International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 12, No. 1, January, 2025, pp. 205-218
Journal homepage: http://iieta.org/journals/mmep

Legendre Operational Differential Matrix for Solving Fuzzy Differential Equations

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https://doi.org/10.18280/mmep.120122

Received: 3 March 2024
Revised: 2 May 2024
Accepted: 10 May 2024
Available online: 25 January 2025

Keywords:
fuzzy differential equation, triangular fuzzy function, fuzzy approximate-analytical solution, Legendre operational differential matrix

ABSTRACT

In this work, we used the Legendre operational differential matrix method based on Tau method to obtain the fuzzy approximate-analytical solutions of the fuzzy differential equations in which the coefficients are triangular fuzzy functions. This method allows for the fuzzy solution of the fuzzy initial (or boundary) value problems to be computed in the form of an infinite fuzzy series. Also, this method enables to approximate the fuzzy exact-analytical solutions with high efficiency, as these solutions can be resorted to if it is not possible to find the exact solutions of these fuzzy problems. We introduced a comparison between the approximate solutions that we computed and the exact solutions of the chosen problem, as we found the absolute error. According to the numerical results, the series solutions that we found are accurate solutions and very close to the exact solutions.

1. INTRODUCTION

In fuzzy differential equations (FDEs), the coefficients may be non-fuzzy variable coefficients or fuzzy variable coefficients. The first type means that the coefficients are real-valued functions while the second type means that the coefficients are fuzzy-valued functions. There are many different types of the fuzzy variable coefficients, the most important of which are the triangular fuzzy function coefficients and the trapezoidal fuzzy function coefficients.

In 2012, the researchers Gasilov et al. [1] introduced a new concept in the fuzzy functions called the triangular fuzzy functions (TFFs). The aforementioned researchers used fuzzy linear transformations method to find the fuzzy exact-analytical solution (FEAS) of the second order linear FDE in which the coefficients are TFF. The researchers continued to study the subject of TFFs and presented different fuzzy methods for solving linear FDE with TFF coefficients, among them we mention: Mondal and Roy [2] used fuzzy Lagrange multiplier method for solving first order linear FDE with TFF coefficients. Eljaoui et al. [3], Patel and Desai [4] and Citil [5] all used fuzzy Laplace transform method to solve different types of second order linear FDE with TFF coefficients. Alikhani and Mostafazadeh [6] added some important observations to the topic of TFFs and then solved first order linear FDE with TFF coefficients using fuzzy cross product method. Jamal et al. [7] studied the existence and the uniqueness results for the fuzzy solution of the first order linear FDE with TFF coefficients, they introduced some important theorems that ensure the fuzzy solution is exist and unique. Moreover, they used fuzzy linear correlated method for solving first order linear FDE.

The above-mentioned methods dealt with the linear case of the FDE with TFF coefficients and did not address the non-linear FDE. Moreover, the fuzzy solutions that have been obtained are exact-analytical solutions, as is well known, the exact-analytical solutions are not always found and sometimes may be difficult. From the above, we can conclude that the current existing methods that use to solve the FDE with TFF coefficients are exact-analytical methods that solve linear equations, as the non-linear FDEs with TFF coefficients have not been solved, moreover, the numerical solutions and the approximate-analytical solutions of these equations have not been obtained. Thus, finding the fuzzy approximate-analytical solution (FAAS) of the FDEs with TFF coefficients is necessary. Therefore, in this work we will search on finding the FAAS of the FDE in which the coefficients are TFF. Also, the FDE that we will deal with will be linear and non-linear, as well as these equations will be with fuzzy initial conditions and fuzzy boundary conditions. The method that we will use is the fuzzy function of Legendre operational differential matrix method (LODMM) based on Tau method.

The beginning of using LODMM based on Tau method dates back to 2014, when the researchers Jung et al. [8] used this method to obtain the approximate-analytical solutions of the second order non-fuzzy differential equations (NFDDs) with initial conditions. Also, in 2019, the researcher Edeo used the same method to obtain the approximate-analytical solutions of the second order NFDDs with boundary conditions [9]. Therefore, during this research, we will expand this method to the fuzzy case so that we can solve the FDEs. The importance of this method lies in that it enables us to find the FAAS with high accuracy and in simple and clear steps. Therefore, the solutions resulting from this method can be an efficient alternative to the exact solutions in case it does not exist.

2. FUNDAMENTAL CONCEPTS IN FUZZY SET THEORY

The fundamental definitions in the fuzzy set theory, which are: fuzzy set, $\alpha-$ level set, fuzzy number, ...etc. can be found in details in studies [9-12]. In this section, we will touch on definitions that are directly related to our work.

Definition (2.1) Triangular Fuzzy Number

Let u_1, u_2 and u_3 are real numbers with $u_1 \leq u_2 \leq u_3$. Then the triangular fuzzy number (TFN) can be written as $\tilde{u} = (u_1, u_2, u_3)$ and it is a fuzzy number with membership function [1]:

$$\mu_{\tilde{u}}(x) = \begin{cases} (x-u_1)/(u_2-u_1) & \text{if } u_1 \leq x \leq u_2 \\ (u_3-x)/(u_3-u_2) & \text{if } u_2 \leq x \leq u_3 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Remark (2.2)

The parametric form of the TFN $\tilde{u} = (u_1, u_2, u_3)$ can be defined as [1]:

$$[\tilde{u}]_\alpha = [[u]^L_\alpha, [u]^U_\alpha] = [(u_2-u_1)\alpha + u_1, (u_2 - u_3)\alpha + u_3]; \forall \alpha \in [0,1] \quad (2)$$

Example (2.3)

The parametric form of the TFN $\tilde{u} = (5, 7, 10)$ can be written as:

$$[\tilde{u}]_\alpha = [[u]^L_\alpha, [u]^U_\alpha] = [2\alpha + 5, -3\alpha + 10]$$

Then, we get:

$$[u]^L_\alpha = 2\alpha + 5$$
$$[u]^U_\alpha = -3\alpha + 10$$

The functions $[u]^L_\alpha$ and $[u]^U_\alpha$ represents the lower bound and the upper bound of parametric form of $\tilde{u}$, respectively.

Definition (2.4) Triangular Fuzzy Function

Let F_a, F_b and $F_c : I \rightarrow R$ for some interval $I \subseteq R$ are continuous real-valued functions such that:

$$F_a(t) \leq F_b(t) \leq F_c(t), \forall t \in I$$

We call the fuzzy set $\tilde{F}(t)$, determined by the membership function [1]:

$$\mu_{\tilde{F}(t)}(x) = \begin{cases} (x - F_a(t))/(F_b(t) - F_a(t)) \text{ if } F_a(t) \leq x \leq F_b(t) \\ (F_c(t) - x)/(F_c(t) - F_b(t)) \text{ if } F_b(t) \leq x \leq F_c(t) \\ 0 \quad \text{otherwise} \end{cases} \quad (3)$$

as TFF for all $x, t \in I$, and it is denoted by: $\tilde{F}(t) = (F_a(t), F_b(t), F_c(t))$.

From the above definition, we can conclude that any fuzzy function produces a TFN for any real input can be described as TFF.

Example (2.5)

The function: $\tilde{F}(t) = (-16t^2 - 21t - 8, -6t^2 + 15t - 6, 12t^2 + 63t + 2)$, $t \in (0,1)$ is a TFF. Since:

If we assume that $F_a(t) = -16t^2 - 21t - 8$, $F_b(t) = -6t^2 + 15t - 6$ and $F_c(t) = 12t^2 + 63t + 2$.

Then, it is clear that the functions F_a, F_b and F_c are continuous real-valued functions and $F_a(t) \leq F_b(t) \leq F_c(t), \forall t \in (0,1)$.

This means that $\tilde{F}(t)$ produces a TFN for any $t \in (0,1)$.

Thus, $\tilde{F}(t)$ is TFF.

Remark (2.6)

The parametric form of the TFF $\tilde{F}(t) = (F_a(t), F_b(t), F_c(t))$ can be defined as [1]:

$$[\tilde{F}(t)]_\alpha = [[F(t)]^L_\alpha, [F(t)]^U_\alpha] = [(F_b(t) - F_a(t))\alpha + F_a(t), (F_b(t) - F_c(t))\alpha + F_c(t)]; \forall \alpha \in [0,1] \text{ and } \forall t \in I \quad (4)$$

Example (2.7)

The parametric form of the TFF $\tilde{F}(t)$ in the Example (2.5) can be written as:

$$[\tilde{F}(t)]_\alpha = [[F(t)]^L_\alpha, [F(t)]^U_\alpha] = [(10t^2 + 36t + 2)\alpha - 16t^2 - 21t - 8, (-18t^2 - 48t - 8)\alpha + 12t^2 + 63t + 2]$$

Then, we get:

$$[F(t)]^L_\alpha = (10t^2 + 36t + 2)\alpha - 16t^2 - 21t - 8$$
$$[F(t)]^U_\alpha = (-18t^2 - 48t - 8)\alpha + 12t^2 + 63t + 2$$

The functions $[F(t)]^L_\alpha$ and $[F(t)]^U_\alpha$ represents the lower bound and the upper bound of parametric form of $\tilde{F}(t)$, respectively. For more details, see studies [1, 2].

3. SHIFTED LEGENDRE POLYNOMIALS

The Legendre polynomials of order r are defined on the interval [-1, 1] and are denoted by $L_r(z)$. These polynomials can be described as [8]:

$$L_0(z) = 1 \quad (5)$$

$$L_1(z) = z \quad (6)$$

$$L_2(z) = \frac{3}{2}z^2 - \frac{1}{2} \quad (7)$$

$$L_3(z) = \frac{5}{2}z^3 - \frac{3}{2}z \quad (8)$$

$$L_4(z) = \frac{35}{8}z^4 - \frac{15}{4}z^2 + \frac{3}{8} \quad (9)$$

$$\vdots$$

$$L_{r+1}(z) = \frac{2r+1}{r+1}zL_r(z) - \frac{r}{r+1}L_{r-1}(z); \quad r = 1,2,3,... \quad (10)$$

In order to use the Legendre polynomials on the interval [0, 1], the so-called shifted Legendre polynomials (SLPs) are defined by introducing z=2t-1.

Let the SLPs $L_r(2t-1)$ be denoted by $p_r(t)$, then $p_r(t)$ can be obtained as follows:

$$p_0(t) = 1 \quad (11)$$

$$p_1(t) = 2t - 1 \quad (12)$$

$$p_2(t) = 6t^2 - 6t + 1 \quad (13)$$

$$p_3(t) = 20t^3 - 30t^2 + 12t - 1 \quad (14)$$

$$p_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t + 1 \quad (15)$$

$$\vdots$$

$$p_{r+1}(t) = \frac{2r+1}{r+1}(2t-1)p_r(t) - \frac{r}{r+1}p_{r-1}(t); \quad r = 1,2,3,\dots \quad (16)$$

In this work, we will use the SLPs as a prime factor to get the FAAS of the FDE. And this will be done based on Tau method, since the basis of Tau method is a definite integral with in the period $[0, 1]$, which is the same period for which the SLPs are defined.

Finding the FAAS of the FDE is based on an infinite fuzzy series. This series is called the solution series, which is a convergent series from which the first terms are taken to approximate the FEAS of FDE. The mathematical formula for this series consists of SLPs and shifted Legendre coefficients (SLCs), which means that the SLP constitute the basic element in finding the desired approximate solution, as we will notice in the next section.

4. DESCRIPTION OF LEGENDRE OPERATIONAL DIFFERENTIAL MATRIX METHOD

To describe LODMM in a simple way, we will consider the following general form of the second order NFDD:

$$x''(t) = f\big(t, x(t), x'(t)\big), t \geq 0 \quad (17)$$

With:

$$x(0) = a, \qquad x'(0) = b \quad (18)$$

The solution x(t) of problem (17) can be approximated as [8, 13]:

$$x(t) = \sum_{r=0}^{\infty} c_r\, p_r(t) \quad (19)$$

where,
$p_r(t)$ are SLPs,
c_r are SLCs.
The coefficients c_r are given by:

$$c_r = (2r+1)\int_0^1 x(t)\, p_r(t)\, dt; r = 0,1,2,\dots \quad (20)$$

Finding the approximate solution x(t) depends mainly on finding the constants c_r as we will notice later.

By considering only the first (m+1) terms of the series solution in Eq. (19), we have:

$$(t) \approx \sum_{r=0}^{m} c_r p_r(t) \quad (21)$$

This means that:

$$x(t) \approx c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + \cdots + c_m\, p_m(t) \quad (22)$$

In matrix form, we get:

$$x(t) \approx C^T W(t) \quad (23)$$

where,
$C^T = [c_0, c_1, c_2, \dots, c_m]$ is SLCs vector,
$W(t) = [p_0(t),\ p_1(t),\ p_2(t), \dots,\ p_m(t)]^T$ is SLPs vector.
The derivative of W(t) is:

$$\frac{d\big(W(t)\big)}{dt} = D^{(1)}W(t) \quad (24)$$

where,
$D^{(1)}$ is $(m+1) \times (m+1)$ operational differential matrix, which is given by:

$$D^{(1)} = \big(d_{ij}\big) = \begin{cases} 4j-2, & \text{if } j = i-k \\ 0, & \text{otherwise} \end{cases} \quad (25)$$

where,

$$k = \begin{cases} 1,3,5,\dots m & \text{, if m is odd} \\ 1,3,5,\dots m-1, & \text{if m is even} \end{cases} \quad (26)$$

In this work, based on many applied problems that we solved by using different values of m, we will consider $m = 4$, since this value of m is suitable for the approximation.

Therefore, form $= 4$, we get:

$$D^{(1)} = \big(d_{ij}\big) = \begin{cases} 4j-2, & \text{if } j = i-1 \text{ or } j = i-3 \\ 0, & \text{otherwise} \end{cases} \quad (27)$$

Thus, the operational differential matrix will be:

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{bmatrix} \quad (28)$$

For the nth order derivative, we obtain:

$$\frac{d^n\big(W(t)\big)}{dt^n} = \big(D^{(1)}\big)^n W(t) = D^{(n)}W(t); \quad n = 1,2,3,\dots \quad (29)$$

where, $\big(D^{(1)}\big)^n$ denotes the matrix powers.
Thus, we find:

$$D^{(2)} = D^{(1)} \times D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 \end{bmatrix} \quad (30)$$

Therefore, we get:

$$x(t) = C^T W(t) \quad (31)$$

This means that:

$$x'(t) = \frac{d\big(x(t)\big)}{dt} = \frac{d\big(C^T W(t)\big)}{dt} = C^T\frac{d\big(W(t)\big)}{dt} \quad (32)$$

This gives:

$$x'(t) = C^T D^{(1)} W(t) \tag{33}$$

Also, we have:

$$x''(t) = \frac{d(x'(t))}{dt} = \frac{d\left(C^T D^{(1)} W(t)\right)}{dt} = C^T \frac{d\left(D^{(1)} W(t)\right)}{dt} \tag{34}$$

This gives:

$$x''(t) = C^T D^{(2)} W(t) \tag{35}$$

where,

$$C^T = [c_0, c_1, c_2, c_3, c_4] \tag{36}$$

$$W(t) = [p_0(t),\ p_1(t),\ p_2(t),\ p_3(t),\ p_4(t)]^T \tag{37}$$

Therefore, from the above description, we will conclude the following:

- From Eq. (31), we find:

$$\begin{aligned} x(t) &= C^T W(t) \\ &= [c_0, c_1, c_2, c_3, c_4][p_0(t), p_1(t),\ p_2(t),\ p_3(t), p_4(t)]^T \end{aligned} \tag{38}$$

This gives:

$$\begin{aligned} x(t) = c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) \\ + c_4\, p_4(t) \end{aligned} \tag{39}$$

- From Eq. (33), we find:

$$x'(t) = C^T D^{(1)} W(t) = [c_0, c_1, c_2, c_3, c_4] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{bmatrix} [p_0(t), p_1(t), p_2(t), p_3(t), p_4(t)]^T \tag{40}$$

This gives:

$$\begin{aligned} x'(t) = 2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) \\ + 10c_3p_2(t) + 14c_4p_3(t) \end{aligned} \tag{41}$$

- From Eq. (35), we find:

$$x''(t) = C^T D^{(2)} W(t) = [c_0, c_1, c_2, c_3, c_4] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 \end{bmatrix} [p_0(t), p_1(t), p_2(t), p_3(t), p_4(t)]^T \tag{42}$$

This gives:

$$\begin{aligned} x''(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) \\ + 140c_4p_2(t) \end{aligned} \tag{43}$$

Now, by using the Eqs. (39), (41) and (43) we can find the residual function R(t) of problem (17) as follows:
From Eq. (17), we have:

$$R(t) = x''(t) - f\big(t, x(t), x'(t)\big) \tag{44}$$

This gives:

$$\begin{aligned} R(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) \\ + 140c_4p_2(t) \\ - f(\, t, c_0\, p_0(t) + c_1\, p_1(t) \\ + c_2\, p_2(t) + c_3\, p_3(t) \\ + c_4\, p_4(t), 2c_1p_0(t) + 2c_3p_0(t) \\ + 6c_2p_1(t) + 6c_4p_1(t) \\ + 10c_3p_2(t) + 14c_4p_3(t)) \end{aligned} \tag{45}$$

Then, we can apply Tau method, which can be defined as:

$$\int_0^1 R(t)\, p_r(t)\, dt = 0; r = 0,1,2, \dots, m-2 \tag{46}$$

For m=4, we get:

$$\int_0^1 R(t)\, p_0(t)\, dt = 0 \tag{47}$$

$$\int_0^1 R(t)\, p_1(t)\, dt = 0 \tag{48}$$

$$\int_0^1 R(t)\, p_2(t)\, dt = 0 \tag{49}$$

From the Eqs. (47)-(49), we get three linear (or non-linear) equations. In addition, two linear equations can be obtained by applying the initial conditions of Eq. (18). Therefore, we will get a system of five linear (or non-linear) equations, and then by solving this system, we will obtain the constants c_0, c_1, c_2, c_3 and c_4.

Through these constants, the approximate-analytical solution of the problem (17) can be obtained, which is:

$$\begin{aligned} x(t) \approx c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) \\ + c_4\, p_4(t) \end{aligned} \tag{50}$$

It is necessary to note that the above mathematical description can be repeated if the NFDD of problem (17) is boundary value problem or if it is higher order initial (or boundary) value problems.

From the above, the solution steps for LODMM based on Tau method can be summarized as follows:

1. We adjust the order of the DE so that it is always an equation of the second order.
2. We use the Eqs. (39), (41) and (43) to obtain the residual function R(t) of the DE in step (1).
3. We use the Eqs. (47)-(49), to get three linear (or non-linear) algebraic equations.

4. We apply the initial conditions (or the boundary conditions) of the DE in step (1) to get two linear algebraic equations.
5. From the steps (3) and (4), we have a system of five linear (or non-linear) algebraic equations.
6. We solve the system in step (5) to find the constants c_0, c_1, c_2, c_3 and c_4.
7. We substitute the constants of step (6) in the Eq. (50) to get the approximate-analytical solution of the DE in step (1).

The advantage of the LODMM based on Tau method over the other methods is therefore:

- It is computational less cost,
- It needs less computational time and effort and
- It has better accuracy.

The theorems that ensure the convergence of the solution and the stability of the errors analysis can be found in studies [8, 13].

5. FUZZY DIFFERENTIAL EQUATIONS WITH TRIANGULAR FUZZY FUNCTION COEFFICIENTS

The general form of the nth-order linear FDE with TFF coefficients is [1, 14]:

$$\tilde{x}^{(n)}(t) + \tilde{a}_{n-1}(t)\tilde{x}^{(n-1)}(t) + \tilde{a}_{n-2}(t)\tilde{x}^{(n-2)}(t) + \cdots + \tilde{a}_1(t)\tilde{x}'(t) + \tilde{a}_0(t)\tilde{x}(t) = \tilde{k}(t), \quad t \in [0,\infty) \subseteq R \tag{51}$$

With:

$$\tilde{x}(0) = x(0;\alpha) = v_0(\alpha) \tag{52}$$

$$\tilde{x}'(0) = x'(0;\alpha) = v_1(\alpha) \tag{53}$$

$$\tilde{x}''(0) = x''(0;\alpha) = v_2(\alpha) \tag{54}$$

$$\vdots$$

$$\tilde{x}^{(n-1)}(0) = x^{(n-1)}(0;\alpha) = v_{n-1}(\alpha) \tag{55}$$

where,

$\tilde{a}_{n-1}(t), \tilde{a}_{n-2}(t), \ldots, \tilde{a}_1(t), \tilde{a}_0(t)$ and $\tilde{k}(t)$ are TFFs.

$v_0(\alpha)$, $v_1(\alpha)$, $v_2(\alpha), \ldots, v_{n-1}(\alpha)$ are TFNs.

Since $\tilde{a}_{n-1}(t), \tilde{a}_{n-2}(t), \ldots, \tilde{a}_1(t), \tilde{a}_0(t)$ and $\tilde{k}(t)$ are TFFs, then we must have:

$$\tilde{a}_{n-1}(t) = a_{n-1}(t;\alpha) = (h_1(t), h_2(t), h_3(t)) \tag{56}$$

$$\tilde{a}_{n-2}(t) = a_{n-2}(t;\alpha) = (h_4(t), h_5(t), h_6(t)) \tag{57}$$

$$\vdots$$

$$\tilde{a}_1(t) = a_1(t;\alpha) = (h_7(t), h_8(t), h_9(t)) \tag{58}$$

$$\tilde{a}_0(t) = a_0(t;\alpha) = (h_{10}(t), h_{11}(t), h_{12}(t)) \tag{59}$$

$$\tilde{k}(t) = k(t;\alpha) = (h_{13}(t), h_{14}(t), h_{15}(t)) \tag{60}$$

where, $h_1(t)$, $h_2(t), \ldots, h_{15}(t)$ are continuous real-valued functions.

Since $v_0(\alpha)$, $v_1(\alpha), v_2(\alpha), \ldots, v_{n-1}(\alpha)$ are TFNs, then we must have:

$$v_0(\alpha) = (b_1, b_2, b_3) \tag{61}$$

$$v_1(\alpha) = (b_4, b_5, b_6) \tag{62}$$

$$v_2(\alpha) = (b_7, b_8, b_9) \tag{63}$$

$$\vdots$$

$$v_{n-1}(\alpha) = (b_{10}, b_{11}, b_{12}) \tag{64}$$

where, b_1, $b_2, \ldots, b_{12}$ are real numbers.

By using the concepts that we introduced in section two, we write the parametric form of the Eqs. (56)-(64) as follows:

$$a_{n-1}(t;\alpha) = [a_{n-1}(t)]_\alpha = [[a_{n-1}(t)]^L_\alpha, [a_{n-1}(t)]^U_\alpha] = [(h_2(t) - h_1(t))\alpha + h_1(t), (h_2(t) - h_3(t))\alpha + h_3(t)] \tag{65}$$

$$a_{n-2}(t;\alpha) = [a_{n-2}(t)]_\alpha = [[a_{n-2}(t)]^L_\alpha, [a_{n-2}(t)]^U_\alpha] = [(h_5(t) - h_4(t))\alpha + h_4(t), (h_5(t) - h_6(t))\alpha + h_6(t)] \tag{66}$$

$$\vdots$$

$$a_1(t;\alpha) = [a_1(t)]_\alpha = [[a_1(t)]^L_\alpha, [a_1(t)]^U_\alpha] = [(h_8(t) - h_7(t))\alpha + h_7(t), (h_8(t) - h_9(t))\alpha + h_9(t)] \tag{67}$$

$$a_0(t;\alpha) = [a_0(t)]_\alpha = [[a_0(t)]^L_\alpha, [a_0(t)]^U_\alpha] = [(h_{11}(t) - h_{10}(t))\alpha + h_{10}(t), (h_{11}(t) - h_{12}(t))\alpha + h_{12}(t)] \tag{68}$$

$$k(t;\alpha) = [k(t)]_\alpha = [[k(t)]^L_\alpha, [k(t)]^U_\alpha] = [(h_{14}(t) - h_{13}(t))\alpha + h_{13}(t), (h_{14}(t) - h_{15}(t))\alpha + h_{15}(t)] \tag{69}$$

$$v_0(\alpha) = [v_0]_\alpha = [[v_0]^L_\alpha, [v_0]^U_\alpha] = [(b_2 - b_1)\alpha + b_1, (b_2 - b_3)\alpha + b_3] \tag{70}$$

$$v_1(\alpha) = [v_1]_\alpha = [[v_1]^L_\alpha, [v_1]^U_\alpha] = [(b_5 - b_4)\alpha + b_4, (b_5 - b_6)\alpha + b_6] \tag{71}$$

$$v_2(\alpha) = [v_2]_\alpha = [[v_2]^L_\alpha, [v_2]^U_\alpha] = [(b_8 - b_7)\alpha + b_7, (b_8 - b_9)\alpha + b_9] \tag{72}$$

$$\vdots$$

$$v_{n-1}(\alpha) = [v_{n-1}]_\alpha = [[v_{n-1}]^L_\alpha, [v_{n-1}]^U_\alpha] = [(b_{11} - b_{10})\alpha + b_{10}, (b_{11} - b_{12})\alpha + b_{12}] \tag{73}$$

There are many main approaches in solving the FDE. The most popular approach is so called the defuzzification. The principal idea in this approach is converting the FDE into a system of NFDEs by using the properties of the α-level sets (For more details, see studies [15-19]).

Therefore, for solving problem (51), we convert it into a system of nth-order linear NFDEs as follows:

$$[x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + a_{n-2}(t)x^{(n-2)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t)]_\alpha = [k(t)]_\alpha \tag{74}$$

With:

$$[x(0)]_\alpha = [v_0]_\alpha \tag{75}$$

$$[x'(0)]_\alpha = [v_1]_\alpha \tag{76}$$

$$[x''(0)]_\alpha = [v_2]_\alpha \tag{77}$$

$$\vdots$$

$$[x^{(n-1)}(0)]_\alpha = [v_{n-1}]_\alpha \tag{78}$$

Then, we get:

$$[x^{(n)}(t)]_\alpha + [a_{n-1}(t)x^{(n-1)}(t)]_\alpha + [a_{n-2}(t)x^{(n-2)}(t)]_\alpha + \ldots + [a_1(t)x'(t)]_\alpha + [a_0(t)x(t)]_\alpha = [k(t)]_\alpha \tag{79}$$

Therefore, we have:

$$[x^{(n)}(t)]_\alpha + [a_{n-1}(t)]_\alpha[x^{(n-1)}(t)]_\alpha + [a_{n-2}(t)]_\alpha[x^{(n-2)}(t)]_\alpha + \cdots + [a_1(t)]_\alpha[x'(t)]_\alpha + [a_0(t)]_\alpha[x(t)]_\alpha = [k(t)]_\alpha \quad (80)$$

Then, we write the lower bound and the upper bound of Eq. (80) as follows:

●) The lower bound of the parametric form

$$[x^{(n)}(t)]_\alpha^L + [a_{n-1}(t)]_\alpha^L[x^{(n-1)}(t)]_\alpha^L + [a_{n-2}(t)]_\alpha^L[x^{(n-2)}(t)]_\alpha^L + \cdots + [a_1(t)]_\alpha^L[x'(t)]_\alpha^L + [a_0(t)]_\alpha^L[x(t)]_\alpha^L = [k(t)]_\alpha^L \quad (81)$$

With:

$$[x(0)]_\alpha^L = [v_0]_\alpha^L \quad (82)$$

$$[x'(0)]_\alpha^L = [v_1]_\alpha^L \quad (83)$$

$$[x''(0)]_\alpha^L = [v_2]_\alpha^L \quad (84)$$

$$\vdots$$

$$[x^{(n-1)}(0)]_\alpha^L = [v_{n-1}]_\alpha^L \quad (85)$$

where,

$$[a_{n-1}(t)]_\alpha^L = (h_2(t) - h_1(t))\alpha + h_1(t) \quad (86)$$

$$[a_{n-2}(t)]_\alpha^L = (h_5(t) - h_4(t))\alpha + h_4(t) \quad (87)$$

$$\vdots$$

$$[a_1(t)]_\alpha^L = (h_8(t) - h_7(t))\alpha + h_7(t) \quad (88)$$

$$[a_0(t)]_\alpha^L = (h_{11}(t) - h_{10}(t))\alpha + h_{10}(t) \quad (89)$$

$$[k(t)]_\alpha^L = (h_{14}(t) - h_{13}(t))\alpha + h_{13}(t) \quad (90)$$

$$[v_0]_\alpha^L = (b_2 - b_1)\alpha + b_1 \quad (91)$$

$$[v_1]_\alpha^L = (b_5 - b_4)\alpha + b_4 \quad (92)$$

$$[v_2]_\alpha^L = (b_8 - b_7)\alpha + b_7 \quad (93)$$

$$\vdots$$

$$[v_{n-1}]_\alpha^L = (b_{11} - b_{10})\alpha + b_{10} \quad (94)$$

Now, by using LODMM that we described in section four, we will solve Eq. (81) subject to the initial conditions in the Eqs. (82)-(85), we can obtain the lower bound of the fuzzy solution of problem (51) which is $[x(t)]_\alpha^L$.

●) The upper bound of the parametric form

$$[x^{(n)}(t)]_\alpha^U + [a_{n-1}(t)]_\alpha^U[x^{(n-1)}(t)]_\alpha^U + [a_{n-2}(t)]_\alpha^U[x^{(n-2)}(t)]_\alpha^U + \cdots + [a_1(t)]_\alpha^U[x'(t)]_\alpha^U + [a_0(t)]_\alpha^U[x(t)]_\alpha^U = [k(t)]_\alpha^U \quad (95)$$

With initial conditions:

$$[x(0)]_\alpha^U = [v_0]_\alpha^U \quad (96)$$

$$[x'(0)]_\alpha^U = [v_1]_\alpha^U \quad (97)$$

$$[x''(0)]_\alpha^U = [v_2]_\alpha^U \quad (98)$$

$$\vdots$$

$$[x^{(n-1)}(0)]_\alpha^U = [v_{n-1}]_\alpha^U \quad (99)$$

where,

$$[a_{n-1}(t)]_\alpha^U = \big(h_2(t) - h_3(t)\big)\alpha + h_3(t) \quad (100)$$

$$[a_{n-2}(t)]_\alpha^U = \big(h_5(t) - h_6(t)\big)\alpha + h_6(t) \quad (101)$$

$$\vdots$$

$$[a_1(t)]_\alpha^U = \big(h_8(t) - h_9(t)\big)\alpha + h_9(t) \quad (102)$$

$$[a_0(t)]_\alpha^U = \big(h_{11}(t) - h_{12}(t)\big)\alpha + h_{12}(t) \quad (103)$$

$$[k(t)]_\alpha^U = \big(h_{14}(t) - h_{15}(t)\big)\alpha + h_{15}(t) \quad (104)$$

$$[v_0]_\alpha^U = (b_2 - b_3)\alpha + b_3 \quad (105)$$

$$[v_1]_\alpha^U = (b_5 - b_6)\alpha + b_6 \quad (106)$$

$$[v_2]_\alpha^U = (b_8 - b_9)\alpha + b_9 \quad (107)$$

$$\vdots$$

$$[v_{n-1}]_\alpha^U = (b_{11} - b_{12})\alpha + b_{12} \quad (108)$$

Now, by using LODMM that we described in section four, we will solve Eq. (95) subject to the initial conditions in the Eqs. (96)-(99), we can obtain the upper bound of the fuzzy solution of problem (51) which is $[x(t)]_\alpha^U$.

Finally, we obtain the fuzzy solution of the problem (51), which is:

$$\tilde{x}(t) = [x(t)]_\alpha = [[x(t)]_\alpha^L, [x(t)]_\alpha^U] \quad (109)$$

The following theorem ensures the existence and uniqueness of the fuzzy solution of the problem (51).

Theorem

Let $\tilde{F}$: $[0,\infty) \times \tilde{E} \times \tilde{E} \times \ldots \times \tilde{E} \to \tilde{E}$ be continuous fuzzy function and assume that there exist real numbers $q_1, q_2, \ldots, q_n > 0$ such that [14]:

$$D\left(\tilde{F}(t, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n), \tilde{F}(t, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)\right) \le \sum_{i=1}^{n} q_i\, D(\tilde{x}_i, \tilde{y}_i)$$

for all $t \in [a,b], \tilde{x}_i, \tilde{y}_i \in \tilde{E}, i = 1,2,\ldots,n$

Then the nth-order FDE described by the problem (51) has a unique solution on [0, ∞).

6. APPLIED EXAMPLES

In this section, we will solve four fuzzy problems. For each problem, we obtain the absolute errors:

$$[error]_\alpha^L = \left|[x_{exact}(t)]_\alpha^L - [x_{app}(t)]_\alpha^L\right| \quad (110)$$

$$[error]_\alpha^U = \left|[x_{exact}(t)]_\alpha^U - [x_{app}(t)]_\alpha^U\right| \quad (111)$$

where,

$[x_{exact}(t)]^L_\alpha$ and $[x_{exact}(t)]^U_\alpha$ are the lower bound and upper bound of the FEAS, respectively.

$[x_{app}(t)]^L_\alpha$ and $[x_{app}(t)]^U_\alpha$ are the lower bound and upper bound of the FAAS, respectively.

To obtain accurate approximate solutions, all numerical results were processed by taking 15 decimal places using MATLAB.

Example (6.1) Consider the FDE

$$\tilde{x}'(t) = (0.5, 1, 1.5)\tilde{x}^2(t) + (0.75, 1, 1.25);\quad t \in [0, 1] \tag{112}$$

With:

$$x(0) = 0 \tag{113}$$

Solution:

The FAAS is:

$$\tilde{x}(t) = [[x(t)]^L_\alpha, [x(t)]^U_\alpha] \tag{114}$$

where,

$$[x(t)]^L_\alpha = [c_0]^L_\alpha p_0(t) + [c_1]^L_\alpha p_1(t) + [c_2]^L_\alpha p_2(t) + [c_3]^L_\alpha p_3(t) + [c_4]^L_\alpha p_4(t) \tag{115}$$

$$[x(t)]^U_\alpha = [c_0]^U_\alpha p_0(t) + [c_1]^U_\alpha p_1(t) + [c_2]^U_\alpha p_2(t) + [c_3]^U_\alpha p_3(t) + [c_4]^U_\alpha p_4(t) \tag{116}$$

First, we write problem (112) in the parametric form as follows:

$$[x'(t)]_\alpha = [0.5\alpha + 0.5, -0.5\alpha + 1.5][x^2(t)]_\alpha + [0.25\alpha + 0.75, -0.25\alpha + 1.25] \tag{117}$$

With:

$$[x(0)]_\alpha = [0, 0] \tag{118}$$

Then, we get the following system:

$$[x'(t)]^L_\alpha = (0.5\alpha + 0.5)[x^2(t)]^L_\alpha + (0.25\alpha + 0.75) \tag{119}$$

$$[x'(t)]^U_\alpha = (-0.5\alpha + 1.5)[x^2(t)]^U_\alpha + (-0.25\alpha + 1.25) \tag{120}$$

With:

$$[x(0)]^L_\alpha = 0 \tag{121}$$

$$[x(0)]^U_\alpha = 0 \tag{122}$$

We will solve problem (112) for α=0.5. In this case, we get the following system:

$$[x'(t)]^L_{0.5} = 0.75[x^2(t)]^L_{0.5} + 0.875 \tag{123}$$

$$[x'(t)]^U_{0.5} = 1.25[x^2(t)]^U_{0.5} + 1.125 \tag{124}$$

With:

$$[x(0)]^L_{0.5} = 0 \tag{125}$$

$$[x(0)]^U_{0.5} = 0 \tag{126}$$

●) The lower bound of the fuzzy solution

By derivation Eq. (123), we have:

$$[x''(t)]^L_{0.5} = 1.5[x'(t)]^L_{0.5}[x(t)]^L_{0.5} \tag{127}$$

With:

$$[x(0)]^L_{0.5} = 0 \tag{128}$$

$$[x'(0)]^L_{0.5} = 0.875 \tag{129}$$

Then, by substituting the Eqs. (11-15) into Eq. (115), we obtain:

$$[x(t)]^L_{0.5} = [c_0]^L_{0.5}(1) + [c_1]^L_{0.5}(2t - 1) + [c_2]^L_{0.5}(6t^2 - 6t + 1) + [c_3]^L_{0.5}(20t^3 - 30t^2 + 12t - 1) + [c_4]^L_{0.5}(70t^4 - 140t^3 + 90t^2 - 20t + 1) \tag{130}$$

Now, we find the lower bound of the fuzzy residual function $[R(t)]^L_{0.5}$:

$$[x''(t)]^L_{0.5} - 1.5[x'(t)]^L_{0.5}[x(t)]^L_{0.5} = 0 \tag{131}$$

This gives:

$$[R(t)]^L_{0.5} = [x''(t)]^L_{0.5} - 1.5[x'(t)]^L_{0.5}[x(t)]^L_{0.5} \tag{132}$$

Thus, by substituting the Eqs. (41), (43) and (115) into Eq. (132), we find:

$$\begin{aligned}[R(t)]^L_{0.5} = {} & 12[c_2]^L_{0.5}\, p_0(t) + 40[c_4]^L_{0.5}\, p_0(t) + 60[c_3]^L_{0.5}\, p_1(t) + 140[c_4]^L_{0.5}\, p_2(t) \\ & - 1.5[\Big(2[c_1]^L_{0.5}\, p_0(t) + 2[c_3]^L_{0.5}\, p_0(t) + 6[c_2]^L_{0.5}\, p_1(t) + 6[c_4]^L_{0.5}\, p_1(t) + 10[c_3]^L_{0.5}\, p_2(t) \\ & + 14[c_4]^L_{0.5}\, p_3(t)\Big)([c_0]^L_{0.5}p_0(t) + [c_1]^L_{0.5}p_1(t) + [c_2]^L_{0.5}p_2(t) + [c_3]^L_{0.5}p_3(t) + [c_4]^L_{0.5}p_4(t))]\end{aligned} \tag{133}$$

Now, we apply the Eqs. (47), (48) and (49) as follows:

- $\int_0^1 [R(t)]^L_{0.5}\, p_0(t)\, dt = 0$

This gives:

$$-3[c_0]^L_{0.5}\,[c_1]^L_{0.5} - 3[c_0]^L_{0.5}\,[c_3]^L_{0.5} - 3[c_2]^L_{0.5}\,[c_1]^L_{0.5} - 3[c_4]^L_{0.5}\,[c_1]^L_{0.5} - 3[c_2]^L_{0.5}\,[c_3]^L_{0.5} - 3[c_3]^L_{0.5}\,[c_4]^L_{0.5} + 12[c_2]^L_{0.5} + 40[c_4]^L_{0.5} = 0 \tag{134}$$

- $\int_0^1 [R(t)]^L_{0.5}\, p_1(t)\, dt = 0$

This gives:

$$-[c_1{}^2]^L_{0.5} - 3[c_0]^L_{0.5}[c_2]^L_{0.5} - 3[c_0]^L_{0.5}[c_4]^L_{0.5} - 3[c_3]^L_{0.5}[c_1]^L_{0.5} - 3[c_2]^L_{0.5}[c_4]^L_{0.5} - \frac{6}{5}[c_2{}^2]^L_{0.5} - \frac{9}{7}[c_3{}^2]^L_{0.5} - \frac{4}{3}[c_4{}^2]^L_{0.5} + 20[c_3]^L_{0.5} = 0 \quad (135)$$

- $\int_0^1 [R(t)]^L_{0.5}\, p_2(t)\, dt = 0$

This gives:

$$-3[c_0]^L_{0.5}[c_3]^L_{0.5} - \frac{9}{5}[c_1]^L_{0.5}[c_2]^L_{0.5} - 3[c_4]^L_{0.5}[c_1]^L_{0.5} - \frac{78}{35}[c_2]^L_{0.5}[c_3]^L_{0.5} - \frac{17}{7}[c_4]^L_{0.5}[c_3]^L_{0.5} + 28[c_4]^L_{0.5} = 0 \quad (136)$$

Moreover, we apply the initial conditions by substituting the Eqs. (128) and (129) in Eq. (130), we find:

$$[c_0]^L_{0.5} - [c_1]^L_{0.5} + [c_2]^L_{0.5} - [c_3]^L_{0.5} + [c_4]^L_{0.5} = 0 \quad (137)$$

$$2[c_1]^L_{0.5} - 6[c_2]^L_{0.5} + 12[c_3]^L_{0.5} - 20[c_4]^L_{0.5} = 0.875 \quad (138)$$

By solving the Eqs. (134)-(138), we get:

$$[c_0]^L_{0.5} = 0.495864567230580$$
$$[c_1]^L_{0.5} = 0.544865800002154$$
$$[c_2]^L_{0.5} = 0.067416854542166$$
$$[c_3]^L_{0.5} = 0.022317863520394$$
$$[c_4]^L_{0.5} = 0.003902241749802$$

Finally, we put the above constants in Eq. (130) to obtain the lower bound of the FAAS of problem (112) at $\alpha = 0.5$, which is:

$$[x(t)]^L_{0.5} = (0.875)t + (0.086166979123356)t^2 - (0.099956574564400)t^3 + (0.273156922486140)t^4 \quad (139)$$

●) The upper bound of the fuzzy solution

By derivation Eq. (124), we get:

$$[x''(t)]^U_{0.5} = 2.5[x'(t)]^U_{0.5}[x(t)]^U_{0.5} \quad (140)$$

With:

$$[X(0)]^U_{0.5} = 0 \quad (141)$$

$$[x'(0)]^U_{0.5} = 1.125 \quad (142)$$

Then, by substituting the Eqs. (11)-(15) in Eq. (116), we obtain:

$$[x(t)]^U_{0.5} = [c_0]^U_{0.5}(1) + [c_1]^U_{0.5}(2t-1) + [c_2]^U_{0.5}(6t^2 - 6t + 1) + [c_3]^U_{0.5}(20t^3 - 30t^2 + 12t - 1) + [c_4]^U_{0.5}(70t^4 - 140t^3 + 90t^2 - 20t + 1) \quad (143)$$

Now, we find the upper bound of the fuzzy residual function $[R(t)]^U_{0.5}$:

$$[x''(t)]^U_{0.5} - 2.5[x'(t)]^U_{0.5}[x(t)]^U_{0.5} = 0 \quad (144)$$

This gives:

$$[R(t)]^U_{0.5} = [x''(t)]^U_{0.5} - 2.5[x'(t)]^U_{0.5}[x(t)]^U_{0.5} \quad (145)$$

Thus, by substituting the Eqs. (41), (43) and (116) in Eq. (145), we find:

$$[R(t)]^U_{0.5} = 12[c_2]^U_{0.5}p_0(t) + 40[c_4]^U_{0.5}p_0(t) + 60[c_3]^U_{0.5}p_1(t) + 140[c_4]^U_{0.5}p_2(t) - 2.5\Big(2[c_1]^U_{0.5}p_0(t) + 2[c_3]^U_{0.5}p_0(t) + 6[c_2]^U_{0.5}p_1(t) + 6[c_4]^U_{0.5}p_1(t) + 10[c_3]^U_{0.5}p_2(t) + 14[c_4]^U_{0.5}p_3(t)\Big)\big([c_0]^U_{0.5}p_0(t) + [c_1]^U_{0.5}p_1(t) + [c_2]^U_{0.5}p_2(t) + [c_3]^U_{0.5}p_3(t) + [c_4]^U_{0.5}p_4(t)\big) \quad (146)$$

Now, we apply the Eqs. (47)-(49) as follows:

- $\int_0^1 [R(t)]^U_{0.5}\, p_0(t)\, dt = 0$

This gives:

$$-5[c_0]^U_{0.5}[c_1]^U_{0.5} - 5[c_0]^U_{0.5}[c_3]^U_{0.5} - 5[c_2]^U_{0.5}[c_1]^U_{0.5} - 5[c_4]^U_{0.5}[c_1]^U_{0.5} - 5[c_2]^U_{0.5}[c_3]^U_{0.5} - 5[c_3]^U_{0.5}[c_4]^U_{0.5} + 12[c_2]^U_{0.5} + 40[c_4]^U_{0.5} = 0 \quad (147)$$

- $\int_0^1 [R(t)]^U_{0.5}\, p_1(t)\, dt = 0$

This gives:

$$-\frac{5}{3}[c_1{}^2]^U_{0.5} - 5[c_0]^U_{0.5}[c_2]^U_{0.5} - 5[c_0]^U_{0.5}[c_4]^U_{0.5} - 5[c_3]^U_{0.5}[c_1]^U_{0.5} - 5[c_2]^U_{0.5}[c_4]^U_{0.5} - 2[c_2{}^2]^U_{0.5} - \frac{15}{7}[c_3{}^2]^U_{0.5} - \frac{20}{9}[c_4{}^2]^U_{0.5} + 20[c_3]^U_{0.5} = 0 \quad (148)$$

- $\int_0^1 [R(t)]^U_{0.5}\, p_2(t)\, dt = 0$

This gives:

$$-5[c_0]^U_{0.5}[c_3]^U_{0.5} - 3[c_1]^U_{0.5}[c_2]^U_{0.5} - 5[c_4]^U_{0.5}[c_1]^U_{0.5} - \frac{26}{7}[c_2]^U_{0.5}[c_3]^U_{0.5} - \frac{85}{21}[c_4]^U_{0.5}[c_3]^U_{0.5} + 28[c_4]^U_{0.5} = 0 \quad (149)$$

Moreover, we apply the initial conditions by substituting the Eqs. (141) and (142) in Eq. (143), we find:

$$[c_0]^U_{0.5} - [c_1]^U_{0.5} + [c_2]^U_{0.5} - [c_3]^U_{0.5} + [c_4]^U_{0.5} = 0 \quad (150)$$

$$2[c_1]^U_{0.5} - 6[c_2]^U_{0.5} + 12[c_3]^U_{0.5} - 20[c_4]^U_{0.5} = 1.125 \quad (151)$$

By solving the Eqs. (147)-(151), we get:

$$[c_0]^U_{0.5} = 0.790314364547047$$
$$[c_1]^U_{0.5} = 0.963777246933006$$
$$[c_2]^U_{0.5} = 0.277516715528700$$
$$[c_3]^U_{0.5} = 0.213066357943577$$
$$[c_4]^U_{0.5} = 0.082012524800837$$

Finally, we put the above constants in Eq. (143) to obtain the upper bound of FAAS of problem (112) at $\alpha = 0.5$, which is:

$$[x(t)]^U_{0.5} = (1.125)t + (2.654236786940220)t^2 - (7.220426313245640)t^3 + (5.740876736058590)t^4 \quad (152)$$

The FEAS at $\alpha = 0.5$ is:

$$[x(t)]_{0.5} = [[x(t)]^L_{0.5}, [x(t)]^U_{0.5}] \quad (153)$$

where,

$$[x(t)]^L_{0.5} = \sqrt{\frac{7}{6}} \tan\left(\sqrt{\frac{21}{32}} t\right) \quad (154)$$

$$[x(t)]^U_{0.5} = \sqrt{\frac{9}{10}} \tan\left(\sqrt{\frac{45}{32}} t\right) \quad (155)$$

In the same way, we can obtain the FAAS of problem (112) at $\alpha = 0.8$, which is:

$$[x(t)]_{0.8} = [[x(t)]^L_{0.8}, [x(t)]^U_{0.8}] \quad (156)$$

where,

$$[x(t)]^L_{0.8} = (0.95)t + (0.236294161886970)t^2 - (0.496519586166660)t^3 + (0.670682205917990)t^4 \quad (157)$$

$$[x(t)]^U_{0.8} = (1.05)t + (0.909874153685058)t^2 - (2.384077355435620)t^3 + (2.217290934190570)t^4 \quad (158)$$

While the FEAS at $\alpha = 0.8$ will be:

$$[x(t)]_{0.8} = [[x(t)]^L_{0.8}, [x(t)]^U_{0.8}] \quad (159)$$

where,

$$[x(t)]^L_{0.8} = \sqrt{\frac{19}{18}} \tan\left(\sqrt{\frac{171}{200}} t\right) \quad (160)$$

$$[x(t)]^U_{0.8} = \sqrt{\frac{21}{22}} \tan\left(\sqrt{\frac{231}{200}} t\right) \quad (161)$$

Below, numerical Tables 1-2 for this example.

Table 1. Results for example (6.1), α=0.5

t	$[\mathbf{x_{app}(t)}]^L_\alpha$	$[\mathbf{error}]^L_\alpha$	$[\mathbf{x_{app}(t)}]^U_\alpha$	$[\mathbf{error}]^U_\alpha$
0	0	0	0	0
0.1	0.088289028908918	5.97 e-4	0.132396029229762	1.94 e-4
0.2	0.178084077644397	1.54 e-3	0.282591463749337	5.33 e-4
0.3	0.269768771680001	1.98 e-3	0.427930901929062	7.54 e-4
0.4	0.364382313103261	1.59 e-3	0.559537046305814	7.24 e-4
0.5	0.463619480615673	5.12 e-4	0.682310703583012	4.31 e-4
0.6	0.569830629532702	8.33 e-4	0.814930784630614	2.99 e-5
0.7	0.686021691783777	1.84 e-3	0.989854304485121	4.77 e-4
0.8	0.815854175912298	1.98 e-3	1.253316382349572	6.96 e-4
0.9	0.963645167075627	1.19 e-3	1.665330241593548	5.68 e-4
1	1.134367327045096	4.64 e-4	2.299687209753170	4.18 e-4

Table 2. Results for example (6.1), α=0.8

t	$[\mathbf{x_{app}(t)}]^L_\alpha$	$[\mathbf{error}]^L_\alpha$	$[\mathbf{x_{app}(t)}]^U_\alpha$	$[\mathbf{error}]^U_\alpha$
0	0	0	0	0
0.0000429	0.040755434838939	4.35 e-10	0.045046674353267	1.67 e-9
0.0000858	0.081511739198974	1.74 e-9	0.090096696660233	6.70 e-9
0.0001287	0.122268912844976	3.91 e-9	0.135150065791779	1.51 e-8
0.0001716	0.163026955541869	6.95 e-9	0.180206780618968	2.68 e-8
0.0002145	0.203785867054631	1.09 e-8	0.225266840013041	4.18 e-8
0.0002574	0.244545647148297	1.56 e-8	0.270330242845420	6.02 e-8
0.0003003	0.285306295587953	2.13 e-8	0.315396987987708	8.20 e-8
0.0003432	0.326067812138742	2.78 e-8	0.360467074311688	1.07 e-7
0.0003861	0.366830196565861	3.52 e-8	0.405540500689323	1.35 e-7
0.000429	0.407593448634561	4.34 e-8	0.450617265992757	1.67 e-7

The researchers in study [19] solved this problem by using the Runge-Kutta method of order sixth for different values of α, t=1 and h=0.02. The absolute error was belonged into [2.02e−5, 9.98e−4]. At α=0.5, they obtained:

$$[error]^L_\alpha = 6.79e-4, [error]^U_\alpha = 1.26e-4$$

While according to our results, at α=0.5 and t=1, we obtained:

$$[error]^L_\alpha = 4.64e-4, [error]^U_\alpha = 4.18e-4$$

Example (6.2): Consider the FDE

$$\tilde{x}''(t) - (0, 2, 3)\tilde{x}'(t) = (4t^2, 5t^2, 7t^2) + 2t; \quad t \in [0, 1] \quad (162)$$

With:

$$\tilde{x}(0) = (1, 2, 2.5) \quad (163)$$

$$\tilde{x}(1) = (3.5, 4, 4.5) \quad (164)$$

Solution

The FAAS is:

$$\tilde{x}(t) = [[x(t)]_\alpha^L, [x(t)]_\alpha^U] \quad (165)$$

where,

$$[x(t)]_\alpha^L = [c_0]_\alpha^L p_0(t) + [c_1]_\alpha^L p_1(t) + [c_2]_\alpha^L p_2(t) + [c_3]_\alpha^L p_3(t) + [c_4]_\alpha^L p_4(t) \quad (166)$$

$$[x(t)]_\alpha^U = [c_0]_\alpha^U p_0(t) + [c_1]_\alpha^U p_1(t) + [c_2]_\alpha^U p_2(t) + [c_3]_\alpha^U p_3(t) + [c_4]_\alpha^U p_4(t) \quad (167)$$

First, we write problem (162) in the parametric form as follows:

$$[x''(t)]_\alpha - [2\alpha, -\alpha + 3][x'(t)]_\alpha = [(\alpha + 4)t^2, (-2\alpha + 7)t^2] + [2t, 2t] \quad (168)$$

With:

$$[x(0)]_\alpha = [\alpha + 1, -0.5\alpha + 2.5] \quad (169)$$

$$[x(1)]_\alpha = [0.5\alpha + 3.5, -0.5\alpha + 4.5] \quad (170)$$

Then, we get:

$$[x''(t)]_\alpha^L - (2\alpha)[x'(t)]_\alpha^L = (\alpha + 4)t^2 + 2t \quad (171)$$

$$[x''(t)]_\alpha^U - (-\alpha + 3)[x'(t)]_\alpha^U = (-2\alpha + 7)t^2 + 2t \quad (172)$$

With:

$$[x(0)]_\alpha^L = \alpha + 1 \quad (173)$$

$$[x(0)]_\alpha^U = -0.5\alpha + 2.5 \quad (174)$$

$$[x(1)]_\alpha^L = 0.5\alpha + 3.5 \quad (175)$$

$$[x(1)]_\alpha^U = -0.5\alpha + 4.5 \quad (176)$$

We will solve problem (162) at α=0.2. Therefore, we have:

$$[x''(t)]_{0.2}^L - 0.4[x'(t)]_{0.2}^L = 4.2t^2 + 2t \quad (177)$$

$$[x''(t)]_{0.2}^U - 2.8[x'(t)]_{0.2}^U = 6.6t^2 + 2t \quad (178)$$

With:

$$[x(0)]_{0.2}^L = 1.2 \quad (179)$$

$$[x(0)]_{0.2}^U = 2.4 \quad (180)$$

$$[x(1)]_{0.2}^L = 3.6 \quad (181)$$

$$[x(1)]_{0.2}^U = 4.4 \quad (182)$$

●) The lower bound of the fuzzy solution

By substituting the Eqs. (11)-(15) in Eq. (166), we obtain:

$$[x(t)]_{0.2}^L = [c_0]_{0.2}^L(1) + [c_1]_{0.2}^L(2t - 1) + [c_2]_{0.2}^L(6t^2 - 6t + 1) + [c_3]_{0.2}^L(20t^3 - 30t^2 + 12t - 1) + [c_4]_{0.2}^L(70t^4 - 140t^3 + 90t^2 - 20t + 1) \quad (183)$$

Now, we find the lower bound of the fuzzy residual function $[R(t)]_{0.2}^L$:

$$[x''(t)]_{0.2}^L - 0.4[x'(t)]_{0.2}^L - 4.2t^2 - 2t = 0 \quad (184)$$

This gives:

$$[R(t)]_{0.2}^L = [x''(t)]_{0.2}^L - 0.4[x'(t)]_{0.2}^L - 4.2t^2 - 2t \quad (185)$$

Thus, by substituting the Eqs. (41) and (43) in Eq. (185), we find:

$$[R(t)]_{0.2}^L = 12[c_2]_{0.2}^L p_0(t) + 40[c_4]_{0.2}^L p_0(t) + 60[c_3]_{0.2}^L p_1(t) + 140[c_4]_{0.2}^L p_2(t) - 0.4[2[c_1]_{0.2}^L p_0(t) + 2[c_3]_{0.2}^L p_0(t) + 6[c_2]_{0.2}^L p_1(t) + 6[c_4]_{0.2}^L p_1(t) + 10[c_3]_{0.2}^L p_2(t) + 14[c_4]_{0.2}^L p_3(t)] - 4.2t^2 - 2t \quad (186)$$

Now, we apply the Eqs. (47)-(49) as follows:

- $\int_0^1 [R(t)]_{0.2}^L\, p_0(t)\, dt = 0$

This gives:

$$12[c_2]_{0.2}^L - 0.8[c_1]_{0.2}^L - 0.8[c_3]_{0.2}^L + 31[c_4]_{0.2}^L - 2.4 = 0 \quad (187)$$

- $\int_0^1 [R(t)]_{0.2}^L\, p_1(t)\, dt = 0$

This gives:

$$-0.8[c_2]_{0.2}^L + 20[c_3]_{0.2}^L + 8.2[c_4]_{0.2}^L + \frac{49}{30} = 0 \quad (188)$$

- $\int_0^1 [R(t)]_{0.2}^L\, p_2(t)\, dt = 0$

This gives:

$$-0.8[c_3]_{0.2}^L + 19[c_4]_{0.2}^L - \frac{7}{50} = 0 \quad (189)$$

Moreover, we apply the boundary conditions by substituting the Eqs. (179) and (181) in Eq. (183), we find:

$$[c_0]_{0.2}^L - [c_1]_{0.2}^L + [c_2]_{0.2}^L - [c_3]_{0.2}^L + [c_4]_{0.2}^L = 1.2 \quad (190)$$

$$[c_0]_{0.2}^L + [c_1]_{0.2}^L + [c_2]_{0.2}^L + [c_3]_{0.2}^L + [c_4]_{0.2}^L = 3.6 \quad (191)$$

By solving the Eqs. (187)-(191), we get:

$$[c_0]_{0.2}^L = 2.126821450933619$$
$$[c_1]_{0.2}^L = 1.272678252580361$$
$$[c_2]_{0.2}^L = 0.268870264960180$$
$$[c_3]_{0.2}^L = -0.072678252580361$$
$$[c_4]_{0.2}^L = 0.004308284106201$$

Finally, we put the above constants in Eq. (183) to obtain the lower bound of the FAAS of problem (162) at α=0.2, which is:

$$[x(t)]_{0.2}^{L} = (1.2) - (0.026169797688710)t + (4.181314736730000)t^2 - (2.056724826475360)t^3 + (0.301579887434070)t^4 \quad (192)$$

●) The upper bound of the fuzzy solution

By substituting the Eqs. (11)-(15) in Eq. (161), we obtain:

$$[x(t)]_{0.2}^{U} = [c_0]_{0.2}^{U}(1) + [c_1]_{0.2}^{U}(2t-1) + [c_2]_{0.2}^{U}(6t^2-6t+1) + [c_3]_{0.2}^{U}(20t^3-30t^2+12t-1) + [c_4]_{0.2}^{U}(70t^4-140t^3+90t^2-20t+1) \quad (193)$$

Now, we find the upper bound of the fuzzy residual function $[R(t)]_{0.2}^{U}$:

$$[x''(t)]_{0.2}^{U} - 2.8[x'(t)]_{0.2}^{U} - 6.6t^2 - 2t = 0 \quad (194)$$

This gives:

$$[R(t)]_{0.2}^{U} = [x''(t)]_{0.2}^{U} - 2.8[x'(t)]_{0.2}^{U} - 6.6t^2 - 2t \quad (195)$$

Thus, by substituting the Eqs. (41) and (43) in Eq. (195), we find:

$$[R(t)]_{0.2}^{U} = 12[c_2]_{0.2}^{U}\, p_0(t) + 40[c_4]_{0.2}^{U}\, p_0(t) + 60[c_3]_{0.2}^{U}\, p_1(t) + 140[c_4]_{0.2}^{U}\, p_2(t) - 2.8[2[c_1]_{0.2}^{U}\, p_0(t) + 2[c_3]_{0.2}^{U}\, p_0(t) + 6[c_2]_{0.2}^{U}\, p_1(t) + 6[c_4]_{0.2}^{U}\, p_1(t) + 10[c_3]_{0.2}^{U}\, p_2(t) + 14[c_4]_{0.2}^{U}\, p_3(t)] - 6.6t^2 - 2t \quad (196)$$

Now, we apply the Eqs. (47)-(49) as follows:

- $\int_0^1 [R(t)]_{0.2}^{U}\, p_0(t)\, dt = 0$

This gives:

$$12[c_2]_{0.2}^{U} - 5.6[c_1]_{0.2}^{U} - 5.6[c_3]_{0.2}^{U} + 40.3[c_4]_{0.2}^{U} - 3.2 = 0 \quad (197)$$

- $\int_0^1 [R(t)]_{0.2}^{U}\, p_1(t)\, dt = 0$

This gives:

$$-5.6[c_2]_{0.2}^{U} - 43[c_3]_{0.2}^{U} - 5.5[c_4]_{0.2}^{U} - \frac{43}{30} = 0 \quad (198)$$

- $\int_0^1 [R(t)]_{0.2}^{U}\, p_2(t)\, dt = 0$

This gives:

$$-5.6[c_3]_{0.2}^{U} + 28[c_4]_{0.2}^{U} - \frac{11}{50} = 0 \quad (199)$$

Moreover, we apply the boundary conditions by substituting the Eqs. (180) and (182) in Eq. (193), we find:

$$[c_0]_{0.2}^{U} - [c_1]_{0.2}^{U} + [c_2]_{0.2}^{U} - [c_3]_{0.2}^{U} + [c_4]_{0.2}^{U} = 2.4 \quad (200)$$

$$[c_0]_{0.2}^{U} + [c_1]_{0.2}^{U} + [c_2]_{0.2}^{U} + [c_3]_{0.2}^{U} + [c_4]_{0.2}^{U} = 4.4 \quad (201)$$

By solving the Eqs. (197)-(201), we get:

$$[c_0]_{0.2}^{U} = 2.621641538498813$$
$$[c_1]_{0.2}^{U} = 1.134745346681082$$
$$[c_2]_{0.2}^{U} = 0.797450387965678$$
$$[c_3]_{0.2}^{U} = -0.134745346681082$$
$$[c_4]_{0.2}^{U} = -0.019091926464490$$

Finally, we put the above constants in Eq. (193) to obtain the upper bound of the FAAS of problem (162) at α=0.2, which is:

$$[x(t)]_{0.2}^{U} = (2.4) - (3.750317265315089)t + (7.108789346422429)t^2 - (0.022037228593040)t^3 - (1.336434852514300)t^4 \quad (202)$$

The FEAS at α=0.2 is:

$$[x(t)]_{0.2} = [[x(t)]_{0.2}^{L}, [x(t)]_{0.2}^{U}] \quad (203)$$

where,

$$[x(t)]_{0.2}^{L} = -\frac{7}{2}t^3 - \frac{115}{4}t^2 - \frac{575}{4}t - (361.5308690588012) + (362.7308690588012)e^{0.4t} \quad (204)$$

$$[x(t)]_{0.2}^{U} = -\frac{11}{14}t^3 - \frac{235}{196}t^2 - \frac{1175}{1372}t + (2.086551078605455) + (0.313448921394545) \quad (205)$$

Below, a numerical Table 3 for this example.

Table 3. Results for example (6.2), α=0.2

T	$[\mathbf{x_{app}(t)}]_{\alpha}^{L}$	$[\mathbf{error}]_{\alpha}^{L}$	$[\mathbf{x_{app}(t)}]_{\alpha}^{U}$	$[\mathbf{error}]_{\alpha}^{U}$
0	1.2	0	2.4	0
0.00000219	1.199999942708197	3.00 e-6	2.399991786839284	8.26 e-6
0.00000438	1.199999885456502	6.00 e-6	2.399983573746756	1.65 e-5
0.00000657	1.199999828244915	8.99 e-6	2.399975360722417	2.48 e-5
0.00000876	1.199999771073435	1.20 e-5	2.399967147766267	3.30 e-5
0.00001095	1.199999713942063	1.50 e-5	2.399958934878307	4.13 e-5
0.00001314	1.199999656850798	1.80 e-5	2.399950722058534	4.96 e-5
0.00001533	1.199999599799640	2.10 e-5	2.399942509306952	5.78 e-5
0.00001752	1.199999542788590	2.40 e-5	2.399934296623557	6.61 e-5
0.00001971	1.199999485817646	2.70 e-5	2.399926084008353	7.43 e-5
0.0000219	1.199999428886809	3.00 e-5	2.399917871461335	8.26 e-5

Example (6.3): Consider the FDE

$$\tilde{x}''(t) = \frac{(\tilde{x}'(t))^2}{(0.2, 0.6, 0.9)};\ t \in [0,1] \quad (206)$$

With:

$$\tilde{x}(0) = \frac{1}{(1.3, 2, 2.8)} \quad (207)$$

$$x(1) = -1 \quad (208)$$

Solution

The FAAS is:

$$\tilde{x}'(t) = [[x(t)]_\alpha^L, [x(t)]_\alpha^U] \quad (209)$$

where,

$$[x(t)]_\alpha^L = [c_0]_\alpha^L p_0(t) + [c_1]_\alpha^L p_1(t) + [c_2]_\alpha^L p_2(t) + [c_3]_\alpha^L p_3(t) + [c_4]_\alpha^L p_4(t) \quad (210)$$

$$[x(t)]_\alpha^U = [c_0]_\alpha^U p_0(t) + [c_1]_\alpha^U p_1(t) + [c_2]_\alpha^U p_2(t) + [c_3]_\alpha^U p_3(t) + [c_4]_\alpha^U p_4(t) \quad (211)$$

In the same way that we have used in the previous examples, we can find FAAS of this problem at α=0.3, which is:

$$[x(t)]_{0.3} = [[x(t)]_{0.3}^L, [x(t)]_{0.3}^U] \quad (212)$$

where,

$$[x(t)]_{0.3}^L = (0.390625) - (1.987660202022536)t + (1.684206221749926)t^2 - (1.798897567337860)t^3 + (0.711726547610470)t^4 \quad (213)$$

$$[x(t)]_{0.3}^U = (0.662251655629139) - (2.142528648228798)t + (2.138256591361530)t^2 - (3.059723346903740)t^3 + (1.401743748141870)t^4 \quad (214)$$

The FEAS at α=0.3 is:

$$[x(t)]_{0.3} = [[x(t)]_{0.3}^L, [x(t)]_{0.3}^U] \quad (215)$$

where,

$$[x(t)]_{0.3}^L = (-0.81)\ln(100t + 21.897154734360726) + (2.890573933139084) \quad (216)$$

$$[x(t)]_{0.3}^U = (-0.32)\ln(25t + 0.139443138240584) + (0.031820176796409) \quad (217)$$

Below, a numerical Table 4 for this example.

Table 4. Result for example (6.3), α=0.3

T	$[x_{app}(t)]_\alpha^L$	$[error]_\alpha^L$	$[x_{app}(t)]_\alpha^U$	$[error]_\alpha^U$
0	0.390625	0	0.662251655629139	0
0.00000112	0.390622773822686	1.92 e-6	0.662249255999735	6.18 e-5
0.00000224	0.390620547649598	3.83 e-6	0.662246856375696	1.24 e-4
0.00000336	0.390618321480735	5.75 e-6	0.662244456757021	1.86 e-4
0.00000448	0.390616095316097	7.67 e-6	0.662242057143710	2.47 e-4
0.0000056	0.390613869155685	9.58 e-6	0.662239657535764	3.09 e-4
0.00000672	0.390611642999498	1.15 e-5	0.662237257933182	3.71 e-4
0.00000784	0.390609416847536	1.34 e-5	0.662234858335965	4.33 e-4
0.00000896	0.390607190699799	1.53 e-5	0.662232458744111	4.94 e-4
0.00001008	0.390604964556288	1.73 e-5	0.662230059157622	5.56 e-4
0.0000112	0.390602738417002	1.92 e-5	0.662227659576497	6.18 e-4

Example (6.4): Consider the FDE

$$\tilde{x}''(t) + \tilde{x}(t) = t;\ t \in [0,1] \quad (218)$$

With:

$$\tilde{x}(0) = (0.9, 1, 1.1) \quad (219)$$

$$\tilde{x}'(0) = (1.8, 2, 2.2) \quad (220)$$

Solution

The FAAS of this problem at α=0.4 is:

$$[x(t)]_{0.4} = [[x(t)]_{0.4}^L, [x(t)]_{0.4}^U] \quad (221)$$

where,

$$[x(t)]_{0.4}^L = (0.94) + (1.88)t - (0.468328835651892)t^2 - (0.154323286870860)t^3 + (0.051028087028100)t^4 \quad (222)$$

$$[x(t)]_{0.4}^U = (1.06) + (2.12)t - (0.527642430400338)t^2 - (0.197240698406060)t^3 + (0.060047128750910)t^4 \quad (223)$$

While the FEAS at α=0.4 is:

$$[x(t)]_{0.4} = [[x(t)]_{0.4}^L, [x(t)]_{0.4}^U] \quad (224)$$

where,

$$[x(t)]_{0.4}^L = (0.88)\sin t + (0.94)\cos t + t \quad (225)$$

$$[x(t)]_{0.4}^U = (1.12)\sin t + (1.06)\cos t + t \quad (226)$$

Below, numerical Tables 5-6 for this example.

Table 5. Result for example (6.4), α=0.4

t	$[\mathbf{x_{app}(t)}]^{\mathbf{L}}_{\alpha}$	$[\mathbf{error}]^{\mathbf{L}}_{\alpha}$	$[\mathbf{x_{app}(t)}]^{\mathbf{U}}_{\alpha}$	$[\mathbf{error}]^{\mathbf{U}}_{\alpha}$
0	0.94	0	1.06	0
0.1	1.123167491165313	1.02 e-5	1.266532339710466	1.45 e-5
0.2	1.296113905218202	2.23 e-5	1.461412452602739	3.22 e-5
0.3	1.458097003550744	2.29 e-5	1.643673064149889	3.38 e-5
0.4	1.608497014963882	1.15 e-5	1.812491012933982	1.78 e-5
0.5	1.746816635667426	5.45 e-6	1.967187250646090	6.87 e-6
0.6	1.872681029280054	1.98 e-5	2.107226842086288	2.85 e-5
0.7	1.985837826829315	2.54 e-5	2.232218965163650	3.76 e-5
0.8	2.086157126751619	2.05 e-5	2.341916910896254	3.10 e-5
0.9	2.173631494892247	9.56 e-6	2.436218083411181	1.46 e-5
1	2.248375964505348	2.67 e-6	2.515163999944512	3.95 e-6

Table 6. Result for example (6.4), α=0.4

T	$[\mathbf{x_{app}(t)}]^{\mathbf{L}}_{\alpha}$	$[\mathbf{error}]^{\mathbf{L}}_{\alpha}$	$[\mathbf{x_{app}(t)}]^{\mathbf{U}}_{\alpha}$	$[\mathbf{error}]^{\mathbf{U}}_{\alpha}$
0	0.94	0	1.06	0
-0.000215	0.939595778353033	7.73 e-11	1.059544175611689	1.09 e-10
-0.000430	0.939191513418270	3.10 e-10	1.059088302454599	4.37 e-10
-0.000645	0.938787205204916	6.97 e-10	1.058632380540495	9.84 e-10
-0.000860	0.938382853722179	1.24 e-9	1.058176409881148	1.75 e-9
-0.001075	0.937978458979273	1.94 e-9	1.057720390488328	2.74 e-9
-0.001290	0.937574020985410	2.80 e-9	1.057264322373812	3.95 e-9
-0.001505	0.937169539749808	3.81 e-9	1.056808205549379	5.38 e-9
-0.001720	0.936765015281685	4.98 e-9	1.056352040026809	7.03 e-9
-0.001935	0.936360447590264	6.31 e-9	1.055895825817887	8.90 e-9
-0.002150	0.935955836684770	7.80 e-9	1.055439562934400	1.10 e-8

Table 7. Comparison between the absolute errors for example (6.4)

Method	$[\mathbf{error}]^{\mathbf{L}}_{\alpha}$	$[\mathbf{error}]^{\mathbf{U}}_{\alpha}$
RKN4	1.85 e-6	2.24 e-6
IRKN3	3.19 e-5	3.52 e-5
IRKN4	4.44 e-8	6.32 e-8
IRKN5	3.37 e-8	4.09 e-8
This research	2.67 e-6	3.95 e-6

The researchers in study [20] solved this problem by using the improved Runge-Kutta Nystrom methods of orders three, four and five for different values of α, t=1 and h=0.1. The absolute error was belonged into [3.13e-8, 3.63e-5]. At α=0.4, Table 7 gives a comparison between the absolute errors in study [20] and in our research.

7. DISCUSSION

Through the applied examples that solved in this work, it can be seen that LODMM based on Tau method has a high efficiency in approximating the exact-analytical solution, as the comparison that we conducted with other approximation methods showed the accuracy of the results that can be obtained when using this method. These results can improve further when increasing the number of terms of the solution series. This means, using a larger value for m, such as m=5, m=6, and so on.

From the solved examples in this work, we can conclude that several factors affect the accuracy of the results, namely:

- The number of terms of the solution series. The more terms in the solution series, the more accurate results will be obtained.
- The value of the variable t. If the value of t is close to the initial value, the results will be more accurate.
- The value of the constant α. In fact, the best value of α cannot be determined, as it changes from one problem to another.
- The mathematical nature of the problem, whether it is linear or non-linear.
- The order of the FDE, whether it is first order or higher order.

It is necessary to note that the lower absolute error is not related to the upper absolute error. In the same problem, with the same value of α and the same value of t, the lower absolute error may be higher than the upper absolute error and vice versa. As for the case of equality between the two errors, it is rare.

8. CONCLUSION

In this work, we have used the fuzzy fiction of LODMM based on Tau method to obtain the FAAS of the FDEs in which the coefficients are TFFs. The approximate solutions that we obtained are accurate solutions and very close to the FEAS. In comparison with the other methods, we can determine many advantages for the LODMM, namely, it is computational less cost, it needs less computational time and effort and it has better accuracy.

For the future works, one can extend and use this method for solving other types of the FDE such as fuzzy fractional differential equations, fuzzy partial differential equations, fuzzy delay differential equations, etc. Also, one can use this method for solving FDEs with other types of the fuzzy function coefficients such as trapezoidal fuzzy function coefficients, exponential fuzzy function coefficients, etc.

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