$$w) + \int_0^x \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)dt)\right]$$

$\frac{\partial E1}{\partial a_i}, \frac{\partial E2}{\partial b_i}$ for all i=0, 1, 2, causing E to be reduced. with each i=0, 1, 2. By rearranging Eq. (20), one can derive the following normal equations, to find fuzzy coefficients $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2$ and $\tilde{b}_0, \tilde{b}_1, \tilde{b}_2$ using Eq. (11) to verify the accuracy of our proposed method as shown in the tables below.

Table 1 displays the fuzzy approximation solutions derived from the Laplace–Sumudu Transform (LST) method for fractional orders $\alpha = 0.5, 0.7,$ and 0.9, as applied to Eq. (13).

Table 1. Numerical results for Example 1, with different values of α at $w = 0.5, 0.75$

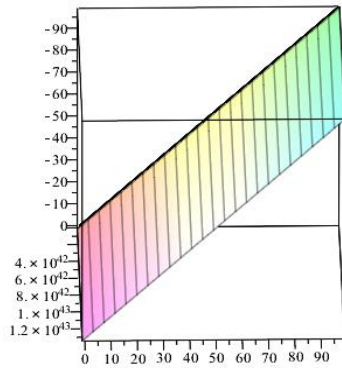| Fuzzy Level Set	Optimal Values	$\underline{y}(x;w)$	$\underline{y}(x;w)$	$\underline{y}(x;w)$
w_i	X_i	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
	0	-500.968815e-3	-502.559529e-3	-503.550231e-3
	0.2	-533.865115e-3	-517.853729e-3	-511.715311e-3
0.5	0.4	-600.995726e-3	-563.848737e-3	-548.020431e-3
	0.6	-702.360646e-3	-640.544552e-3	-612.465590e-3
	0.8	-837.959876e-3	-747.941174e-3	-705.050790e-3
	0	-250.484407e-3	-251.279764e-3	-251.775115e-3
	0.2	-266.932558e-3	-258.926865e-3	-255.857655e-3
0.75	0.4	-300.497863e-3	-281.924368e-3	-274.010215e-3
	0.6	-351.180323e-3	-320.272276e-3	-306.232795e-3
	0.8	-418.979938e-3	-373.970587e-3	-352.525395e-3
Fuzzy Level Set	**Optimal Values**	$\overline{y}(x;w)$	$\overline{y}(x;w)$	$\overline{y}(x;w)$
w_i	X_i	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
	0	500.968815e-3	502.559529e-3	503.550231e-3
	0.2	533.865115e-3	517.853729e-3	511.715311e-3
0.5	0.4	600.995726e-3	563.848737e-3	548.020431e-3
	0.6	702.360646e-3	640.544552e-3	612.465590e-3
	0.8	837.959876e-3	747.941174e-3	705.050790e-3
	0	250.484407e-3	251.279764e-3	251.775115e-3
	0.2	266.932558e-3	258.926865e-3	255.857655e-3
0.75	0.4	300.497863e-3	281.924368e-3	274.010215e-3
	0.6	351.180323e-3	320.272276e-3	306.232795e-3
	0.8	418.979938e-3	373.970587e-3	352.525395e-3

Figure 2. Exact and approximate fuzzy solutions of V-FIDEs of Eq. (13) when $w = 1$

The preliminary estimate employed for this equation is $\tilde{y}_0(x) = [\underline{y}(x), \overline{y}(x)]$, as elaborated in Section 5. The values of α are derived from Eq. (15), and by adjusting I^α for $\alpha \in [0,1]$, the convergence domain of the solution is investigated. To assess the convergence behavior, we quantitatively examined the absolute and relative errors between the approximate and exact fuzzy solutions over the domain. For example, for $x = 1$, the average absolute error across α-levels was determined to be less than 10^{-3}, signifying exceptional precision. The relative error consistently stayed below 1% for all evaluated values of α, with the lowest error recorded at $\alpha = 0.7$, indicating optimal convergence at this parameter. Figure 2 depicts the lower and upper bound solutions $\overline{y}(x)$ and $\underline{y}(x)$ for the eighth-order fuzzy Volterra linear fractional integro-differential equation associated with Eq. (15), displayed at $w = 1$. The proximity and consistency of these

bounds across various α-levels validate the method's robustness and define the effective convergence zone of the LST methodology. The results indicate that the LST method yields dependable and precise approximations for FFIDEs when suitable fractional orders are used. The precision is enhanced with smooth kernel functions and uniform α-level decomposition.

It also shows the valid region of the fuzzy convergence parameters $\tilde{y}_0(x) = [\underline{y}(x), \overline{y}(x)]$. Hence, the order of the fuzzy fractional LST solution can be seen clearly, as can the region of valid convergence control parameter (α) values and the line segment at $\alpha = 1$ for α [0,1].

Furthermore, using a distinct value for ω and various α would thus result in a triangle shape for the graph. When the value of $w = 0, 0.3, 0.6, 0.8$, $\alpha = 0.75$ we would have created a single shape in Figure 3.

Eq. (5) delineates a triangular fuzzy number, which underpins the fuzzy beginning conditions employed in this work. The fuzzy approximate solutions are shown graphically in Figure 3, with special attention to the triangle form that is depicted there. This form directly embodies the theoretical characteristics of the fuzzy solution framework, particularly the application of triangular fuzzy numbers and α-cuts. The triangle arrangement verifies that the fuzziness disseminates in accordance with the established features of fuzzy sets. A comprehensive examination provides insights into the exact and approximate solutions for various values of www and α, emphasizing their respective similarities and distinctions. The graphical findings validate that the suggested method produces fuzzy approximate solutions that nearly correspond to the exact solution. The strong similarity supports the accuracy of the triangular fuzzy representation and shows that the chosen method works well within the fuzzy theoretical framework.
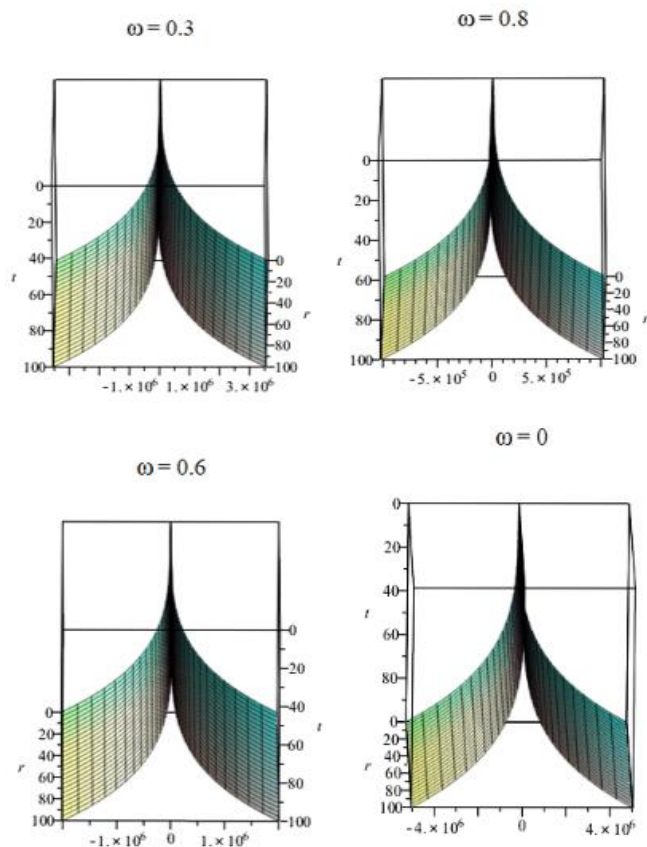

Figure 3. Fuzzy approximate solution where $w = 0, 0.3, 0.6, 0.8$

Example (6.2): [36] Given the non-linear V-FIDEs

$$\mathfrak{D}^{\alpha}\tilde{y}(x) = \frac{25\,x^{6/5}}{3\Gamma\left(1/5\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(1/5\right)}$$
$$+ \int_0^t (t-s)[\tilde{y}(t)]^2 d\,s,\ 0 < \alpha \tag{21}$$
$$\leq 1, x \in [0,1]$$

From the Definition 2.6 of the fuzzy number, we can defuzzify the initial condition let $[\tilde{0}]_\alpha$ and by Definition 2.5 triangular fuzzy number such that $\forall \alpha \in [0,1]$, we have non-linear V-FFIDEs:

$$\mathfrak{D}^{\alpha}\tilde{y}(x;\omega) = \frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)}$$
$$-[w - 1, 1 - w] + \int_0^t (t-s)[\tilde{y}(t;\omega)]^2 ds, \tag{22}$$
$$0 < \alpha \leq 1, x \in [0,1]$$

with fuzzy initial condition: $(\tilde{y}(0))^w = [w - 1, 1 - w]$.
The exact solution of Eq. (21) when $\alpha = 1$:

$$[y(x)]^\omega = [w - 1, 1 - w]\,(x^2 - x)$$

From Section 5, the kth order deformation equation of Eq. (11) and operating I^α on the both sides of Eq. (22) and plugging property Eq. (4) of defined (2).

$$\overline{y}\,(x;w) =$$
$$I^\alpha \begin{pmatrix} \frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} - \\ (w - 1) + \int_0^t (t-s)[\overline{y}\,(t;\omega)]^2 d\,s \end{pmatrix}$$
$$\underline{y}\,(x;w) = \tag{23}$$
$$I^\alpha \begin{pmatrix} \frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} - \\ (1 - w) + \int_0^t (t-s)[\underline{y}\,(t;\omega)]^2 d\,s \end{pmatrix}$$

By applied Shifted Legendre Polynomials in Section 3 yields.

Hence, in order to minimize $E1, E2$, it is necessary to determine the values of a_i, b_i that will yield the optimal estimate for the solution of the fuzzy fractional integro differential Eq. (25). In order to obtain the least value of $E1, E2$, we need to calculate the partial derivatives of $^c\mathfrak{D}^\alpha y(x)$ with respect to α, where α is a value between 0 and 1.

$$\sum_{i=0}^2 a_i \bar{\mathcal{P}}_{l_{i+1}}(x;w) - I^\alpha \left(\frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12}\right.$$
$$+ \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} - (w - 1) + \int_0^t (t-s)[\overline{y}\,(t;\omega)]^2 d\,s \bigg) = 0$$
$$\sum_{i=0}^2 b_i \underline{\mathcal{P}}_{l_{i+1}}(x;w) - I^\alpha \left(\frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12}\right. \tag{24}$$
$$+ \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} - (1 - w) + \int_0^t (t-s)\left[\underline{y}\,(t;\omega)\right]^2 d\,s \bigg) = 0$$

Thus, we minimized Eq. (24) as

$$E1(\underline{a_0}, \underline{a_1}, \underline{a_2})^\omega = \sum_{i=0}^2 \underline{a_i}\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)$$
$$-I^\alpha \left(\begin{array}{c} \frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} - (w - 1) \\ + \int_0^1 (t-s)\left(\sum_{i=0}^2 a_i\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)^2 d\,t \end{array} \right)$$
$$\tag{25}$$
$$E2(\overline{b_0}, \overline{b_1}, \overline{b_2})^\omega = \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w)$$
$$-I^\alpha \left(\begin{array}{c} \frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} - (1 - w) \\ + \int_0^1 (t-s)\left(\sum_{i=0}^2 b_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)\right)^2 d\,t \end{array} \right)$$

$$\begin{cases} \dfrac{\partial E}{\partial a_i} = 0, i = 0, 1, 2 \\ \dfrac{\partial E}{\partial b_i} = 0, i = 0, 1, 2 \end{cases} \tag{26}$$

$$\frac{\partial E}{\partial \underline{a_i}} = \sum_{i=0}^2 \underline{a_i}\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)I^\alpha$$
$$\frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)}$$
$$- \left(+ \int_0^1 (t-s)\left(\sum_{i=0}^2 \underline{a_i}\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)\right)^2 d\,t \right)$$
$$\tag{27}$$
$$\frac{\partial E}{\partial \overline{b_i}} == \sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w)$$
$$\frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)}$$
$$-I^\alpha \left(+ \int_0^1 (t-s)\left(\sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w)\right)^2 d\,t \right)$$

Applying Eq. (26) in Eq. (27) we get $\frac{\partial E1}{\partial a_i}, \frac{\partial E2}{\partial b_i}$ for all $i = 0,1,2$ causing E to be reduced.

By manipulating Eq. (28), we may deduce the subsequent normal equations, which allow us to determine the coefficients $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2$ and $\tilde{b}_0, \tilde{b}_1, \tilde{b}_2$. To assess the precision of our proposed approach, we will employ Eq. (22) and present the results in the figures and tables provided below. Therefore, we can identify the convergence region for I^α by plotting the curves of the eighth order fuzzy Volterra linear fractional integro-differential equation LST lower bound solution $\underline{y}(x)$ and upper bound solution $\overline{y}(x)$ for Eq. (22) at $w = 1$ in Figure 4.

$$\left(\frac{\partial E1}{\partial(a_0, a_1, a_2)}\right)^\omega =$$
$$2\left[\sum_{i=0}^2 \underline{a_i}\,\underline{\mathcal{P}}_{l_{i+1}}(t;w) - I^\alpha \left(\frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} + \right.\right.$$
$$\left.\left. \int_0^1 (t-s)\left(\sum_{i=0}^2 a_i\,\bar{\mathcal{P}}_{l_{i+1}}(t;w)\right)^2 d\,t \right)\right]$$
$$\left(\frac{\partial E2}{\partial(b_0, b_1, a_2)}\right)^\omega = \tag{28}$$
$$2\left[\sum_{i=0}^2 \overline{b_i}\,\bar{\mathcal{P}}_{l_{i+1}}(x;w) - I^\alpha \left(\frac{25\,x^{\frac{6}{5}}}{3\Gamma\left(\frac{1}{5}\right)} + \frac{x^6}{30} - \frac{x^5}{10} + \frac{x^4}{12} + \frac{5\sqrt[5]{x}}{3\Gamma\left(\frac{1}{5}\right)} + \right.\right.$$
$$\left.\left. \int_0^1 (t-s)\left(\sum_{i=0}^2 b_i\,\underline{\mathcal{P}}_{l_{i+1}}(t;w)\right)^2 d\,t \right)\right]$$

As stated in Section 5, the chosen starting approximation for Eq. (22) is $\tilde{y}_0(x) = [\underline{y}(x), \overline{y}(x)]$. The values of (α) can be determined from Eq. (24) in Section 5 by testing all the values of I^α at $\alpha = 0.5$ for $\alpha \in [0, 1]$. This allows us to identify the convergence region for $I^{0.5}$. Figure 4 shows the curves of the lower bound solution $\underline{y}(x)$ and the upper bound solution $\overline{y}(x)$ for Eq. (23) of the fuzzy Volterra linear fractional integro-differential equation LST with eight roots.

Figure 4 illustrates the graphical representation of the fuzzy solution bounds for Eq. (23), which is a fuzzy linear fractional Volterra integro-differential equation solved using the (LST). The two curves represent the lower bound solution $\overline{y}(x)$ the upper bound solution $\underline{y}(x)$. These bounds encapsulate the fuzzy solution interval and are obtained under the condition of using eight distinct roots within the LST-based approach.

Table 2 shows the fuzzy approximate solutions by using LST for fractional orders of $\alpha = 0.5, 0.7, 0.9$ for Eq. (22).
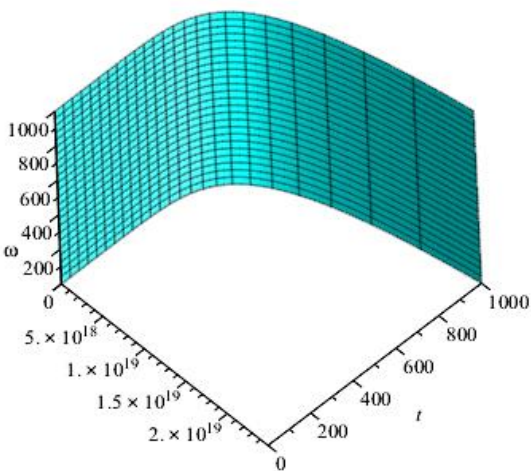

Figure 4. Exact and first order solution of V-FFIDEs of Eq. (22) when $w = 1$ and , $t \in [0,1]$

Table 2. Numerical results for Example 2, with different values of α at $w = 0.5, 0.75$

| Fuzzy Level Set	Optimal Values	$\underline{y}(x; w)$	$\underline{y}(x; w)$	$\underline{y}(x; w)$
w_i	X_i	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
	0	-587.454e-3	-504.332e-3	-481.161e-3
	0.2	-752.866e-3	-674.418e-3	-636.916e-3
0.5	0.4	-790.050e-3	-747.114e-3	-715.350e-3
	0.6	-699.006e-3	-722.421e-3	-716.462e-3
	0.8	-479.734e-3	-600.337e-3	-640.253e-3
	0	-339.598e-3	-256.879e-3	-233.569e-3
	0.2	-503.510e-3	-424.063e-3	-386.395e-3
0.75	0.4	-548.282e-3	-500.812e-3	-467.614e-3
	0.6	-473.916e-3	-487.127e-3	-477.225e-3
	0.8	-280.410e-3	-383.006e-3	-415.228e-3
Fuzzy Level Set	**Optimal Values**	$\overline{y}(x; w)$	$\overline{y}(x; w)$	$\overline{y}(x; w)$
w_i	X_i	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.9$
	0	408.751e-3	491.556e-3	515.407e-3
	0.2	246.893e-3	326.820e-3	364.247e-3
0.5	0.4	198.874e-3	250.255e-3	283.884e-3
	0.6	264.692e-3	261.861e-3	274.319e-3
	0.8	444.350e-3	361.637e-3	335.550e-3
	0	158.341e-3	240.989e-3	264.671e-3
	0.2	-3.582e-3	76.578e-3	114.199e-3
0.75	0.4	-53.772e-3	-2.100e-3	32.021e-3
	0.6	7.770e-3	4.956e-3	18.135e-3
	0.8	181.043e-3	97.746e-3	72.541e-3
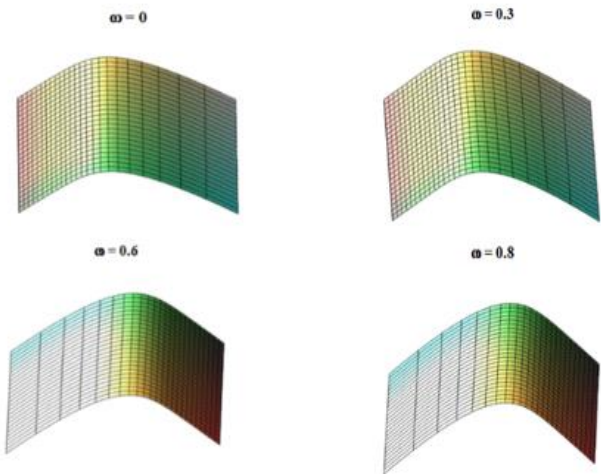
Figure 5. Lower and upper approximate solution were w=0, 0.3, 0.6, 0.8

According to Section 5, the selected initial guess of Eq. (23) is $\tilde{y}_0(x) = [\underline{y}(x), \overline{y}(x)]$. From Section 5, the values of (α) can be obtained from Eq. (24) and by testing all the values of I^α for $\alpha \in [0,1]$.

Furthermore, if we vary the value of w, the resulting drawing will have a triangular shape, as depicted in Figure 5.

Figure 5 illustrates the fuzzy approximate solutions represented as Triangular Fuzzy Numbers, as defined in Definition 5. The results are shown for different values of the parameters w and α. The comparison clearly indicates that the proposed method provides approximations that closely match the exact solution. Moreover, the similarity between the approximate solutions for different parameter values confirms the stability and accuracy of the suggested approach.

7. CONCLUSIONS

This paper presented an improved method for estimating

analytical solutions to fuzzy fractional integro-differential equations (FFIDEs) via the least squares technique (LST). The main findings show that the least squares technique provides very accurate approximate solutions for both linear and nonlinear first-order fuzzy fractional integro-differential equations across different fractional orders, while still keeping the fuzzy uncertainty intact using α-cuts. The convergence of the series solution was theoretically validated by Caputo's first-order derivative. Notwithstanding its encouraging efficacy, the approach possesses specific constraints. The present analysis concentrates exclusively on first-order FFIDEs and presumes triangular fuzzy numbers. More work is needed to see if the method can be used for higher-order systems and different types of fuzzy numbers, like trapezoidal or interval-valued functions. Future research might look into extending LST to work with multi-dimensional or linked FFIDEs, and combining it with other methods to make calculations faster and more efficient. Furthermore, comparison analyses with alternative contemporary semi-analytical approaches may assist in further substantiating the advantages and disadvantages of the suggested methodology. The LST is an efficient, rapid-converging, and dependable method for addressing FFIDEs, which are important for future investigations in applied mathematics, engineering, and control theory where fuzzy uncertainty and fractional dynamics intersect.

ACKNOWLEDGMENT

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