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II-Bitopology, II-Induced Topology and II-Separation Axioms on Locally Finite Graphs

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ABSTRACT

The research aims to achieve three goals, the first is to introduce a bi-topological space on locally finite graph by using the recent two well-known topologies which are independent topology and incidence topology associate from the same graph. The second goal is to propose a new induced topology from the two well-known topologies mentioned above which are associated with any locally finite graph, we named it II-induced topology, then some properties of this new bi-topological space and II-induced topology are investigated. Third goal is the II-separation axioms of II-induced topology and some of its properties are defined and studied with some examples. Our motivation is to give a foundation step for studying various properties and features of some different and certain graphs with their corresponding bi-topological space and II-induced topology. Some results are obtained, such as II-induced topology must be a locally finite graph and not contains isolated vertex, also observations about the specification and type of II-induced topology with certain types of graphs, which achieves the discrete topology and which does not, in addition to the fact that II-induced topology fulfills Alexandroff topology.

1. INTRODUCTION

Numerous applications have utilized the relationship between topology and graph to generate many new types of topologies generated by graph, because of the importance of topological graph theory as it is part of graph theory that has a great role and illustrious history in mathematics. The sources [1-9] contain many and various basics on graph theory and topological graph theory with their applications.

On the basis of vertices or edges, some topological models are developed or based, in the undirected graphs and directed graphs. For any locally finite without isolated vertex, Amiri et al. [10] started using a graphic topology. In 2018, Kilicman and Abdulkalek [11] defined a sub-basis on a set of vertices of a topology which they named it an Incidence Topology for any simple graph not contain isolated vertex. In 2020, a new definition also of the concept "Sub-basis family" formed independent topology on any un-digraph (could comprise one or more isolated vertex) via using the idea of non- adjacent vertices of any certain vertex, it is introduced by Hassan and Abed [12]. In 2022, Hassan and Zainy [13] presented the new type of topology which is named independent compatible edges on the set of edges. Again in 2022, Ali and Hassan [14] created an independent incompatible edges topology based on digraphs with some applications. In 2023, non-incidence topology was founded by Hassan and Jafar [15], the references [16-18] for some topologies with graphs can be found.

The motivating insight behind bitopology in graph theory is that various geometric problems depend not on exact shape (graph) of the objects involved, but rather on the two ways they are put together.

So, we will collect the incidence topology proposed by Kilicman and Abdulkalek [11] with the independent topology proposed by Hassan and Abed [12] to establish a new type of bitopology, which is useful in some biomathematical applications in future work.

So, we have two objectives for this work: The first, introducing a definition of new type of bi-topological space founded from two well-known topologies on graphs (locally finite, i.e. A graph for which every vertex has finite degree). The second, proposing a new induced topology from the two well-known topologies (named as II-induced topology) on graphs. Our motivation is to give a foundation step for studying various properties and features of some different and certain graphs with their corresponding bi-topological space and II-induced topology.

The research gap is our need for a topology that is stronger than other topologies on graphs, our work addresses this gap taking the intersection of two topologies on graph, that will generate a stronger version of topology. Biomathematical field and decision making is the potential or proposed main application field.

In Section 2 of this article, we review some preliminaries and foundations of our topic including the definitions of two well-known topologies on graph (locally finite). Section 3 is dedicated to main results of our new bi-topology on graphs (locally finite), and our new definition (II-induced topology. Section 4 is devoted to some foundation outcomes of

Alexandroff topology. The last Section is conclusions of our bi-topology and our induced topology are presented.

2. PRELIMINARIES

Basic definitions and introductions to bitopology and graph theory are covered in this part. These ideas are all often utilized and can be found in references [1, 2, 19].

Typically, a graph consists of two sets, $\mathfrak{S} = (\Lambda, \mathfrak{L})$, where Λ represents the set of vertices and $\mathfrak{L}$ represents the set of edges, an edge of the form $\vartheta = (\varpi, \varpi)$ is loop. Parallel edges are those with the identical end vertices. If a graph contains no parallel edges or loop, it is considered simple. If the vertices ϖ and ρ are linked via the same edge then they are said to be adjacent vertices. All of these ideas are well-known and are available in books mentioned above.

We use the symbols K_n for the complete graph with ~~n~~ vertices and C_n is cycle graph on n vertices and P_m is a path on m edges and $K_{n1,n2}$ is a whole bipartite graph of size partite $n1$ and $n2$.

The family of open subsets of the any non-empty set $\mathcal{X}$ is named a topology when the following conditions are hold: $\mathcal{X}, \emptyset \in \mathcal{T}$, for every $\mathrm{H}, \mathrm{G} \in \mathcal{T}, \mathrm{H} \cap \mathrm{G} \in \mathcal{T}$ and $\cup_{\mathrm{i} \in \Delta} \mathrm{G}_\mathrm{i} \in \mathcal{T}$ for all sub-combination G_i of $\mathcal{T}$, then $(\mathcal{X}, \mathcal{T})$ is named a topological space, an open set is a sub-set of $\mathcal{X}$ which is satisfied the conditions. Indiscrete topology is defined as $\mathcal{T} = \{\emptyset, \mathcal{X}\}$ on $\mathcal{X}$ while discrete topology is def. $\mathcal{T} = \mathrm{P}(\mathcal{X})$ on $\mathcal{X}$.

A bi-topological space is any non-empty set $\mathcal{X}$ combined with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\mathcal{X}$, written as $(\mathcal{X}, \mathcal{T}_1, \mathcal{T}_2)$. The intersection of two topologies is induced a new topology.

In this work we are depended on two well-known topological spaces associated with un-directed graphs $\mathfrak{S} = (\Lambda, \mathfrak{L})$, the Independent and Incidence, which are topologies defined as follows:

The graph $\mathfrak{S}$ may contain one or more isolated vertex, let $\mathrm{S}_{\mathrm{I}d} = \{\mathrm{I}_\varpi : \varpi \in \Lambda\}$ s. t. I_ϖ is represent the set of all not adjacent vertices with a vertex ϖ. we have $\Lambda = \cup_{\varpi \in \Lambda} \mathrm{I}_\varpi$. Hence $\mathrm{S}_{\mathrm{I}d}$ represents a sub-basis for the topology $\mathcal{T}_{\mathrm{I}d}$ on Λ, so $\mathcal{T}_{\mathrm{I}d}$ named independent topology.

Now, let I_ϑ be the set of incidence vertices of the edge ϑ. And $S_{\mathrm{I}c}$ is defined as follows:

$S_{\mathrm{I}c} = \{I_\vartheta | \vartheta \in \mathfrak{L}\}$, where $\mathfrak{S}$ is a locally finite graph, we have $\Lambda = \cup_{\vartheta \in \mathfrak{L}} I_\vartheta$. Hence $S_{\mathrm{I}c}$ represents a sub-basis of a topology $\mathcal{T}_{\mathrm{I}c}$ on $\mathcal{V}$, so $\mathcal{T}_{\mathrm{I}c}$ is named Incidence topology.

3. A NEW BI-TOPOLOGY ON LOCALLY FINITE GRAPH

The two well-known topologies associated with undirected graphs (proposed recently), which are independent topology [12] and incidence topology [11], as we defined in previous part in this paper.

The reader can notice that the independent topological space defined on un-directed graphs which contain one, more or not contain isolated vertices, and sub-basis for a topology $\mathcal{T}_{\mathrm{I}d}$ is defined as a family of sets to each vertex which non-adjacent to that vertex. Whereas, the Incidence topology defined on a graph (locally finite, i.e., a graph for which every vertex has finite degree), and sub-basis for a topology $\mathcal{T}_{Ie}$ is defined as a family of sets to each incident vertices with that edge.

So that, any graph $\mathfrak{S} = (\Lambda, \mathfrak{L})$ throughout this paper is a locally finite graph.

After that we will explain that if intersect the two topologies will induce a new topology. The intersection interprets the common sets from two topology, that is mean the common graphic qualities.

Remark 3.1

(1) Since the topological space can represent a bi-topological space. Then if $(\Lambda, \mathcal{T}_{\mathrm{I}d})$ is an independent topological space, therefore $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}d})$ represents a bi-topological space.

(2) The topologies $\mathcal{T}_{\mathrm{I}d}$ (independent topology) and $\mathcal{T}_{\mathrm{I}c}$ (Incident topology) on Λ gives the bi-topological space $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$.

Remark 3.2

Since the bi-topological space can induce a topological space in many different ways. E.g., the intersection of two topologies will induce a new topology.

Now, we are going to associate the set Λ with two well-known topologies $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$ to establish a new bi-topological space, which is written as $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$.

Example 3.3

$\mathfrak{S} = (\Lambda, \mathfrak{L})$ is a locally finite graph as shown in Figure 1 below, s.t. $\Lambda = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$, $\mathfrak{L} = \{\vartheta_1, \vartheta_2\}$. Then:

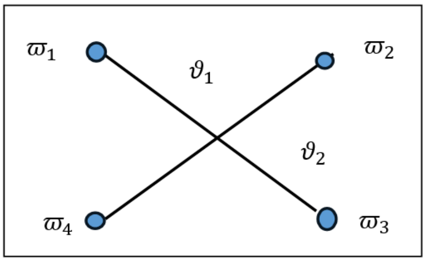

Figure 1. Locally finite graph

The set of all not adjacent vertices with a vertex ϖ_1 and ϖ_3 is $\{\varpi_2, \varpi_4\}$, and the set of all not adjacent vertices with a vertex ϖ_2 and ϖ_4 is $\{\varpi_1, \varpi_3\}$, then the sub-base of independent topology is $\mathrm{S}_{\mathrm{I}d} = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_4\}\}$, and by taking finite intersection, the base $\beta_{\mathrm{I}d}$ is produced $\{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_4\}, \emptyset\}$. Then, utilizing each union generate the independent topology $\mathcal{T}_{\mathrm{I}d}$ as $\mathcal{T}_{\mathrm{I}d} = \{\emptyset, \Lambda, \{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_4\}\}$. $(\Lambda, \mathcal{T}_{\mathrm{I}d})$ represents a topological space, then$(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}d})$ represents a bi-topological space.

The set of incidence vertices of the edge ϑ_1 is $\{\varpi_1, \varpi_3\}$, and the set of incidence vertices of the edge ϑ_2 is $\{\varpi_2, \varpi_4\}$, then the sub-base of incidence topology is $\mathrm{S}_{\mathrm{Ic}} = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_4\}\}$, and by taking finite intersection, the base β_{Ic} is produced $\{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_4\}, \emptyset\}$. Then, utilizing each union generate the incidence topology $\mathcal{T}_{\mathrm{I}c}$ as $\mathcal{T}_{\mathrm{I}c} = \{\emptyset, \Lambda, \{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_4\}\}$. $(\Lambda, \mathcal{T}_{\mathrm{I}c})$ represents a topological space, then $(\Lambda, \mathcal{T}_{\mathrm{I}c}, \mathcal{T}_{\mathrm{I}c})$ represents a bi-topological space. Clearly, $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ represents a bi-topological space.

4. II-INDUCED TOPOLOGICAL SPACE

In this part of our paper, we will define and study a new bi-topological space, that is by using the two well-known topological spaces mentioned above as follows:

Definition 4.1

$(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ is a bi-topological space, then $(\Lambda, \mathcal{T}_{\mathrm{II}})$ represents a topological space, where $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$ is called the II-induced topological space from the bi-topological space

$(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ on locally finite graphs.

Definition 4.2

$(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ is a bi-topological space, and the set $\gamma \subseteq \Lambda$. γ is named II-bi-open set iff γ is open in II-induced topology ($\gamma \in \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$).

Definition 4.3

$(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ is a bi-topological space. and the set $\mathbb{M} \subseteq \Lambda$ is named II-bi-closed set iff M is closed in II-induced topology (the complement of $M \in \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$).

Example 4.4

$\mathfrak{S} = (\Lambda, \mathfrak{L})$ is a locally finite graph as in Figure 2, such that: $\Lambda = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$, $\mathfrak{L} = \{\vartheta_1, \vartheta_2, \vartheta_3\}$. Then,

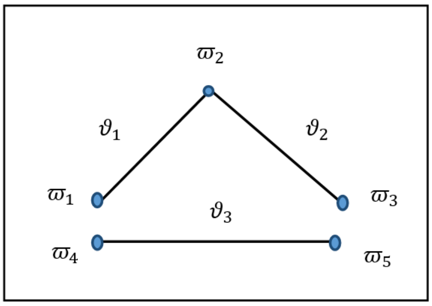

Figure 2. Locally finite graph

Independent topology is:

$\mathcal{T}_{\mathrm{I}d} = \{\emptyset, \Lambda, \{\varpi_1\}, \{\varpi_3\}, \{\varpi_3, \varpi_4, \varpi_5\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}\}$.

And incident topology is:

$\mathcal{T}_{\mathrm{I}c} = \{\emptyset, \Lambda, \{\varpi_2\}, \{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_4, \varpi_5\}, \{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}, \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}\}$.

Since $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$, then II-induced topology: $\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}\}$.

Then, $\emptyset, \Lambda, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}$ are II-bi-open sets, and $\Lambda, \emptyset, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_4, \varpi_5\}$ are II-bi-closed sets (resp.).

Therefore $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$ on Λ give us a new bi-topological space $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$.

Clear that $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$ are non-similar, and they induce $\mathcal{T}_{\mathrm{II}}$ topology on vertices set Λ of a graph $\mathfrak{S} = (\Lambda, \mathfrak{L})$.

Remark 4.5

Independent topology $\mathcal{T}_{\mathrm{I}d}$ of $\mathfrak{S} = (\Lambda, \mathfrak{L})$ is discrete iff $I_\varpi \nsubseteq I_\psi \wedge I_\psi \nsubseteq I_\varpi$ for all different vertices $\varpi, \psi \in \Lambda$ [12], and the incidence topology $\mathcal{T}_{\mathrm{I}c}$ of a graph $\mathfrak{S}$ will be discrete if $d(\varpi) > 2$ for all $\varpi \in \Lambda$ [6].

Thus, if a graph $\mathfrak{S}$ satisfies the above terms, then it is surely that $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$ are identical (both are discrete).

Remark 4.6

Clear that II-induced topological space $(\Lambda, \mathcal{T}_{\mathrm{II}})$ to a cycle graph C_n, $n \geq 4$ is discrete.

That is because the incidence topology $\mathcal{T}_{\mathrm{I}c}$ is discrete when $n \geq 3$ [11] and the independent topology $\mathcal{T}_{\mathrm{I}d}$ is discrete when $n \geq 4$ [12].

Proof: Since the II-induced topology $\mathcal{T}_{\mathrm{II}}$ arises from the intersection of the topologies Independent $\mathcal{T}_{\mathrm{I}d}$ and incidence $\mathcal{T}_{\mathrm{I}c}$, then all the open sets will exist in the intersection (in the II-induced topological space $(\Lambda, \mathcal{T}_{\mathrm{II}})$, thus the proof of above remark will be verified directly.

Remark 4.7

The II-induced topology $\mathcal{T}_{\mathrm{II}}$ of P_m path on m edges is not discrete topology.

That is because, Incidence topology $\mathcal{T}_{\mathrm{I}c}$ on P_m is not discrete since P_m consists of two vertices incident with one edge which is not open [11].

And independent topology $\mathcal{T}_{\mathrm{I}d}$ on P_m doesn't discrete since the set consists of only two vertices of degree 1 is open [12].

As we will show in the following example.

Example 4.8

$\mathfrak{S} = (\Lambda, \mathfrak{L})$ is a P_3 (path on 3 edges) as in Figure 3, such that $\Lambda = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$, $\mathfrak{L} = \{\vartheta_1, \vartheta_2, \vartheta_3\}$. Then,

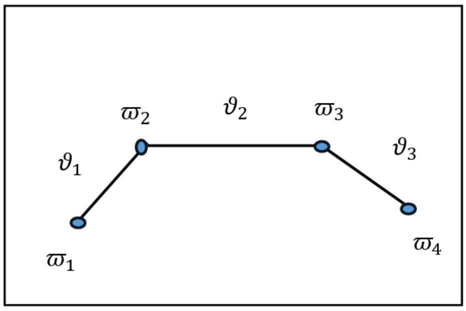

Figure 3. P_3 (path on 3 edges)

The set of all not adjacent vertices with a vertex ϖ_1 is $\{\varpi_3, \varpi_4\}$, and the set of all not adjacent vertices with a vertex ϖ_2 is $\{\varpi_4\}$ and the set of all not adjacent vertices with a vertex ϖ_3 is $\{\varpi_1\}$ and and the set of all not adjacent vertices with a vertex $\{\varpi_4\}$ is $\{\varpi_1, \varpi_2\}$, then the sub-base of Independent Topology is $S_{\mathrm{I}d} = \{\{\varpi_1\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_3, \varpi_4\}\}$, and by taking finite intersection, the base β_{Id} is produced $\{\{\varpi_1\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_3, \varpi_4\}, \emptyset\}$. Then, utilizing each union generate The independent topology $\mathcal{T}_{\mathrm{I}d}$ as the followin $\mathcal{T}_{\mathrm{I}d} = \{\emptyset, \Lambda, \{\varpi_1\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_4\}, \{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_4\}, \{\varpi_1, \varpi_3, \varpi_4\}$, clearly $(\Lambda, \mathcal{T}_{\mathrm{I}d})$ represents a topological space.

Now, the set of incidence vertices of the edge ϑ_1 is $\{\varpi_1, \varpi_2\}$, and the set of incidence vertices of the edge ϑ_2 is $\{\varpi_2, \varpi_3\}$, and the set of incidence vertices of the edge ϑ_3 is $\{\varpi_3, \varpi_4\}$.

Then the sub-base of incidence topology is $S_{\mathrm{I}c} = \{\{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_3, \varpi_4\}\}$, and by taking finite intersection, the base β_{Ic} is produced $\{\{\varpi_2\}, \{\varpi_3\}\{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_3, \varpi_4\}, \emptyset\}$.

Then, utilizing each union generate the incidence topology $\mathcal{T}_{\mathrm{I}c}$ as $\mathcal{T}_{\mathrm{I}c} = \{\emptyset, \Lambda, \{\varpi_2\}, \{\varpi_3\}, \{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_2, , \varpi_3, \varpi_4\}\}$, clearly $(\Lambda, \mathcal{T}_{\mathrm{I}c})$ represents a topological space.

Then, $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$ and the II-induced topology $\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, \{\varpi_1, \varpi_2\}, \{\varpi_3, \varpi_4\}\}$.

It is clear that the II -induced topology $\mathcal{T}_{\mathrm{II}}$ of P_m (path on ~~m~~ edges) is not discrete topology.

Remark 4.9

The II-induced topology $\mathcal{T}_{\mathrm{II}}$ of a complete bipartite graph $K_{n1,n2}$ is discrete topology. That is because, independent topology $\mathcal{T}_{\mathrm{I}d}$ and the incidence topology $\mathcal{T}_{\mathrm{I}c}$ are discrete at the complete bipartite graph, so that the intersection between them will produce a discrete topology [11, 12].

Proof: Since the II-induced topology $\mathcal{T}_{\mathrm{II}}$ arises from the intersection of the topologies independent $\mathcal{T}_{\mathrm{I}d}$ and incidence $\mathcal{T}_{\mathrm{I}c}$, then the proof of above remark will be verified directly.

Remark 4.10

The II-induced topology $\mathcal{T}_{\mathrm{II}}$ of uncomplete bipartite graph $K_{n1,n2}$ is not discrete topology. That is because, the independent topology $\mathcal{T}_{\mathrm{I}d}$ and the Incidence topology $\mathcal{T}_{\mathrm{I}c}$ are not discrete topology at the un-complete bipartite graph, so that the intersection between them will not produce a discrete topology

[11, 12].

Proof: Since the II-induced topology $\mathcal{T}_{\mathrm{II}}$ arises from the intersection of the topologies Independent $\mathcal{T}_{\mathrm{I}d}$ and Incidence $\mathcal{T}_{\mathrm{I}c}$, then the proof of above remark will be verified directly.

Remark 4.11

The II-induced topology $\mathcal{T}_{\mathrm{II}}$ of a tree graph (locally finite, i.e., a graph for which every vertex has finite degree) is not discrete topology.

That is because, the Independent $\mathcal{T}_{\mathrm{I}d}$ and incidence $\mathcal{T}_{\mathrm{I}c}$ are not discrete topology at a graph of tree [11, 12].

As we will show in the following example:

Example 4.12

$\mathfrak{S} = (\Lambda, \mathfrak{L})$ is a tree graph as in Figure 4 below, such that: $\Lambda = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}$, $\mathfrak{L} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\}$. Then,

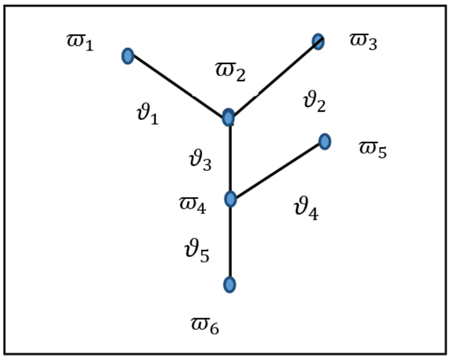

Figure 4. Tree graph

The sets of all not adjacent vertices with a vertex $\Lambda = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}$ forms the sub-base, then the sub-base of Independent Topology is

$$S_{Id} = \{\{\varpi_3, \varpi_4, \varpi_5, \varpi_6\}, \{\varpi_5, \varpi_6\}, \{\varpi_1, \varpi_4, v_5, \varpi_6\}, \{\varpi_1, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}\}.$$

By taking the finitely intersection the base β_{Id} is obtained as follows:

$$\{\emptyset, \{\varpi_3, \varpi_4, \varpi_5, \varpi_6\}, \{\varpi_5, \varpi_6\}, \{\varpi_1, \varpi_4, v_5, \varpi_6\}, , \{\varpi_1\}, \{\varpi_1, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}, \{\varpi_3\}, \{\varpi_6\}, \{\varpi_5\}, \{\varpi_1, \varpi_5\}, \{\varpi_4, \varpi_5, \varpi_6\}, \{\varpi_3, \varpi_6\}, \{\varpi_3, \varpi_5\}, \{\varpi_1, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3\}\}.$$

Utilizing each union generate the independent topology $\mathcal{T}_{\mathrm{Id}}$ as the following:

$$\mathcal{T}_{\mathrm{Id}} = \{\emptyset, \{\varpi_3, \varpi_4, \varpi_5, \varpi_6\}, \{\varpi_5, \varpi_6\}, \{\varpi_1, \varpi_4, v_5, \varpi_6\}, \{\varpi_1\}, \{\varpi_1, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}, \{\varpi_3\}, \{\varpi_6\}, \{\varpi_5\}, \{\varpi_1, \varpi_5\}, \{\varpi_4, \varpi_5, \varpi_6\}, \{\varpi_3, \varpi_6\}, \{\varpi_3, \varpi_5\}, \{\varpi_1, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_5, \varpi_6\}, \{\varpi_1, v_5, \varpi_6\}\},$$

clearly$(\Lambda, \mathcal{T}_{\mathrm{Id}})$ represents a topological space.

Now, the sets of incidence vertices of the all edges $\mathfrak{L} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\}$ forms the sub-base, then the sub-base of incidence topology is $\mathrm{S}_{\mathrm{Ic}} = \{\{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_4\}, \{\varpi_4, \varpi_5\}, \{\varpi_4, \varpi_6\}\}$. And by taking finite intersection, the base β_{Ic} is produced:

$$\{\{\varpi_2\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_4\}, \{\varpi_4, \varpi_5\}, \{\varpi_4, \varpi_6\}, \varphi\}$$

Then, utilizing each union generate the incidence topology $\mathcal{T}_{\mathrm{Ic}}$ as the following:

$$\mathcal{T}_{\mathrm{Ic}} = \{\emptyset, \Lambda, \{\varpi_2\}, \{\varpi_4\}, \{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_4\}, \{\varpi_4, \varpi_5\}, \{\varpi_4, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_2, \varpi_3, \varpi_4\}, \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}, \{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_2, \varpi_4, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_4, \varpi_6\}, \{\varpi_1, \varpi_2, \varpi_4\}, \{\varpi_2, \varpi_3, \varpi_4, \varpi_6\}, \{\varpi_4, \varpi_5, \varpi_6\}, \{\varpi_2, \varpi_4, \varpi_5, \varpi_6\}\},$$

clearly $(\Lambda, \mathcal{T}_{\mathrm{Ic}})$ represents a topological space.

Then, $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{Id}} \cap \mathcal{T}_{\mathrm{Ic}}$, and the II-induced topology: $\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, \{\varpi_1, \varpi_2, \varpi_3\}\}$.

Now, it is clear that the II-induced topology $\mathcal{T}_{\mathrm{II}}$ of a tree graph is not discrete topology.

Remark 4.13

The II-induced topology $\mathcal{T}_{\mathrm{II}}$ on a graph which has two or more components does not represent discrete topology. That is because, independent $\mathcal{T}_{\mathrm{I}d}$ and incidence $\mathcal{T}_{\mathrm{I}c}$ do not represent discrete topology at the graph of two or more components [11, 12].

The II-induced topology of two components graph is equal to the as in the Independent Topology sub-base. i.e., as the following:

$$\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, \{set\ of\ vertices\ of\ first\ component\}, \{set\ of\ vertices\ of\ second\ component\}\}$$

For example, the reader can see the Example 4.4. above, which is two components graph and $\mathcal{T}_{\mathrm{I}d} = \{\emptyset, \Lambda, \{\varpi_1\}, \{\varpi_3\}, \{\varpi_3, \varpi_4, \varpi_5\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}\}$, clearly not discrete.

And incident topology: $\mathcal{T}_{\mathrm{I}c} = \{\emptyset, \Lambda, \{\varpi_2\}, \{\varpi_1, \varpi_2\}, \{\varpi_2, \varpi_3\}, \{\varpi_4, \varpi_5\}, \{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}, \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}\}$, clearly not discrete.

Since $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$, $\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}\}$, then II-induced topology also clearly not discrete.

And the II-induced topology of three components graph is equal to {Λ, the sets of vertices of the components and the arbitrary unions of these sets (vertices of components sets), i.e. The II-induced topology of three components graph is as the following, i.e., $\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, Co_i, \bigcup_{i=1}^{n} Co_i\}$, where Co_i is abbreviation of the word of component, and i=1, 2, …, n such that n≥3.

After the view of all above remarks and examples we conclude that:

The potential advantages of using the II-induced topology over existing topologies are that in most cases is discrete (in certain types of graphs), and disadvantages of using the II-induced topology are that it takes the characteristics of a more specific (less general) of existing topologies.

5. THE ALEXANDROFF SPACE $(\mathcal{V}, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$

Alexandroff spaces turn into more substantial in account of their utilization in the topic of digital topology.

It represents a topological space, where the arbitrary intersection of any open sets is also open or the arbitrary union of any closed sets is also closed [20].

In this part of research, we are interested by a bi-topological space $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ is an Alexandroff bi-topological space in the graph $\mathfrak{S} = (\Lambda, \mathfrak{L})$, which achieved the powerful condition namely, the arbitrary sets intersection of a sub-basis sets is open in their topologies, we will define it as follows:

Definition 5.1

Independent topology $\mathcal{T}_{\mathrm{I}d}$ and Incidence topology $\mathcal{T}_{\mathrm{I}c}$ on the set Λ will form the bi-topological space $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$. The bi-topological space is named as an Alexandroff bi-topological space iff the arbitrary intersection of any members of the two topologies $S_{\mathrm{I}d}$ and $S_{\mathrm{I}c}$ are open set at $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$ resp., where $S_{\mathrm{I}d}$ is represent a sub-basis of the Independent topology $\mathcal{T}_{\mathrm{I}d}$, and $S_{\mathrm{I}c}$ is a sub-basis of Incidence topology $\mathcal{T}_{\mathrm{I}c}$.

Remark 5.2

Independent topological space $(\Lambda, \mathcal{T}_{\mathrm{I}d})$ represents an Alexandroff [12], and the incidence $(\Lambda, \mathcal{T}_{\mathrm{I}c})$ is also represents an Alexandroff space [11].

Proposition 5.3

If a graph $\mathfrak{S} = (\Lambda, \mathfrak{L})$ is locally finite, and $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ is a bi-topological space, then $(\Lambda, \mathcal{T}_{\mathrm{II}})$ represents an Alexandroff topological space, such that $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{I}d} \cap \mathcal{T}_{\mathrm{I}c}$ (i.e., $\mathcal{T}_{\mathrm{II}}$ is II-induced topology of $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$).

Proof: It is very easy, that by applying the intersection of two Alexandroff topological spaces $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$.

6. SEPARATION AXIOMS ON II -INDUCED TOPOLOGICAL SPACE

In this part, definitions of the separation axioms of II-induced topology from bi-topology on un-directed graphs (locally finite graphs) are introduced. And their basic properties are investigated with respect to two well-known topologies $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{\mathrm{I}c}$, the II-T$_i$; (i=0, 1, 2, 3, 4) spaces and their notions of II-normal and II-regular are defined and discussed, and we will introduce some illustrative examples. Also we will give a fundamental properties of locally finite graphs by their corresponding II-separation axioms of bi-topological spaces.

6.1 II-T$_i$: (*i*=0,1,2) spaces of bi-topological spaces on locally finite graphs

Generally, the separation axioms expressing how rich the population of open set is. And graph properties connect to fulfillment of separation axioms in the topology via the measure of tightness is the extent to which envelope can separate the subset of vertices connection from other subsets of vertices connection.

Definition 6.1.1 Let $(\Lambda, \mathcal{T}_{\mathrm{II}})$ be an II-induced topological space from bi-topological space $(\Lambda, \mathcal{T}_{\mathrm{I}d}, \mathcal{T}_{\mathrm{I}c})$ over Λ, then $(\Lambda, \mathcal{T}_{\mathrm{II}})$ is said to be II -T$_0$-space if for each pair of distinct vertices $\varpi_1, \varpi_2 \in \Lambda$, there is II -bi-open sets W_1, W_2 s. t. '$\varpi_1 \in W_1$, $\varpi_2 \notin W_1$' or '$\varpi_1 \notin W_2$, $\varpi_2 \in W_2$'.

Example 6.1.2 $\mathfrak{S} = (\Lambda, \mathfrak{L})$ is a graph as in Figure 5, such that: $\Lambda = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$, $\mathfrak{L} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6\}$. Then,

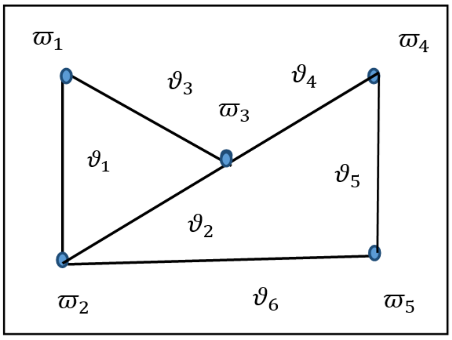

Figure 5. Undirected graph

The sub-base of independent topology is the following set: $S_{\mathrm{I}d} = \{\{\varpi_4\}, \{\varpi_5\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}\}$, taking the finitely intersection the base $\beta_{\mathrm{I}d}$ obtained: $\{\emptyset, \{\varpi_1\}, \{\varpi_4\}, \{\varpi_5\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}\}$.

Then via taking the unions:

$\mathcal{T}_{\mathrm{I}d} = \{\emptyset, \Lambda, \{\varpi_1\}, \{\varpi_4\}, \{\varpi_5\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_1, \varpi_4\}, \{\varpi_1, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_5\}, \{\varpi_1, \varpi_3, \varpi_4\}, \{\varpi_1, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_3, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}\}$

$\varpi_1, \varpi_2 \in \Lambda$, $\varpi_1 \neq \varpi_2$, there is an $\mathcal{T}_{\mathrm{I}d}$ open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi_1\}$, $\varpi_2 \notin \{\varpi_1\}$.

$\varpi_2, \varpi_3 \in \Lambda$, $\varpi_2 \neq \varpi_3$, there is an $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1, \varpi_3\}$ such that $\varpi_3 \in \{\varpi_1, \varpi_3\}$, $\varpi_2 \notin \{\varpi_1, \varpi_3\}$

$\varpi_3, \varpi_4 \in \Lambda$, $\varpi_3 \neq \varpi_4$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_4\}$ such that $\varpi_4 \in \{\varpi_4\}$, $\varpi_3 \notin \{\varpi_4\}$.

$\varpi_4, \varpi_5 \in \Lambda$, $\varpi_4 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{I}d}$-open sets $\{\varpi_4\}$ such that $\varpi_4 \in \{\varpi_4\}$, $\varpi_5 \notin \{\varpi_4\}$.

$\varpi_1, \varpi_3 \in \Lambda$, $\varpi_1 \neq \varpi_3$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi_1\}$, $\varpi_3 \notin \{\varpi\}$.

$\varpi_1, \varpi_4 \in \Lambda$, $\varpi_1 \neq \varpi_4$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi\}$, $\varpi_4 \notin \{\varpi_1\}$

$\varpi_1, \varpi_5 \in \Lambda$, $\varpi_1 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi_1\}$, $\varpi_5 \notin \{\varpi_1\}$

$\varpi_2, \varpi_4 \in \Lambda$, $\varpi_2 \neq \varpi_4$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1, \varpi_2\}$ such that $\varpi_2 \in \{\varpi_1, \varpi_2\}$, $\varpi_4 \notin \{\varpi_1, \varpi_2\}$.

$\varpi_2, \varpi_5 \in \Lambda$, $\varpi_2 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1, \varpi_2\}$ such that $\varpi_2 \in \{\varpi_1, \varpi_2\}$, $\varpi_5 \notin \{\varpi_1, \varpi_2\}$.

$\varpi_3, \varpi_5 \in \Lambda$, $\varpi_3 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{I}d}$-open set $\{\varpi_1, \varpi_3\}$ such that $\varpi_3 \in \{\varpi_1, \varpi_3\}$, $\varpi_5 \notin \{\varpi_1, \varpi_3\}$.

Then it is clear, the independent topology $\mathcal{T}_{\mathrm{I}d}$ is represents a T$_0$-space.

The sub-base of incident topology is the following set:

$S_{\mathrm{I}c} = \{\{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_5\}, \{\varpi_3, \varpi_4\}, \{\varpi_4, \varpi_5\}\}$

By taking finitely intersection the base β_{Ic} obtained,

$\{\emptyset, \{\varpi_1\}, \{\varpi_3\}, \{\varpi_2\}, \{\varpi_4\}, \{\varpi_5\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_5\}, \{\varpi_3, \varpi_4\}\{\varpi_4, \varpi_5\}\}$

Then via taking all unions;

$\mathcal{T}_{\mathrm{Ic}} = \{\Lambda, \emptyset, \{\varpi_1\}, \{\varpi_3\}, \{\varpi_2\}, \{\varpi_4\}, \{\varpi_5\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}, \{\varpi_2, \varpi_5\}, \{\varpi_3, \varpi_4\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_5\}, \{\varpi_1, \varpi_4\}, \{\varpi_1, \varpi_5\}, \{\varpi_2, \varpi_4\}, \{\varpi_3, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_5\}, \{\varpi_1, \varpi_3, \varpi_4\}, \{\varpi_1, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_3, \varpi_5\}, \{\varpi_3, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}, \{\varpi_2, \varpi_3, \varpi_5\} \{\varpi_2, \varpi_3, \varpi_4\}, \{ \varpi_2, \varpi_4, \varpi_5\}, \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}, , \{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}\}$

Now,

$\varpi_1, \varpi_2 \in \Lambda$, $\varpi_1 \neq \varpi_2$, there is an $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi_1\}$, $\varpi_2 \notin \{\varpi_1\}$.

$\varpi_2, \varpi_3 \in \Lambda$, $\varpi_2 \neq \varpi_3$, there is an $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_1\}$ such that $\varpi_3 \in \{\varpi_1\}$, $\varpi_2 \notin \{\varpi_1\}$.

$\varpi_3, \varpi_4 \in \Lambda$, $\varpi_3 \neq \varpi_4$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_4\}$ such that $\varpi_4 \in \{\varpi_4\}$, $\varpi_3 \notin \{\varpi_4\}$.

$\varpi_4, \varpi_5 \in \Lambda$, $\varpi_4 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{Ic}}$-open sets $\{\varpi_4\}$ such that $\varpi_4 \in \{\varpi_4\}$, $\varpi_5 \notin \{\varpi_4\}$.

$\varpi_1, \varpi_3 \in \Lambda$, $\varpi_1 \neq \varpi_3$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi_1\}$, $\varpi_3 \notin \{\varpi\}$.

$\varpi_1, \varpi_4 \in \Lambda$, $\varpi_1 \neq \varpi_4$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi\}$, $\varpi_4 \notin \{\varpi_1\}$.

$\varpi_1, \varpi_5 \in \Lambda$, $\varpi_1 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_1\}$ such that $\varpi_1 \in \{\varpi_1\}$, $\varpi_5 \notin \{\varpi_1\}$.

$\varpi_2, \varpi_4 \in \Lambda$, $\varpi_2 \neq \varpi_4$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{ \varpi_2\}$ such that $\varpi_2 \in \{\varpi_2\}$, $\varpi_4 \notin \{ \varpi_2\}$.

$\varpi_2, \varpi_5 \in \Lambda$, $\varpi_2 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{ \varpi_2\}$ such that $\varpi_2 \in \{\varpi_2\}$, $\varpi_5 \notin \{ \varpi_2\}$.

$\varpi_3, \varpi_5 \in \Lambda$, $\varpi_3 \neq \varpi_5$, there is $\mathcal{T}_{\mathrm{Ic}}$-open set $\{\varpi_3\}$ such that $\varpi_3 \in \{\varpi_3\}$, $\varpi_5 \notin \{ \varpi_3\}$.

Clearly, the incidence topology $\mathcal{T}_{\mathrm{Ic}}$ represents a T$_0$-space.

Then the II-induced topology $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{\mathrm{Id}} \cap \mathcal{T}_{\mathrm{Ic}}$: $\mathcal{T}_{\mathrm{II}} = \{\emptyset, \Lambda, \{\varpi_1\}, \{\varpi_4\}, \{\varpi_5\}, \{\varpi_1, \varpi_2\}, \{\varpi_1, \varpi_3\}, \{\varpi_4, \varpi_5\}, \{\varpi_1, \varpi_4\}, \{\varpi_1, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_4\}, \{\varpi_1, \varpi_2, \varpi_5\}, \{\varpi_1, \varpi_3, \varpi_4\}, \{\varpi_1, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3\}, \{\varpi_1, \varpi_3, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\},$

$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\},\{\varpi_1,\varpi_3,\varpi_4,\varpi_5\},\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}\}$

It is clear that for every distinct vertex in II-induced topology, there exist II -bi-open set such that one of them contain one vertex but not the other.

Then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ represents a II-T_0-space.

In this example we conclude that the independent $\mathcal{T}_{\mathrm{Id}}$ is the most dominant and the most powerful because it controls the the intersection of the other topology.

Definition 6.1.3. Let $(\Lambda,\mathcal{T}_{\mathrm{II}})$ be II -induced topological space from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over Λ, then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ is said to be II-T_1-space if for each pair of distinct vertices $\varpi_1,\varpi_2 \in \Lambda$, there is II-bi-open sets W_1, W_2 s. t. '$\varpi_1 \in W_1, \varpi_2 \notin W_1$' and '$\varpi_1 \notin W_2, \varpi_2 \in W_2$'.

Definition 6.1.4. Let $(\Lambda,\mathcal{T}_{\mathrm{II}})$ be II-induced topological space from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over Λ, then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ is said to be II-T_2-space if for each pair of distinct vertices $\varpi_1,\varpi_2 \in \Lambda$, there is II-bi-open sets W_1, W_2 s. t. '$\varpi_1 \in W_1, \varpi_2 \in W_2$ and $W_1 \cap W_2 = \emptyset$'.

Remark 6.1.5. Only if the two topologies are discrete, the $(\Lambda,\mathcal{T}_{\mathrm{II}})$ will be II-T_1 and II-T_2 as shows the example below.

6.2 II-regular, II-normal and II-T$_i$: (i=3,4) spaces

In this section, we define II-T_3 and II-T_4 spaces using ordinary points and characterize II-regular and II-normal spaces on II-induced topological space $(\Lambda,\mathcal{T}_{\mathrm{II}})$ from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over the vertices set Λ.

Definition 6.2.1. Let $(\Lambda,\mathcal{T}_{\mathrm{II}})$ be II-induced topological space from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over Λ, then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ is said to be II-regular-space if for each II-bi-closed set F in Λ, and each $\varpi \notin F$, there is an II-bi-open sets W_1, W_2 s. t. '$\varpi \in W_1$, $F \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$'.

Definition 6.2.2. Let $(\Lambda,\mathcal{T}_{\mathrm{II}})$ be II-induced topological space from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over Λ, then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ is said to be II-T_3-space iff it is an II-regular and II-T_1-space.

Definition 6.2.3. Let $(\Lambda,\mathcal{T}_{\mathrm{II}})$ be II-induced topological space from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over Λ, then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ is said to be II-normal space if for all pair of II-bi-closed sets F_1 and F_2 in Λ s. t. $F_1 \cap F_2 = \emptyset$, there is an II-bi-open sets W_1, W_2 s. t. '$F_1 \subseteq W_1$, $F_2 \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$'.

Definition 6.2.4. Let $(\Lambda,\mathcal{T}_{\mathrm{II}})$ be II-induced topological space from bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{Id}},\mathcal{T}_{\mathrm{Ic}})$ over Λ, then $(\Lambda,\mathcal{T}_{\mathrm{II}})$ is said to be II-T_4-space iff it is an II-normal and II-T_1-space.

Example 6.2.5. $\mathfrak{S} = (\Lambda,\mathfrak{L})$ is a cycle $\mathbb{C}_3$ graph as in Figure 6, such that: $\Lambda = \{\varpi_1,\varpi_2,\varpi_3\}$, $\mathfrak{L} = \{\vartheta_1,\vartheta_2,\vartheta_3\}$.

The sub-base of independent topology is the set: $S_{\mathrm{Id}} = \{\{\varpi_1\},\{\varpi_2\},\{\varpi_3\}\}$ taking the finitely intersection the base β_{Id} obtained, $\{\emptyset,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\}\}$.

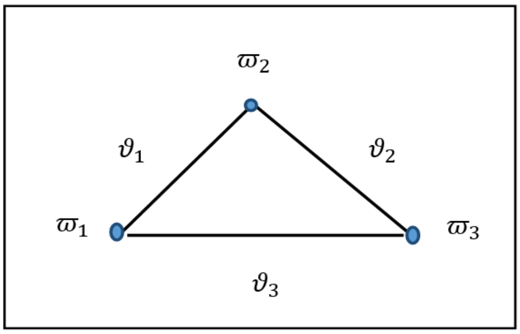

Figure 6. A cycle $\mathbb{C}_3$ graph

Then via taking the unions: independent topology is $\mathcal{T}_{Id} = \{\emptyset,\Lambda,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\},\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}\}$.

It is clear that $\mathcal{T}_{Id}$ is discrete.

The sub-base of incident topology is the set $S_{\mathrm{Ic}} = \{\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}\}$.

Taking the finitely intersection the base β_{Ic} obtained $\{\emptyset,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\},\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}\}$.

And, incident topology is $\mathcal{T}_{Ic} = \{\emptyset,\Lambda,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\},\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}\}$.

It is clear that $\mathcal{T}_{\mathrm{Ic}}$ is discrete.

Since $\mathcal{T}_{\mathrm{II}} = \mathcal{T}_{Id} \cap \mathcal{T}_{\mathrm{Ic}}$, then II-induced topology $\mathcal{T}_{\mathrm{II}} = \{\emptyset,\Lambda,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\},\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}\}$, then $\emptyset,\Lambda,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\},\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}$ are II-bi-open sets, and $\emptyset,\Lambda,\{\varpi_1\},\{\varpi_2\},\{\varpi_3\},\{\varpi_1,\varpi_2\},\{\varpi_2,\varpi_3\},\{\varpi_1,\varpi_3\}$ are II-bi-closed sets.

Therefore $\mathcal{T}_{Id}$ and $\mathcal{T}_{Ic}$ on Λ give us a discrete bi-topological space $(\Lambda,\mathcal{T}_{\mathrm{I}d},\mathcal{T}_{\mathrm{I}c})$.

Clear that $\mathcal{T}_{\mathrm{I}d}$ and $\mathcal{T}_{Ic}$ are discrete, and they induce $\mathcal{T}_{\mathrm{II}}$ topology, which is also a discrete topology, thus $\mathcal{T}_{\mathrm{II}}$ investigates the II-T_2-space and II-T_2-space, also it investigates the II-regular-space and II-normal-space. And sequentially it is II-T_3-space and II-T_4-space.

Therefore $\mathcal{T}_{\mathrm{II}}$ of the cycle graph investigates the II-T_3 and II-T_4 axioms.

7. CONCLUSIONS

In this our paper an assembly between bi-topology and graph has been founded. An II-induced topology has been introduced, the new bi-topological spaces of the two well-known topologies (independent and incidence) with any certain and known locally finite graphs, i.e. A graph for which every vertex has finite degree. have been associated with some properties and separation axioms. Some results are obtained, such as II-induced topology must be a locally finite graph and not contains isolated vertex, also observations about the specification and type of II-induced topology with certain types of graphs, in addition that II-induced topology fulfills Alexandroff topology.

Suggestions for future research are to define a new kind of a tritopology. The potential applications of the new bi-topology and II-induced topology is in decision making. Also, we suggest as a potential application of this work, in the network communication (the network devices).

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