International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 12, No. 3, March, 2025, pp. 1031-1042
Journal homepage: http://iieta.org/journals/mmep

Alpha Power Type II-G Family: Adding a Power Parameter of Distributions

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https://doi.org/10.18280/mmep.120330

Received: 16 August 2024
Revised: 18 December 2024
Accepted: 26 December 2024
Available online: 31 March 2025

Keywords:
Alpha Power-G family, exponential distribution, APIIE distribution, Weibull distribution, APIIW distribution

ABSTRACT

This paper introduces a new family of distributions named the Alpha Power Type II-G (APII-G) family, which emerges as a groundbreaking modeling strategy for examining data governed by univariate continuous distributions. This family aims to enhance the modeling capabilities of continuous prior distributions to better fit the data utilizing a new function encompassing the additional parameter power. The innovative methodology implemented encompasses two continuous distributions: firstly, the one-parameter exponential distribution, which engendered a fresh two-parameter, Alpha Power II Exponential (APIIE) distribution, and secondly, the two-parameter Weibull distribution, which yielded a new three-parameter, Alpha Power II Weibull (APIIW) distribution. Moreover, a scrutiny of the characteristics and statistical functions, and the estimations of the parameters of the two distributions. The efficacy of these estimators is substantiated through simulation studies and finding the mean square error (MSE) and bias values of the estimators compared to sample sizes. It has been empirically proven that the two suggested models outperformed the asymptotic distributions they were compared against using multiple goodness-fit criteria as Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected AIC (CAIC) and Hannan-Quinn information criterion (HQIC) on authentic datasets, The values of these criteria appeared to be the lowest for the two new distributions, which means that the new distributions are the best, especially in the context of the given data.

1. INTRODUCTION

In the field of statistical analysis, many distributions have been formulated that extend over many years of research. Among these distributions are the typical normal distribution and the exponential distribution, in addition to the Gamma distribution, Weibull distribution, Gumbel distribution, Lomax distribution, and others. These distributions are intended for use in a variety of fields, such as survival analysis, ecology, medicine, actuarial science, reliability engineering, hydrology, social sciences, and more. Efforts are constantly being made to improve these lifetime distributions to better fit specific real datasets, moving away from traditional approaches. The incorporation of an additional parameter frequently facilitates enhanced control over the characteristics of the distribution, including aspects such as skewness, kurtosis, or tail behavior. This adaptability is crucial when attempting to model empirical data that demonstrates attributes inadequately represented by more elementary distributions. Empirical datasets are typically derived from processes with intricacies that are challenging to encapsulate using simpler models. By introducing a parameter, the model is enabled to adjust to these intricacies, resulting in improved goodness-of-fit evaluations. The newly introduced parameter may occasionally signify a significant physical or probabilistic characteristic of the phenomenon being modeled. This augmentation contributes to the interpretability of the model; for instance, in survival analysis, the incorporation of a shape parameter into a baseline hazard function permits the modeling of increasing, constant, or decreasing hazard rates over time. Also, incorporating a new complementary parameter to the distribution may improve its accuracy for a variety of data, and lead to improved predictive capabilities. Integrating an additional parameter into the conventional baseline distributions has given rise to numerous distinct distribution families Recently. A method introduced by Mahdavi and Kundu [1] involved the first presentation of the Alpha Power technique for producing novel distributions, subsequently adopted by various researchers for the creation of multiple distributions using this approach. This method was dependent on finding the distribution function CDF $F(x,\alpha)$ of the first type Alpha Power distribution as following:

$$F(x,\alpha)=\frac{\alpha^{G(x)}-1}{\alpha-1},0<\alpha,\alpha\neq 1 \tag{1}$$

where, the function $G(x)$ denotes to the CDF of the baseline distribution, potentially affected by the parameter vector. Mahdavi and Kundu [1] introduced an extra one parameter to the exponential distribution, deriving various properties and demonstrating its practical application through data analysis. Arashi et al. [2] extended the beta-generating technique to multivariate distributions, constructing a new family of distributions with Dirichlet-generated marginals and demonstrating their applicability through simulated and real data analysis. Hassan et al. [3] proposed Weibull-Lindley distribution by a technique that adds one parameter to the baseline Weibull distributions. Farooq et al. [4] generalized the method of generating continuous distributions by nesting one model within another, which includes famous distributions like Beta, Kumaraswami, and Gamma as special cases, thereby enhancing the modeling of complex systems. Kalt [5] developed a generator for new distributions by adding a parameter by using the survival function.

This study focuses on enhancing the flexibility of a specific set of distribution functions by introducing an additional parameter. The newly introduced group is referred to as the APII-G family. A number of some properties of distribution functions within this category are examined in this paper. Then, this family is applied to create two distributions, the first is a two-parameter distribution by integrating the new group into the one-parameter exponential distribution. The resulting distribution from the exponential distribution exhibits several useful properties. The paper also delves into the estimators of the unknown coefficients of the resulting distribution. The second is a three-parameter distribution constructed by applying the new family with the two-parameter Weibull distribution and studying its statistic properties and the estimators of the three parameters. These two distributions were chosen for generalization due to their importance in wide applications in engineering research and others in studies [6-9]. Moreover, an evaluation of the two distributions for some sets of real, data is included, along with comparisons with other similar, distributions by using the goodness-of-fit criteria.

2. NEW FAMILY WITH SOME PROPERTIES

Modern classes of distributions stemming from a straightforward, innovative, and well-founded transformation of the baseline distribution are not yet common. Within this manuscript, we put forth a potential candidate in which we present a transformation that is contingent upon the Alpha Power of baseline distribution generated APII-G family, as specified by the subsequent CDF. The general formula for CDF of the new family (APII-G) is defined by

$$F(x,\alpha)=\frac{\left(1+G(x)\right)^{\alpha}-1}{2^{\alpha}-1},\alpha>0 \tag{2}$$

where, the function $G(x)$ denotes to the CDF of the baseline distributions, potentially affected by the parameter vector, designated as R. In the field of mathematical functions, it is noted that the function $G(x)$ represents the cumulative distribution function obtained from existing distributions. This specific function is mathematically expressed by Eq. (2), which not only indicates the cumulative distribution function related to APII-G but also provides insight into its properties. In cases where $x_1 > x_2$ is true, it can be inferred that $G(x_1) > G(x_2)$, so that $F(x_1;\varphi,\alpha) > F(x_2;\varphi,\alpha)$ is a direct result, thus establishing an important relationship within the mathematical framework. Therefore, it can be concluded that the function $F(x;\varphi,\alpha)$ fulfills the properties of distribution functions of being monotonically increasing. Also, the limit of $F(x:\varphi,\alpha)$ satisfied the following:

$$\lim_{x\to-\infty}F(x;\varphi,\alpha)=\lim_{G(x)\to 0}F(x;\varphi,\alpha)=0$$
$$\lim_{x\to\infty}F(x;\varphi,\alpha)=\lim_{G(x)\to 1}F(x;\varphi,\alpha)=1$$

To find the density function, we derive the Eq. (1) and we obtain:

$$f(x,\alpha)=\frac{\alpha\left(1+G(x)\right)^{\alpha-1}g(x)}{2^{\alpha}-1},\alpha>0 \tag{3}$$

Which fulfills the basic condition for the probability density function as follows:

$$\int_{-\infty}^{\infty}f(x,\alpha)dx=\int_{-\infty}^{\infty}\frac{\alpha\left(1+G(x)\right)^{\alpha-1}g(x)}{2^{\alpha}-1}dx$$
$$=\frac{1}{2^{\alpha}-1}[(2)^{\alpha}-1]=1 \tag{4}$$

When the well definition of the general form of the transformation has been established, as indicated by Eq. (2), which serves as a representation of the cumulative distribution function, it is important to highlight that through this formulation, the corresponding density function can be derived. The application of this constant transformation is then directed towards two specific distributions, the exponential and the Weibull, by replacing $G(x)$ in the transformation with the cumulative distribution function related to these respective distributions. This substitution is the basis for creating the new distributions for this family because it allows a comprehensive examination of the statistical properties and functions that define the new transformed distributions. When the value of new alpha parameter in the new distributions is equal to 1, it will remain the same as the original distribution that was used.

3. APIIE DISTRIBUTION

In this section, we will apply the new APII-G family with the one-parameter exponential distribution, where we will obtain a two-parameter new distribution which is called APIIE distribution, by replacing $G(x)$ in Eq. (2) with the CDF of the exponential distribution, then the CDF of APIIE distribution is defined as follows:

$$F(x;\alpha,\lambda)=\frac{\left(2-e^{\frac{-x}{\lambda}}\right)^{\alpha}-1}{2^{\alpha}-1},x>0,\alpha,\lambda>0 \tag{5}$$

And the pdf is defined as the following formula:

$$f(x,\alpha,\lambda)=\frac{\alpha\left(2-e^{\frac{-x}{\lambda}}\right)^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda(2^{\alpha}-1)},x>0;\alpha,\lambda>0 \tag{6}$$

Some plots of the CDF $F(x;\alpha,\lambda)$ and pdf $f(x,\alpha,\lambda)$ of the (APIIE) model, which is plotted for some different value of the parameters α and λ in Figure 1 and Figure 2, respectively.

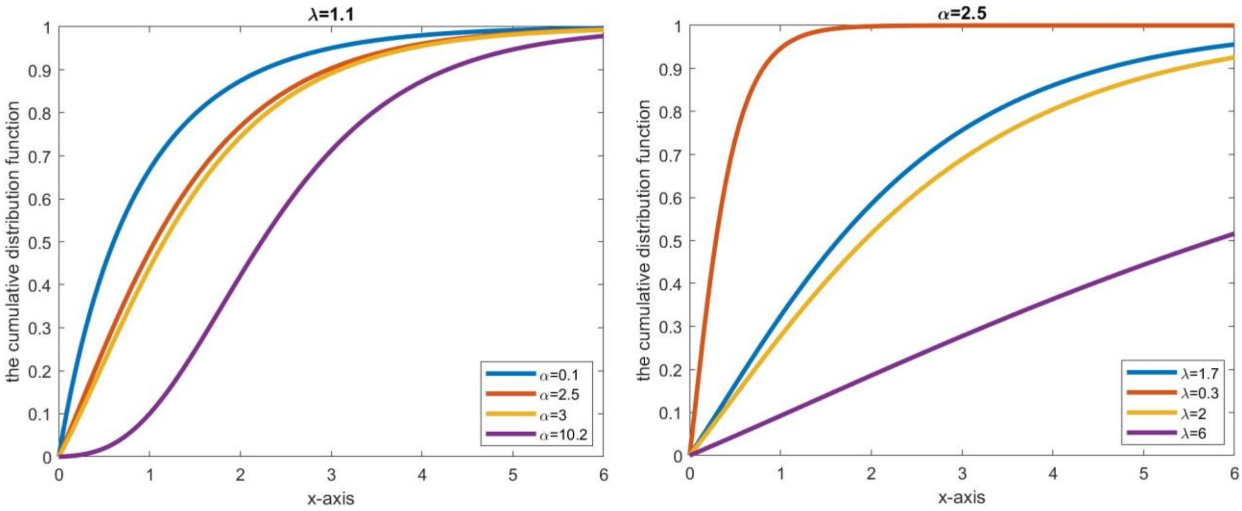

Figure 1. The CDF of APIIE

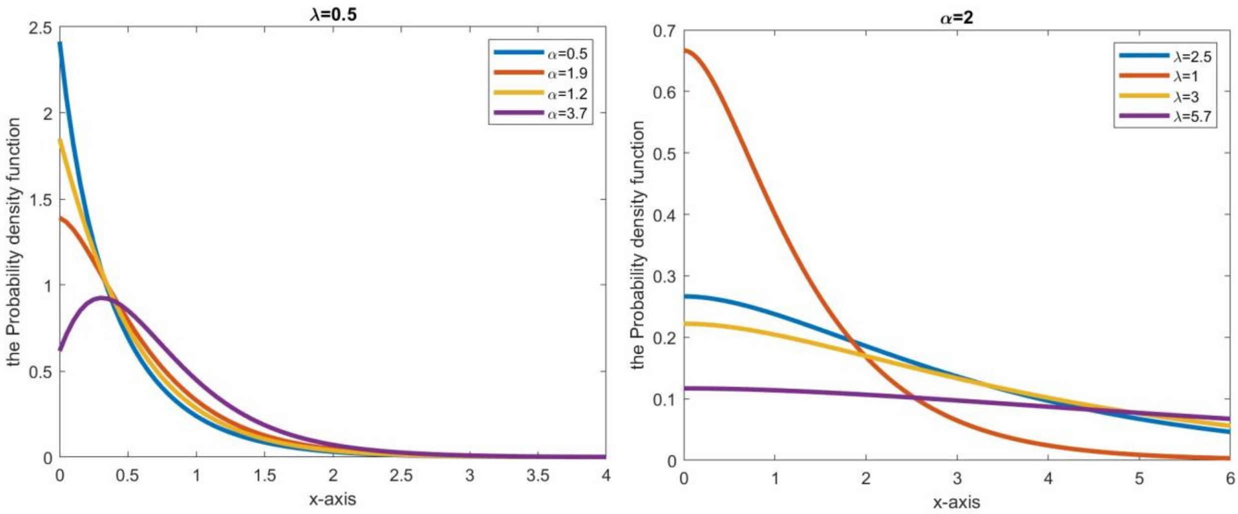

Figure 2. The pdf of APIIE

3.1 The statistical properties of the APIIE distribution

We present the functions of reliability, reversed hazard, hazard rate, and the cumulative of the hazard rate of APIIE distribution [10]. The survival function of random variable $X \sim APIIE(\alpha, \lambda)$ is defined by the following:

$$\bar{F}(x;\alpha,\lambda) = 1 - F(x;\alpha,\lambda) = \frac{2^{\alpha} - \left(2 - e^{\frac{-x}{\lambda}}\right)^{\alpha}}{2^{\alpha} - 1} \tag{7}$$

The function of reverse hazard $r(x\colon \alpha, \lambda)$ to the APIIE distribution is defined by the following:

$$r(x\colon \alpha,\lambda) = \frac{f(x\colon \alpha,\lambda)}{F(x\colon \alpha,\lambda)} = \frac{\alpha\left(2 - e^{\frac{-x}{\lambda}}\right)^{\alpha-1} e^{\frac{-x}{\lambda}}}{\lambda\left(\left(2 - e^{\frac{-x}{\lambda}}\right)^{\alpha} - 1\right)} \tag{8}$$

The function of hazard function $h(x\colon \alpha, \lambda)$ to the APIIE distribution is defined by the following:

$$h(x\colon \alpha,\lambda) = \frac{f(x\colon \alpha,\lambda)}{\bar{F}(x\colon \alpha,\lambda)} = \frac{\alpha\left(2 - e^{\frac{-x}{\lambda}}\right)^{\alpha-1} e^{\frac{-x}{\lambda}}}{\lambda\left(2^{\alpha} - \left(2 - e^{\frac{-x}{\lambda}}\right)^{\alpha}\right)} \tag{9}$$

And the cumulative hazard $H(x\colon \alpha, \lambda)$ of the APIIE is defined by the following:

$$H(x\colon \alpha,\lambda) = -Ln[1 - F(x\colon \alpha,\lambda)] = -Ln\left[\frac{2^{\alpha} - \left(2 - e^{\frac{-x}{\lambda}}\right)^{\alpha}}{2^{\alpha} - 1}\right] \tag{10}$$

3.2 The moments of APIIE distribution

We will define the r^{th} moment (at the point of origin) for APIIE distribution by the following theorem.

Theorem:
If X is variable to APIIE distribution, then moment of the order r (about the point of origin) is defined by the following:

$$\grave{\mu}_r = \frac{\alpha \lambda^r r!}{(2^{\alpha} - 1)} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha - 1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - n)!\, n!\, (n+1)^{(r+1)}} \tag{11}$$

Proof:
By definition the moment of the order r (about the point of origin) $\grave{\mu}_r$ to the variable X of APIIE distribution:

$$\acute{\mu}_r = E(x^r) = \int_0^\infty x^r f(x,\alpha,\lambda)\,dx, r = 1,2,3,\dots$$
$$\acute{\mu}_r = \frac{\alpha}{\lambda(2^\alpha-1)}\int_0^\infty x^r\left(2-e^{\frac{-x}{\lambda}}\right)^{\alpha-1} e^{\frac{-x}{\lambda}}\,dx \quad (12)$$

Let $e^{\frac{-x}{\lambda}} = y$, and if $x \to o$ then $y \to 1$; if $x \to \infty$ then $y \to 0, dx = \frac{-\lambda}{y}\,dy$, then

$$\acute{\mu}_r = \frac{\alpha\lambda^r}{(2^\alpha-1)}\int_0^1 (-lny)^r\,(2-y)^{\alpha-1}\,dy \quad (13)$$

Now, Taking the Maclaurin series of formula $(2-y)^{\alpha-1}$, we obtain the following series:

$$(2-y)^{\alpha-1} = \sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-n)!\,n!}y^n \quad (14)$$

We replace Eq. (14) with Eq. (13)

$$\acute{\mu}_r = \frac{\alpha\,\lambda^r}{(2^\alpha-1)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-\mathrm{n})!\,n!}\int_0^1(-lny)^r\,y^n dy \quad (15)$$

Using the infinitive form in reference [11] and comparing it with Eq. (15), the integration result becomes as follows:

$$\acute{\mu}_r = \frac{\alpha\,\lambda^r r!}{(2^\alpha-1)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-n)!\,n!\,(n+1)^{(r+1)}} \quad (16)$$

Now, by using theorem (3.2.1) we calculate the mean and variance as follows:

$$E(x) = \frac{\alpha\,\lambda}{(2^\alpha-1)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-\mathrm{n})!\,n!\,(n+1)^2} \quad (17)$$

$$Var(x) = \frac{\alpha\lambda^2 2!}{(2^\alpha-1)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-\mathrm{n})!\,n!\,(n+1)^3} - \frac{\alpha^2\lambda^2}{(2^\alpha-1)^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-n)!\,n!\,(n+1)^2}\right)^2 \quad (18)$$

The moment-generating function (mgf), denoted by the symbol $M_x(t)$, of APIIE can be find by the following:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x,\alpha,\lambda)\,dx \quad (19)$$

By Taylor series for e^{tx} yields, which is $e^{tx} = \sum_{s=0}^{\infty}\frac{t^s x^s}{s!}$.

$$M_x(t) = \sum_{s=0}^{\infty}\frac{t^s}{s!}\int_0^\infty x^s f(x,\alpha,\lambda)\,dx = \sum_{s=0}^{\infty}\frac{t^s}{s!}\acute{\mu}_s \quad (20)$$

And by substituting Eq. (16) into Eq. (16) then

$$M_x(t) = \frac{\alpha}{(2^\alpha-1)}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda^s t^s(-1)^n(\alpha-1)!\,2^{\alpha-(n+1)}}{(\alpha-1-\mathrm{n})!\,n!\,(n+1)^{(s+1)}} \quad (21)$$

3.3 Maximum Likelihood Estimation (MLE) of APIIE distribution

In this subsection, we find MLE of the two parameters α and λ. If $x_1, x_2, x_3, \dots, x_n$ denote a sample of APIIE, the function of likelihood is defined by

$$L(\alpha,\lambda;x) = \prod_{i=1}^{n} f(x_i:\alpha,\lambda) = \prod_{i=1}^{n}\frac{\alpha\left(2-e^{\frac{-x}{\lambda}}\right)^{\alpha-1} e^{\frac{-x}{\lambda}}}{\lambda(2^\alpha-1)} = \frac{\alpha^n\prod_{=1}^{n}\left(2-e^{\frac{-x_i}{\lambda}}\right)^{\alpha-1} e^{\frac{-1}{\lambda}\sum_{=1}^{n}x_i}}{\lambda^n(2^\alpha-1)^n} \quad (22)$$

and the logarithm of $L(\alpha,\lambda;x)$ will be as

$$ln\big(L(\alpha,\lambda;x)\big) = n\,ln\,\alpha - n\,ln\,(2^\alpha-1 - n\,ln\,\lambda - \frac{1}{\lambda}\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n} ln\left(2-e^{-\frac{x_i}{\lambda}}\right) \quad (23)$$

by taking derivatives of $ln(L(\alpha,\lambda;x))$ to the parameter α and λ respectfully, then we get

$$\frac{\partial\, ln\big(L(\alpha,\lambda;x)\big)}{\partial\lambda} = \frac{-n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}x_i - \frac{(\alpha-1)}{\lambda^2}\sum_{i=1}^{n}\frac{x_i e^{\frac{-x_i}{\lambda}}}{\left(2-e^{-\frac{x_i}{\lambda}}\right)} \quad (24)$$

$$\frac{\partial\, ln(L(\alpha,\lambda;x))}{\partial\alpha} = \frac{n}{\alpha} - \frac{n2^\alpha ln2}{2^\alpha-1} + \sum_{i=1}^{n} ln(2-e^{-\frac{x_i}{\lambda}}) \quad (25)$$

By setting Eq. (24) equal to zero, we get

$$\lambda - \frac{1}{n}\sum_{i=1}^{n}x_i + \frac{(\alpha-1)}{n}\sum_{i=1}^{n}\frac{x_i e^{\frac{-x_i}{\lambda}}}{\left(2-e^{-\frac{x_i}{\lambda}}\right)} = 0 \quad (26)$$

Also, by setting Eq. (25) equal to zero, we get

$$\frac{n}{\alpha} - \frac{n2^\alpha ln2}{2^\alpha-1} + \sum_{i=1}^{n} ln(2-e^{-\frac{x_i}{\lambda}}) = 0 \quad (27)$$

By solving Eq. (26) and Eq. (27), by numerical methods, then obtain the MLE for both parameters α and λ.

3.4 Order statistics of APIIE

If $Y_1, Y_2, \dots, Y_n$ denotes the order statistic to the random sample $X_1, X_2, \dots, X_n$ taken from the population distributing by the APIIE distribution with the CDF $F_X(x)$ and pdf $f_X(x)$, then the pdf of random variable Y_j of the order j is defined by following:

$$f_{Y_j}(x) = \frac{n!}{(j-1)!\,(n-j)!}\left(\frac{\left(2-e^{-\frac{x_j}{\lambda}}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{j-1} \left(\frac{2^{\alpha}-\left(2-e^{-\frac{x_j}{\lambda}}\right)^{\alpha}}{2^{\alpha}-1}\right)^{n-j} \frac{\alpha\left(2-e^{-\frac{x_j}{\lambda}}\right)^{\alpha-1} e^{-\frac{x_j}{\lambda}}}{\lambda(2^{\alpha}-1)} \quad (28)$$

Also, if $1 \le i \le j \le n$ then the joint pdf of Y_i and Y_j with the order random variable u and v, is

$$f_{Y_i,Y_j}(u,v) = \frac{n!}{(n-j)!\,(j-i-1)!\,(i-1)!} \quad (29)$$

with $0 < u < v < \infty$**.** The j.p.d.f of n order random variables $Y_1, Y_2, \dots, Y_n$ can extract more than one ranking statistics through our use of similar functions. Thus, the joint pdf for all order statistics $f_{Y_1, Y_2, \dots, Y_n}(x_1, \dots, x_n)$, taken from the APIIE distribution, as the following:

$$f_{X_{(1)},\dots,X_{(n)}}(x_1,\dots,x_n) = n!\,\frac{\alpha^n \prod_{=1}^{n}\left(2-e^{\frac{-x_i}{\lambda}}\right)^{\alpha-1}}{\lambda^n (2^{\alpha}-1)^n}\, e^{\frac{-1}{\lambda}\sum_{=1}^{n} x_i} \quad (30)$$

3.5 Simulation studies of the APIIE distribution

In order to understand and interpret experimentally the adopted estimation method MLE that was studied in this section of the study, we will employ the simulation approach to examine the MLEs of two parameters of the APIIE distribution. This part includes a description of the Monte Carlo simulation experiment for the research in terms of the sample sizes which generated when the number of iterations of the simulation 1000. We also present the simulation test results obtained where we used statistical R software with the "BFGS mathed" to apply this simulation. The stages of building simulation experiments contain some important stages, which are the stage of choosing the sample size, where n was chosen: (n=30, 50, 75, 100, 150, 250, 500), the stage of choosing values for the parameters for three experiments, the default values for the parameters (α,λ)=(1.5, 2.5), (α,λ)=(1.5, 5) and (α,λ)=(2.5, 8). The stage of generating appropriate data for the APIIE distribution by employing the inverse cumulative distribution function on uniformly distributed random variables, calculating the MLE for the two parameters (α,λ) and following the calculation of the MSEs and the bias.

$$\text{MSE}(\psi) = \frac{1}{1000}\sum_{h=1}^{1000}\left(\hat{\psi}_h - \psi\right)^2$$
$$Bias\ (\psi) = \frac{1}{1000}\sum_{h=1}^{1000}\left(\hat{\psi}_h - \psi\right)$$

where, ψ represents one of the parameters α, λ. Following the simulation, the outcomes are shown in Figure 3.

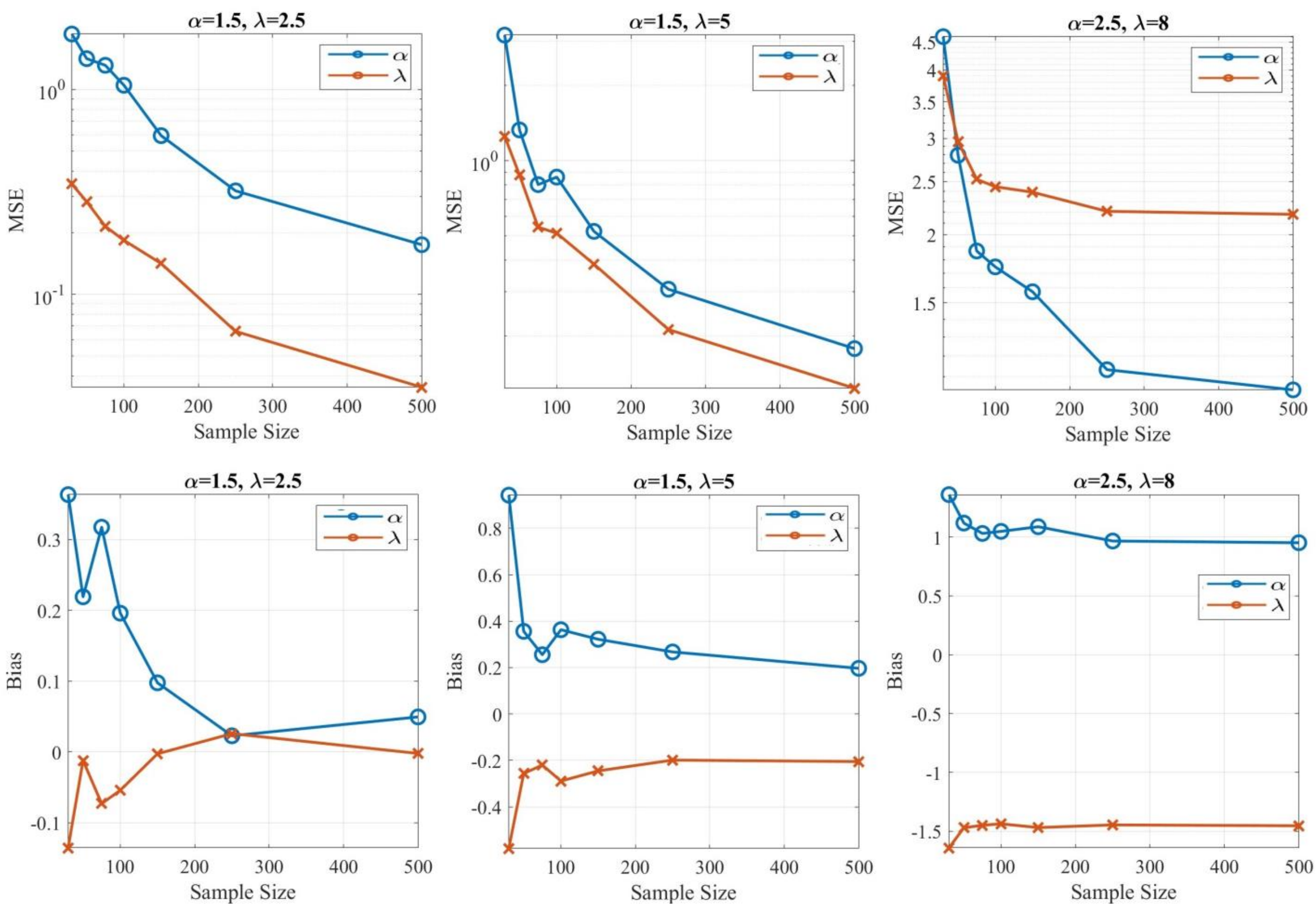

Figure 3. The MSE and bias of the APIIE distribution for different parameters values

3.6 Application APIIE distribution with real data sets

In this section, our focus will be directed towards the examination and interpretation of an authentic data collection that serves to show the advantages of new family associated with the application of the aforementioned APIIE distribution methodology. To gauge the relevance of the model, multiple information criteria were determined. These requirements featured the AIC, which weighs the trade-off between model fit and complexity. Additionally, the CAIC was computed to address potential issues with small sample sizes. The BIC underwent scrutiny to penalize models with a significant number of parameters, demonstrating a preference for simpler models. Finally, the HQIC was integrated in the assessment, which is a modification of the AIC that considers the number of observations in the dataset. AIC, CAIC, and BIC are common criteria for comparing the fit of models to real data. These criteria are defined as follows:

$$AIC = -2lnL_M + 2h_p$$
$$BIC = -2lnL_M + h_p ln\, n$$
$$CAIC = -2lnL_M + h_p(ln\ n + 1)$$
$$HQIC = -2lnL_M + 2h_p ln\,(ln\,(n))$$

where, L_M is maximized of the likelihood function, h_p is number of parameters estimated and *n* is size of data sample.

The first dataset under consideration has been sourced from the research conducted by Bjerkedal in the year 1960, encompassing information pertaining to the survival durations (measured in days) of a total of 72 guinea pigs that were deliberately infected with highly pathogenic tubercle bacilli. This compilation of data is outlined as shown in Table 1.

These data were recently used by Elgarhy et al. [12]. Comparing the distributions chosen by Elgarhy with the APIIE distribution. It turns out that the APIIE distribution is better than the other test distributions, as shown in Table 2.

The second dataset provides a comprehensive depiction of the failures and service durations pertaining to a specific model of windshield, as documented in reference [13]. The dataset in question meticulously outlines the service durations related to a particular model of windshield. This dataset, comprised of 63 instances of Aircraft Windshield service times, is meticulously itemized and presented in a structured format. This data is outlined as shown in Table 3.

This data was recently used by Almarashi et al [14]. They compared the TIHLE distribution with the exponential distribution, and when we studied this evidence on the new distribution, it turned out to be most suitable for data than the other distributions, as shown in Table 4.

The third dataset employed by Hinkley [15], functions as the central element of focus in the present study. This particular dataset comprises a series of thirty consecutive measurements of March precipitation, delineated in inches, pertaining to the Minneapolis/St. Paul region. The information encompassed within this dataset provides significant and noteworthy perspectives on the various patterns and tendencies observed in rainfall within this specific geographical vicinity. This data is outlined as shown in Table 5.

These data were recently used by Sapkota et al. [16]. To demonstrate the performance of the ATE distribution, a number of well-known distributions were chosen for a comparative evaluation. Among these distributions are the Gompertz distribution (GZD), Exponential power distribution (EPD), the Marshall-Olkin Ext-Exponential distribution (MOEED), and the Exponential extension (NHED). This compilation of data is outlined as in Table 6.

According to the resulting criteria values in Table 2, Table 4 and Table 6, we see that the APIIE model is the best according to the criteria of fit that was used compared the other test distributions with which it was compared in the tables.

Table 1. Data on survival durations of guinea pigs

0.1	0.33	0.44	0.56	0.59	0.72	0.74
0.77	0.92	0.93	0.96	1	1	1.02
1.05	1.07	0.07	0.08	1.08	1.08	1.09
1.12	1.13	1.15	1.16	1.2	1.21	1.22
1.22	1.24	1.3	1.34	1.36	1.39	1.44
1.46	1.53	1.59	1.6	1.63	1.63	1.68
1.71	1.72	1.76	1.83	1.95	1.96	1.97
2.02	2.13	2.15	2.16	2.22	2.3	2.31
2.4	2.45	2.51	2.53	2.54	2.54	2.78
2.93	3.27	3.42	3.47	3.61	4.02	4.32
4.58	5.55					

Table 2. Goodness for first dataset

Distribution	AIC	CAIC	BIC	HQIC
EWED	225.041	226.641	224.47	228.666
EED	308.551	308.725	308.266	310.364
WED	298.659	299.012	298.231	301.378
RED	289.026	289.199	288.74	290.838
APII-ED	206.824	206.998	211.377	208.637

Table 3. Data on failures of windshields

0.046	1.436	2.592	0.14	1.492	2.6	0.15
1.58	2.67	0.248	1.719	2.717	0.28	1.794
2.819	0.313	1.915	2.82	0.389	1.92	2.878
0.487	1.963	2.95	0.622	1.978	3.003	0.9
2.053	3.102	0.952	2.065	3.304	0.966	2.117
3.483	1.003	2.137	3.5	1.01	2.141	3.622
1.085	2.163	3.655	1.092	2.183	3.695	1.152
2.24	4.015	1.183	2.341	4.628	1.244	2.435
4.806	1.249	2.464	4.881	1.262	2.543	5.14

Table 4. Goodness for second dataset

Distribution	AIC	CAIC	BIC	HQIC
ED	222.597	223.196	226.883	224.283
TIHLED	211.706	211.906	211.305	213.392
APII-ED	206.113	206.313	210.40002	207.799

Table 5. Data on precipitation to the Minneapolis

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37
2.20	3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32
0.59	0.81	2.81	1.87	1.18	1.35	4.75	2.48	0.96
1.89	0.90	2.05						

Table 6. Goodness for third dataset

Distribution	AIC	BIC	CAIC	HQIC
ATED	82.4562	85.2585	82.9006	83.3527
EPD	84.9537	87.7561	85.3982	85.8502
MOEED	82.7540	85.5564	83.1984	83.6505
GZD	86.1523	88.9547	86.5967	87.0488
NHED	86.8436	89.6459	87.2880	87.7401
APII-ED	80.7381	83.5405	81.1825	81.6346

4. APIIW DISTRIBUTION

In this section, we will apply the new APII-G family with the two-parameter Weibull distribution, where we will obtain a new distribution with three parameters which is called APIIW distribution, by replacing $G(x)$ in Eq. (2) with the CDF of the Weibull distribution, as follows:

$$F(x;\lambda,\alpha,\kappa)=\frac{\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha}-1}{2^{\alpha}-1},(\lambda,\alpha,\kappa,x)>0 \tag{31}$$

And the pdf of the APIIW is defined as the following formula:

$$f(x;\lambda,\alpha,\kappa)=\frac{\alpha k\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}x^{k-1}}{\lambda^{k}(2^{\alpha}-1)} \tag{32}$$
$$(x;\lambda,\alpha,\kappa)>0$$

Some plots of the CDF $f(x;\lambda,\alpha,\kappa)$ and PDF $f(x;\lambda,\alpha,\kappa)$ of the APIIW model, which is plotted for some different value of the parameters α, κ and λ are sketched in Figure 4 and Figure 5. respectively.

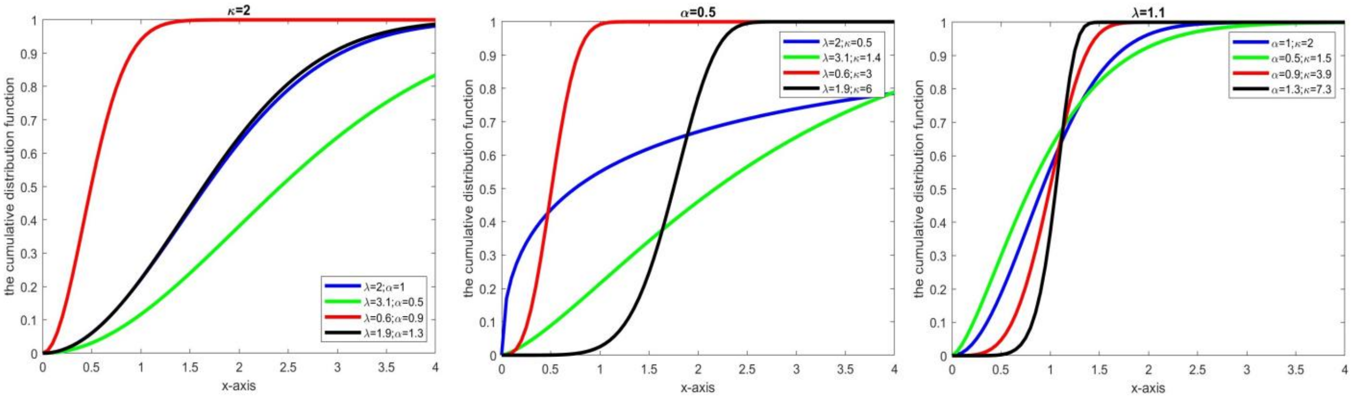

Figure 4. The CDF of APIIW

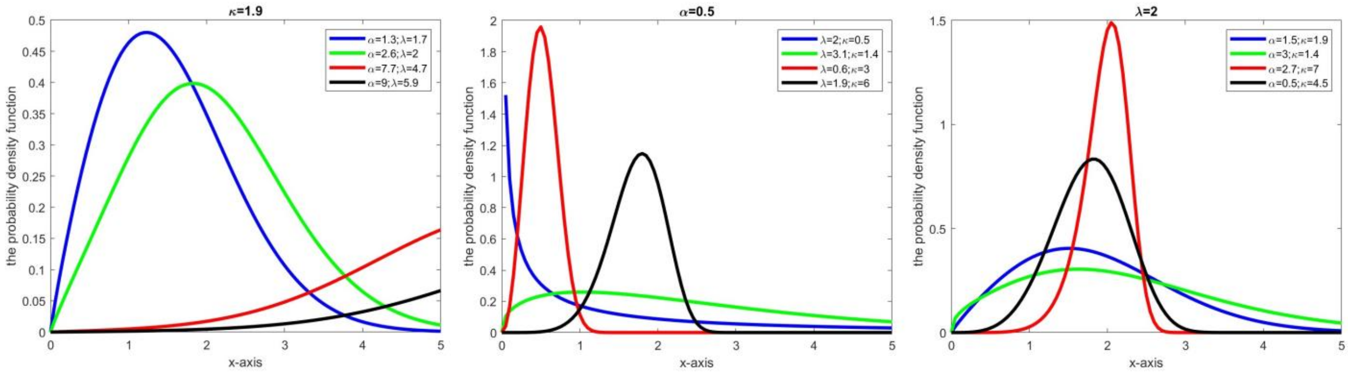

Figure 5. The pdf of APIIW

4.1 The statistical properties of APIIW distribution

We present the functions of reliability, reversed hazard, hazard rate, and the cumulative of the hazard rate of APIIW distribution [17].

The survival function of random variable $X \sim APIIW(\alpha,\lambda,k)$ is defined as follows:

$$\bar{F}(x;\alpha,\lambda,k)=1-F(x;\alpha,\lambda,k)=\frac{2^{\alpha}-\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha}}{2^{\alpha}-1} \tag{33}$$

The function of reverse hazard $r(x:\alpha,\lambda)$ to the APIIW distribution is defined by the following:

$$r(x:\alpha,\lambda,k)=\frac{f(x:\alpha,\lambda,k)}{F(x:\alpha,\lambda,k)}=\frac{\alpha k\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}x^{k-1}}{\lambda^{k}\left(\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha}-1\right)} \tag{34}$$

The function of hazard $h(x:\alpha,\lambda,k)$ of the APIIW distribution is defined by the following:

$$h(x:\alpha,\lambda,k)=\frac{f(x:\alpha,\lambda,k)}{\bar{F}(x:\alpha,\lambda,k)}=\frac{\alpha k\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}x^{k-1}}{\lambda^{k}\left(2^{\alpha}-\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha}\right)} \tag{35}$$

And the function of cumulative hazard $H(x:\alpha,\lambda,k)$ of the APIIW distribution is defined by the following:

$$H(x:\alpha,\lambda,k)=-Ln[1-F(x:\alpha,\lambda,k)]$$
$$=-Ln\left[\frac{2^{\alpha}-\left(2-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{\alpha}}{2^{\alpha}-1}\right] \tag{36}$$

4.2 Moments of APIIW distribution

We will define the r^{th} moment (at the point of origin) for APIIW distribution by the following theorem.

Theorem:
If *X* is a variable to the APIIW distribution, then moment of the order r (about the point of origin) is defined by the following.

$$\acute{\mu}_r = \frac{\alpha\lambda^r}{(2^\alpha - 1)} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - \mathrm{n})!\, n!} \frac{\Gamma(\frac{r+k}{k})}{(n+1)^{\frac{r+k}{k}}} \quad (37)$$

Proof:
By definition the moment of the order r (about the point of origin) $\acute{\mu}_r$ to the variable *X* of APIIW distribution.

$$\acute{\mu}_r = E(x^r) = \int_0^\infty x^r f(x{:}\,\alpha,\lambda,k)\, dx$$

$$= \int_0^\infty \frac{\alpha k \left(2 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^k} x^{r+k-1}}{\lambda^k (2^\alpha - 1)} dx \quad (38)$$

$$= \frac{\alpha k}{\lambda^k (2^\alpha - 1)} \int_0^\infty \left(2 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^k} x^{r+k-1}\, dx$$

$$x, \alpha, \lambda, k > 0, r = 1,2,3, \ldots$$

Let $\boldsymbol{e}^{-\left(\frac{x}{\lambda}\right)^{\boldsymbol{k}}} = y$, if $x \to o$ then $y \to 1$, and if $x \to \infty$ then $y \to 0, dx = \frac{-\lambda}{ky} (-lny)^{\frac{1}{k}-1} dy$. So

$$\acute{\mu}_r = \frac{\alpha\lambda^r}{(2^\alpha - 1)} \int_0^1 (-lny)^{\frac{r+k}{k}-1} (2-y)^{\alpha-1}\, dy \quad (39)$$

Now, Taking the Maclaurin series of formula $(2-y)^{\alpha-1}$, we obtain the following series:

$$(2-y)^{\alpha-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - \mathrm{n})!\, n!} y^n \quad (40)$$

We replace $(2-y)^{\alpha-1}$ in Eq. (39) with Eq. (40)

$$\acute{\mu}_r = \frac{\alpha\, \lambda^r}{(2^\alpha - 1)} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - \mathrm{n})!\, n!} \int_0^1 (-lny)^{\frac{r+k}{k}-1} y^n dy \quad (41)$$

Using the infinitive form in reference [11] and comparing it with Eq. (40), the integration result becomes as follows:

$$\acute{\mu}_r = \frac{\alpha\lambda^r (\alpha-1)!}{(2^\alpha - 1)} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{\alpha-(n+1)}}{(\alpha - 1 - n)!\, n!} \frac{\Gamma(\frac{r+k}{k})}{(n+1)^{\frac{r+k}{k}}} \quad (42)$$

Now, by using theorem (4.2.1) we calculate the mean and variance as follows.

$$E(x) = \frac{\alpha\lambda}{(2^\alpha - 1)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1+k}{k})(-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - \mathrm{n})!\, n!\, (n+1)^{\frac{1+k}{k}}} \quad (43)$$

$$Var(x) = \frac{\alpha\lambda^2}{(2^\alpha - 1)} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - \mathrm{n})!\, n!} \frac{\Gamma\left(\frac{2+k}{k}\right)}{(n+1)^{\frac{2+k}{k}}} - \frac{\alpha^2\lambda^2}{(2^\alpha - 1)^2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)}}{(\alpha - 1 - \mathrm{n})!\, n!} \frac{\Gamma(\frac{1+k}{k})}{(n+1)^{\frac{1+k}{k}}} \right)^2 \quad (44)$$

The moment generating function mgf, denoted by the symbol $M_x(t)$, of APIIW distribution can be find by the following:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x{:}\,\alpha,\lambda,k) \quad (45)$$

where, $f(x{:}\,\alpha,\lambda,k)\, dx$ is pdf of APIIW distribution. By Taylor series for e^{tx} yields, which is $e^{tx} = \sum_{s=0}^{\infty} \frac{t^s x^s}{s!}$.

$$M_x(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \int_0^\infty x^s f(x{:}\,\alpha,\lambda,k)\, dx = \sum_{s=0}^{\infty} \frac{t^s}{s!} \acute{\mu}_s \quad (46)$$

And by substituting Eq. (42) into Eq. (46) then

$$M_x(t) = \frac{\alpha}{(2^\alpha - 1)} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{\lambda^s t^s (-1)^n (\alpha-1)!\, 2^{\alpha-(n+1)} \Gamma(\frac{s+k}{k})}{s!\, (\alpha - 1 - \mathrm{n})!\, n!\, (n+1)^{\frac{s+k}{k}}} \quad (47)$$

4.3 Maximum Likelihood Estimation of APIIW

In this subsection, we find MLE of the two parameters α, λ and k. If $x_1, x_2, x_3 \ldots, x_n$ denote a sample of APIIW, the function of likelihood is defined by references [18, 19].

$$L(\alpha,\lambda,k;x) = \prod_{i=1}^{n} f(x_i{:}\,\alpha,\lambda,k)$$

$$= \prod_{i=1}^{n} \frac{\alpha k \left(2 - e^{-\left(\frac{x_i}{\lambda}\right)^k}\right)^{\alpha-1} e^{-\left(\frac{x_i}{\lambda}\right)^k} x^{k-1}}{\lambda^k (2^\alpha - 1)} \quad (48)$$

$$= \frac{\alpha^n k^n \prod_{i=1}^{n} \left(2 - e^{-\left(\frac{x_i}{\lambda}\right)^k}\right)^{\alpha-1} \prod_{i=1}^{n} x_i^{\,k-1}\, e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\lambda}\right)^k}}{\lambda^{nk} (2^\alpha - 1)^n}$$

Then the logarithm of the likelihood in Eq. (48) will have the following:

$$ln\, L(\alpha,\lambda,k;x) = \mathrm{n} \ln \alpha + \mathrm{n} \ln k - n \ln(\, 2^\alpha - 1) - \mathrm{n}\, \mathrm{k} \ln \lambda - \sum_{i=1}^{n} \left(\frac{x_i}{\lambda}\right)^k + (\mathrm{k}-1) \sum_{i=1}^{n} \ln(x_i) + (\alpha - 1) \sum_{i=1}^{n} ln\left(2 - e^{-\left(\frac{x_i}{\lambda}\right)^k}\right) \quad (49)$$

by taking the partial derivatives of $ln\,L(\alpha,\lambda,k;x))$ with respect to the parameter α , λ and k respectful, as follows:

$$\frac{\partial\, ln\,L(\alpha,\lambda,k;x)}{\partial\lambda}=\frac{-kn}{\lambda}+\frac{k}{\lambda^{k+2}}\sum_{i=1}^{n}(x_i)^k-\frac{(\alpha-1)k}{\lambda^{k+2}}\sum_{i=1}^{n}\frac{(x_i)^k e^{-\left(\frac{x_i}{\lambda}\right)^k}}{\left(2-e^{-\left(\frac{x_i}{\lambda}\right)^k}\right)} \quad (50)$$

By setting Eq. (50) equal to zero, we get

$$\frac{-kn}{\lambda}+\frac{k}{\lambda^{k+2}}\sum_{i=1}^{n}(x_i)^k-\frac{(\alpha-1)k}{\lambda^{k+2}}\sum_{i=1}^{n}\frac{(x_i)^k e^{-\left(\frac{x_i}{\lambda}\right)^k}}{\left(2-e^{-\left(\frac{x_i}{\lambda}\right)^k}\right)}=0 \quad (51)$$

$$\frac{\partial\, ln(L(\alpha,\lambda,k;x))}{\partial\alpha}=\frac{n}{\alpha}-\frac{n2^{\alpha}ln2}{2^{\alpha}-1}+\sum_{i=1}^{n}ln(2-e^{-\left(\frac{x_i}{\lambda}\right)^k}) \quad (52)$$

Also, by setting Eq. (52) equal to zero, we get

$$\frac{n}{\alpha}-\frac{n2^{\alpha}ln2}{2^{\alpha}-1}+\sum_{i=1}^{n}ln(2-e^{-\left(\frac{x_i}{\lambda}\right)^k})=0 \quad (53)$$

$$\frac{\partial\, ln\big(L(\alpha,\lambda,k;x)\big)}{\partial k}=\frac{n}{k}+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i}{\lambda}\right)^k}\left(\frac{x_i}{\lambda}\right)^k\ln\left(\frac{x_i}{\lambda}\right)}{2-e^{-\left(\frac{x_i}{\lambda}\right)^k}}+\sum_{i=1}^{n}ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k\ln\left(\frac{x_i}{\lambda}\right)-nln(\lambda) \quad (54)$$

Also, by setting Eq. (54) equal to zero, we get

$$\frac{n}{k}+(\alpha-1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i}{\lambda}\right)^k}\left(\frac{x_i}{\lambda}\right)^k\ln\left(\frac{x_i}{\lambda}\right)}{2-e^{-\left(\frac{x_i}{\lambda}\right)^k}}+\sum_{i=1}^{n}ln(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k\ln\left(\frac{x_i}{\lambda}\right)-nln(\lambda)=0 \quad (55)$$

By solving Eq. (51), Eq. (53) and Eq. (55), by using numerical methods, then obtain the MLE for all the parameters $\alpha,\ \lambda$ and k.

4.4 Order statistics of APIIW distribution

If $Y_1, Y_2, \dots, Y_n$ denotes the order statistic to the random sample $X_1, X_2, \dots, X_n$ taken from the population distributing by the APIIW distribution with the CDF $F_X(x)$ and pdf $f_X(x)$, then the pdf of random variable Y_j of the order j is defined by following:

$$f_{Y_j}(x)=\frac{n!\,\alpha k\left(2-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^k}x^{k-1}}{(j-1)!\,(n-j)!\,\lambda^k(2^{\alpha}-1)}\left(\frac{\left(2-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{j-1}\left(\frac{2^{\alpha}-\left(2-e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha}}{2^{\alpha}-1}\right)^{n-j} \quad (56)$$

And, if $1\le i\le j\le n$ then the joint pdf of Y_i and Y_j with the order random variable u and v, is

$$f_{Y_i,Y_j}(u,v)f_{X_{(i)},X_{(j)}}(u,v)=\frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left(\frac{\left(2-e^{-\left(\frac{u}{\lambda}\right)^k}\right)^{\alpha}-1}{2^{\alpha}-1}\right)\left(\frac{\left(2-e^{-\left(\frac{v}{\lambda}\right)^k}\right)^{\alpha}-1}{2^{\alpha}-1}-\frac{\left(2-e^{-\left(\frac{u}{\lambda}\right)^k}\right)^{\alpha}-1}{2^{\alpha}-1}\right)^{j-i-1^{i-1}}\left(\frac{2^{\alpha}-\left(2-e^{-\left(\frac{v}{\lambda}\right)^k}\right)^{\alpha}}{2^{\alpha}-1}\right)^{n-j}\frac{\alpha k\left(2-e^{-\left(\frac{u}{\lambda}\right)^k}\right)^{\alpha-1}e^{-\left(\frac{u}{\lambda}\right)^k}u^{k-1}}{\lambda^k(2^{\alpha}-1)}\frac{\alpha k\left(2-e^{-\left(\frac{v}{\lambda}\right)^k}\right)^{\alpha-1}e^{-\left(\frac{v}{\lambda}\right)^k}v^{k-1}}{\lambda^k(2^{\alpha}-1)},0<u<v<\infty \quad (57)$$

The j.p.d.f of n order random variables $Y_1, Y_2, \dots, Y_n$ can extract more than one ranking statistics through our use of similar functions. Thus, the joint pdf for all order statistics $f_{Y_1,Y_2,\dots,Y_n}(x_{1,\dots},x_n)$, taken from the APIIW distribution, as the following

$$f_{Y_1,Y_2\dots,Y_n}\big(x_{1,\dots},x_n\big)=\begin{cases} n!\dfrac{\alpha^n k^n\prod_{=1}^{n}\left(2-e^{-\left(\frac{x_i}{\lambda}\right)^k}\right)^{\alpha-1}\prod_{i=1}^{n}x_i^{\,k-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k}}{\lambda^{nk}(2^{\alpha}-1)^n} \\ 0\ otherwise \\ 0<x_1<\dots<x_n<\infty \end{cases} \quad (58)$$

4.5 Simulation studies of the APIIW distribution

In order to understand and interpret experimentally the adopted estimation method MLE that was studied in this section of the study, we will employ the simulation approach to examine the MLEs of the all parameters of the APIIW distribution. This part includes a description of the Monte Carlo simulation experiment for the research in terms of the sizes of samples which generated when the number of iterations of the simulation 1000. We also present the simulation test results obtained where we used statistical R software to apply the simulation method. The stages of building simulation experiments contain some important stages, which are the stage of choosing the sample size, where n was chosen: (n = 30, 50, 75, 100, 150, 250, 500), the stage of choosing values for the parameters for three experiments, the default values for the parameters (α,λ,k)=(0.5,4,0.5), (α,λ,k)=(1.5,0.5,1.5) and (α,λ,k)=(2.5,1.5,4). The stage of generating appropriate data for the APII-W distribution and calculating the MLE for the three parameters (α,λ,k) and following the calculation of the MSE and the bias. Following the simulation, the outcomes are shown in Figure 6.

4.6 Application APIIW distribution with real data

In this section, as in the Section 3.6, our focus will be directed towards the examination and interpretation of an authentic data collection that serves to show the advantages of the application of the aforementioned APIIW distribution methodology on real datasets. To gauge the relevance of the model, multiple information criteria were determined. These requirements featured the AIC, which weighs the trade-off between model fit and complexity. Additionally, the CAIC was computed to address potential issues with small sample sizes, the BIC underwent scrutiny to penalize models with a significant number of parameters, demonstrating a preference for simpler models. Finally, the HQIC was integrated in the assessment, which is a modification of the AIC that considers the number of observations in the dataset.

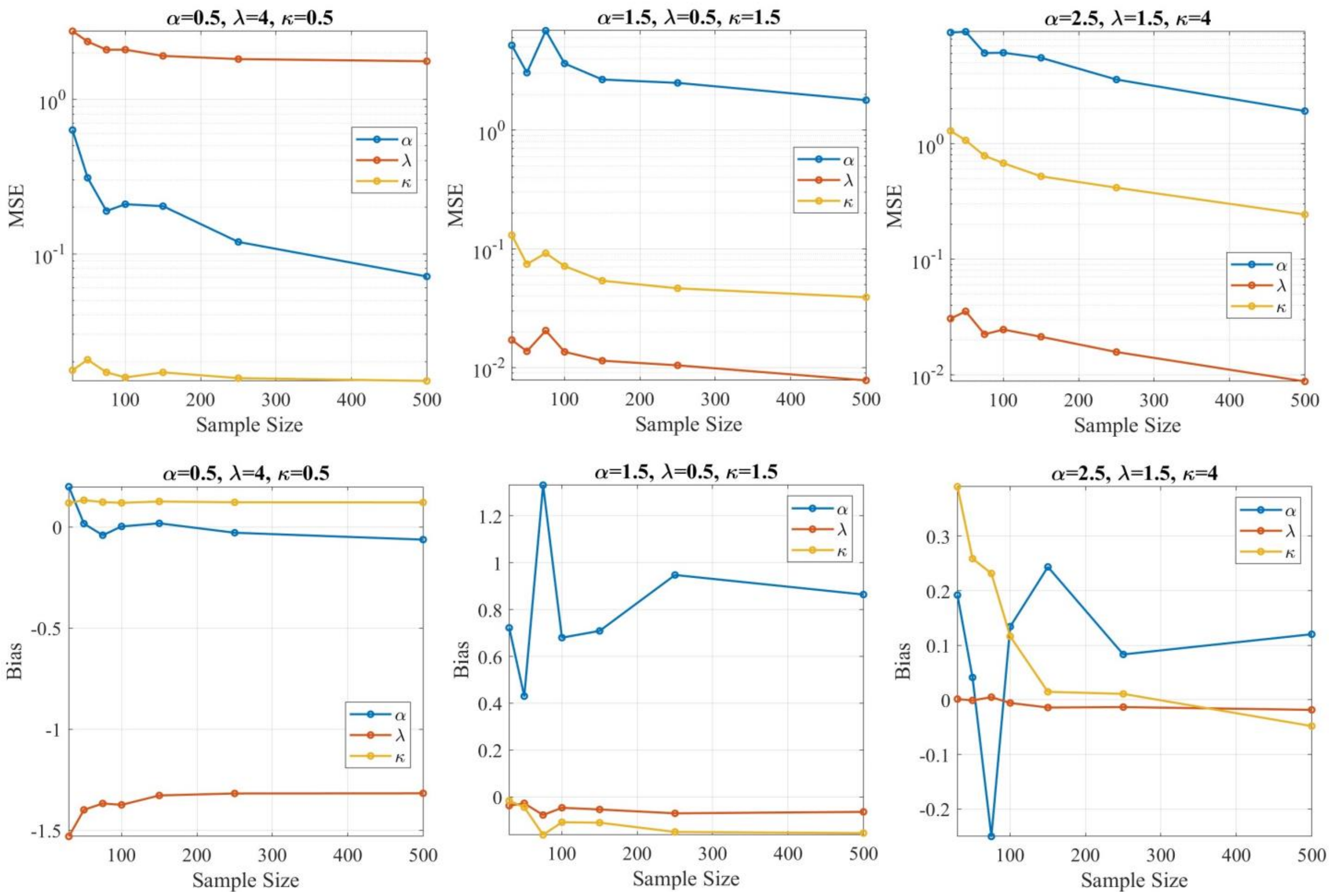

Figure 6. The MSE and the bias of the APIIW distribution for different parameters values

The first authentic dataset illustrates an uncensored dataset derived from Nichols and Padgett's research on the breaking stress of the carbon fibers (measured in Gba) [20]. This dataset is presented as shown in Table 7.

These data were recently used by Hassan and Hemeda [21]. By comparing the distributions, they chose with the APIIW distribution, we found that the new APIIW distribution is better than the other distributions, as shown in Table 8.

The second dataset showcases the durations of remission (expressed in months) for a specific group of 128 patients diagnosed with bladder cancer, as detailed by Lee and Wang in 2003. This dataset is shown in Table 9.

Table 7. Data on the stress of the carbon fibres

3.7	2.74	2.73	2.5	3.6	3.11	3.27	2.87	1.47
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15
4.9	3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67
2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31
2.85	3.56	3.15	2.55	2.59	2.38	2.77	1.92	3.68
2.97	1.36	0.98	2.67	4.91	3.68	1.84	1.59	3.19
1.57	0.81	5.56	1.73	1.59	2.00	2.48	0.85	1.61
2.79	4.70	2.03	1.61	2.21	1.89	2.88	2.82	2.05
3.65								

Table 8. Goodness for first dataset

Distribution	AIC	CAIC	BIC	HQIC
AWBXIID	1018.17	1024.43	1027.40	1020.9
AWUD	1021.24	1026.06	1032.40	1024.0
EMWD	1927.80	1935.70	1939.81	1930.6
TEMWD	1452.15	1487.27	1534.74	1454.9
APII-WD	211.237	211.580	218.149	213.99

Table 9. Data on duration of remission of bladder cancer

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63
0.20	2.23	3.52	4.98	6.97	9.02	13.29	0.40
2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69
4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
1.46	11.79	18.10	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	12.02	2.02	3.31	4.51
6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
2.07	21.73	3.36	6.93	8.65	12.63	22.69	

These data were recently used by Almetwally [22]. So, when we studied this evidence on the new APII distribution, it turned out to be better than the other two distributions, as shown in Table 10.

The third dataset comprises 76 observations pertaining to the endurance limits of fatigue fracture in Kevlar 373/epxy under constant pressure at a stress level of 90%, until all specimens experienced failure. This dataset is presented as shown in Table 11.

These data were recently used by Selim [23]. The above-mentioned dataset was utilized by him for fitting to the inv. generated power Weibull (IGPWD), inv. Naderajah-Haghighi (INHD), inv. Weibull (IWD), and inv. exponential (IED). So when we studied this evidence on the new APIIW distribution, it turned out to be better than the other two distributions, as shown in Table 12.

Based on the results in Tables 4-6, we see that the APIIW model is the best according to the criteria of fit to the data that was used compared to the other distributions with which it was compared in the tables.

Table 10. Goodness for the second dataset

Distribution	AIC	CAIC	BIC	HQIC
IED	922.765	922.796	925.617	923.923
IWD	892.002	892.098	897.706	894.319
INHD	866.118	866.214	871.822	868.436
IRD	1550.683	1550.715	1553.535	1551.842
IGPWD	859.819	860.013	868.375	863.296
APIIWD	826.604	826.798	835.160	830.080

Table 11. Data on endurance limits of fatigue fracture

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451
0.4763	0.5650	0.5671	0.6566	0.6748	0.6751
0.6753	0.7696	0.8375	0.8391	0.8425	0.8645
0.8851	0.9113	0.9120	0.9836	1.0483	1.0596
1.0773	1.1733	1.2570	1.2766	1.2985	1.3211
1.3503	1.3551	1.4595	1.4880	1.5728	1.5733
1.7083	1.7263	1.7460	1.7630	1.7746	1.8275
1.8375	1.8503	1.8808	1.8878	1.8881	1.9316
1.9558	2.0048	2.0408	2.0903	2.1093	2.1330
2.2100	2.2460	2.2878	2.3203	2.3470	2.3513
2.4951	2.5260	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005
5.4435	5.5295	6.5541	9.0960		

Table 12. Goodness for the third dataset

Distribution	AIC	BIC	CAIC	HQIC
IED	328.203	330.5337	328.257	329.1344
IRD	693.8294	696.1601	693.8834	694.7608
IWD	311.0787	315.7401	311.2431	312.9416
INHD	293.0930	297.7545	293.2574	294.9560
IGPWD	270.1234	277.1156	270.4568	272.9179
APIIWD	247.7863	254.7785	248.1197	250.5808

5. CONCLUSIONS

The significance of extended distributions was initially acknowledged within the domain of financial sciences and subsequently recognized in various other applied disciplines, including engineering and medical sciences. In order to accommodate data within these fields, a multitude of methodologies has been developed. Within this framework, we have examined a two-parameter heavy-tailed model, designated as the Alpha Power Type II Exponential distribution, alongside a three-parameter heavy-tailed model, referred to as the Alpha Power Type II Weibull distribution, The two new models serve as specific cases of a novel family approach that facilitates closed-form expressions for certain fundamental mathematical and associated properties. The introduced class is termed the APII-G family. The efficacy of the proposed family of distributions has been substantiated through the analysis of six distinct data sets originating from the domains of medical, engineering, and financial sciences, demonstrating that the two models exhibit superior performance relative to established heavy-tailed distribution alternatives. The family developed within this research represents a promising methodological advancement for the modeling of data in the context of distribution theory, and may prove beneficial for scholars engaged with such data sets. Consequently, the novel two models may function as a formidable competitive alternative to existing models in the field.

Future work includes the following aspects:

(1) A bivariate extension of the Alpha Power Type II Exponential distribution;

(2) A bivariate extension of the Alpha Power Type II Weibull distribution;

(3) Using the APII-G family to expand the Rayleigh distribution;

(4) Using the APII-G family to expand the continuous uniform distribution;

(5) Using the APII-G family to expand the Gamma distribution;

(6) Modeling engineering data with APII-G family extension.

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