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Mathematical Modelling of Geophysical Fluid Flow: The Condition for Deep Water Stratification

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ABSTRACT

Fluid dynamics in stratified deep water is a complex and intriguing field that plays a vital role in understanding various natural and engineered systems. The deep water stratification occurs when the vertical amplitude is equal to the horizontal amplitude of surface waves though not in the same direction. The variation in density and pressure owing to salinity, temperature, and other factors significantly affects fluid behavior. This often leads to stratification which is further simplified when subjected to conditions at boundary, which can occur in extreme natural settings or through controlled experimental conditions. The interplay between these factors results in unique and often counterintuitive fluid dynamics, necessitating a thorough examination to predict and interpret the behavior of such systems accurately. This study aims to delve into the mathematical modeling of geophysical fluid flow in stratified deep water, specifically focusing on the boundary condition necessary for the establishment of stratified deep water regime. The boundary condition can influence the stratification and the dynamics of fluid flow, leading to novel phenomena that are not observed under normal shallow water conditions. By understanding these interactions at boundary condition, this research can provide valuable insights with significant implications for fields such as climate science, oceanography and engineering applications involving fluid dynamics. The primary focus of this research is to show how vertical amplitude of a surface wave is equal to the horizontal amplitude and its effect on deep water stratification.

1. INTRODUCTION

Our work sufficiently established a very strong relationship between the vertical amplitude and the horizontal amplitude of deep water waves under stratified condition with effect of gravity modification, area relatively new and research promising with understanding of impact of deep water stratification [1]. By studying the condition for equality of amplitude of deep water stratification we gain insights into how these factor influences the propagation of waves and how energy is transferred within the ocean [2]. Analysis of current shear indicate that the downward flux is supported by tidal mixing while the upwards flux is dominated by wind driven near-inertial shear [3]. Unlike other models that have discussed stratification in deep water none has sufficiently explained the important of amplitude regime nor incorporated the concept of reduced gravity into their model; in view of this research gap, one of the main reasons and goals of this work is to develop a very accurate model and numerical simulations that can predict the behavior of waves in deep water with understanding of the amplitude regime and its variation effect [4]. Overall, our model incorporating amplitude is aimed to advance our understanding of the complex interaction between waves, the seafloor [5, 6] and the ocean as a whole with deep understanding of deep water formulation [7], and it can contribute to the development of new technologies and a robust strategy for offshore engineering [8], with its environmental assessment, thereby improving deep water ventilation [9]. With the numerical values of both amplitudes being equal, we extended the equations of shallow water to deep water and developed condition for deep water stratification within the thermocline regime. Geophysical fluid flow is governed by the laws of fluid dynamics [3] but needs to equally account for the additional effects of the earth's rotation and density stratification within the medium [10]. The derivation of the equations for stratified geophysical flow involves the application of the principles of fluid mechanics, thermodynamics, and conservation laws to the specific conditions of stratified flows in the Earth's atmosphere and oceans [11].

The equations that describe such flows are known as the primitive equations, and they are derived from the fundamental equations of fluid dynamics, namely the Navier-Stokes equations and the first law of thermodynamics [12]. The result is interesting as confirm in tidal simulation of open channel [13].

The conservation of mass for a fluid element in a stratified flow leads to the equation of continuity, which represents the conservation of mass within the fluid. In a non-divergent flow (incompressible flow), the equation of continuity simplifies to $\nabla(\rho v) = 0$, where ρ is the fluid density and the density varies in each layer of the fluid parcel, v is the velocity vector, and ∇ stands for divergence operator [14].

The conservation of momentum for a fluid element in a stratified flow leads to the Navier-Stokes equations [2], which describe the motion of the fluid. In the context of geophysical flows, the Coriolis force due to the Earth's rotation and the pressure gradient force are important factors [15]. Deep Water Stratification and its effects with gravity modification [7]. The impact of deep water stratification on ocean currents [16] and the patterns of circulation with density changes owing to modified gravity [17].

When the flow is stratified, buoyancy forces due to variations in density also play a crucial role [11]. By combining the equations of continuity, momentum, and energy, while considering the specific characteristics of stratified geophysical flows, a set of primitive equations is derived [18].

These equations typically include the momentum equations in terms of velocity components [19], the thermodynamic equation for temperature, and the equation for the evolution of density or buoyancy [20-22]. Finite Volume Methods for Hyperbolic Problems [1] highlight the importance of noting that the derivation and formulation of the primitive equations for stratified geophysical flow is a complex process involving a comprehensive understanding of fluid dynamics [23, 24], thermodynamics, and geophysical phenomena.

The resulting equations provide a fundamental framework for modeling and understanding the behavior of stratified flows in deep water and effect of modified gravity on it [25, 26].

2. ASSUMPTIONS OF THE MODEL

We wish to formulate some basic and necessary assumptions to aid the mathematical formulation of our model.

The following are the assumptions for the model:

The fluid is incompressible with continuous density stratification.

In classical geophysical flows the depth of deep water is infinite, so the vertical length scale (h) and the horizontal length scale (L) guarantee deep water regime when

$$\frac{L}{h} \ll 1 \ (deep\ water\ assumption) \text{ but fails when } \frac{L}{h} \gg 1$$

On a large scale this implies that the flow is predominantly horizontal with the vertical acceleration. This means that the horizontal length scale, which is typically measured as the distance between wave crests or troughs, is generally smaller than the vertical length scale, which is the depth of the water affected by the wave.

In deep water, the motion of water particles becomes circular when the vertical amplitude is equal to the horizontal amplitude of the surface wave.

The cartesian coordinates *x*, *y* and *z* will be used, in Cartesian coordinates for deep water waves, the z-axis typically measures the vertical direction, the x-axis is the horizontal direction, the y-axis represents another horizontal direction and the z-axis represents the vertical direction up-down.

The velocity components in the directions of increasing x, y and z will be denoted by u, v and w.

Take the (*x*, *y*) horizontal plane as being parallel to the surface of the still water, and the depth of the water at a given point as $h = (x, y, t) > 0$.

We denoted depth-average velocity in the x direction as $U = u(x, y, t)$ and the depth-average velocity in the y- direction as $v = v\,(x; y; t\,)$. While the plane (*z*=0) can be chosen arbitrarily, it is usually positioned at mean water level.

Measuring down from this plane to the transition zone which is the thermocline, the point where the circular orbit of the deep water particles decrease at depth $z = -\zeta(x, y)$. The equation $z = -\zeta(x, y)$ is the equation for the bottom surface at which the diameter of the orbital path is zero at any instant. The interaction of deep water flow at this thermocline regime which at this instant serves as the bottom condition is the layer of the ocean where the temperature changes most rapidly with varying depth.

3. MATHEMATICAL FORMULATION OF THE PROBLEM

The deep water stratification can be formulated using various physical and mathematical principles. A common approach is to use the concept of density variation or potential temperature to describe the stratification of the water column. Now for a model formulation for deep water stratification:

The conservation of momentum equation can be used to describe the vertical movement of water masses in response to density differences. This equation can be quite complex and is often solved using a numerical model.

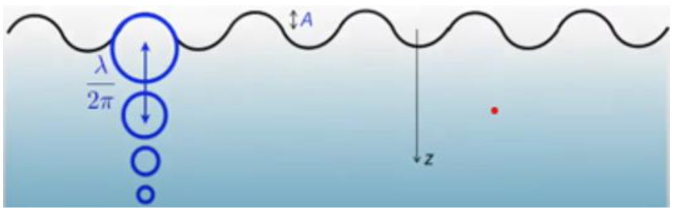

Figure 1. Exponential decay in circular motion of deep water

Figure 1 shows the exponential decay of circular trajectory of deep water at the point the vertical amplitude is equal to the horizontal amplitude thereby guaranteeing deep water condition.

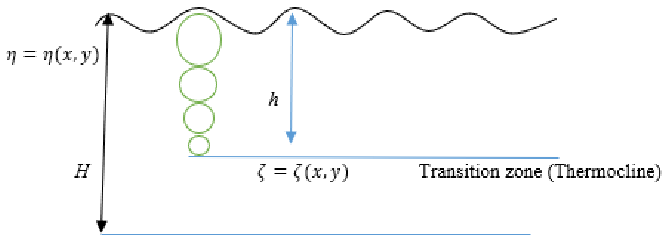

Figure 2. Geometry of the flow problem in deep water with infinite depth at the thermocline regime

Figure 2 illustrates the zone where the circular orbit dissipates and shows $\eta = \eta(x, y)$ representing the progressive surface wave in deep water, with $\zeta = \zeta(x, y)$ defined as the bottom boundary condition.

Consider perturbing the deep water's surface of amplitude $\eta(x, y, t)$. So $z = \eta(x, y, t)$ is the instantaneous positon of the actual water surface measured from the plane ($z = 0$). Where η is the surface elevation of the deep water.

3.1 The fluid satisfies Laplacian equation $\nabla^2\emptyset = 0$ and continuity equation

$$\nabla . \vec{q} = 0 \tag{1}$$

There are two boundary condition that will define the regime of deep water stratification and they are free surface and bottom surface. The free surface boundary condition include: Kinematic condition and dynamic condition and the bottom surface condition is at the thermocline regime. Eq. (1) is for momentum conservation which account for the body of deep water and the continuity equation that account for the mass of the deep water.

On the free surface: $pressure = p_{atmosphere} = p_{atm}$

Applying Bernoulli's equation of motion

$$\frac{\partial \phi}{\partial t} + {}^1/_2\left(\phi_x^2 + \phi_y^2\right) + \frac{P}{\rho} + gy = const = 0(W.L.G) \tag{2}$$

$$\gamma = p_{atm} \text{ at } y = \eta(x, t)$$
$$\frac{\partial \phi}{\partial t} + {}^1/_2\left(\phi_x^2 + \phi_y^2\right) + gh = \frac{-p_{atm}}{\rho}$$

Dynamic free surface condition

$$y = \eta(x, t)$$

Kinematic condition

$$\frac{D}{Dt}(y - \eta(x.t)) = 0, \text{ on } y = \eta(x, t)$$

The equation then simplifies to

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\frac{D}{Dt}(y - \eta), \eta = \eta(x, t) \Longrightarrow \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v \tag{3}$$

$$y = \eta(x, t) \text{ and } u = \phi_x, v = \phi_y \tag{4}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \phi_x\frac{\partial \eta}{\partial x}$$

Kinetic free surface condition

$$y = \eta(x, t)$$

where, $\phi_x^2, \phi_y^2, \phi_x, \frac{\partial \eta}{\partial x}, \phi(x, y, t)$ and $\phi(x, \eta, t)$ are the two boundary conditions non-linear for 2-dimension.

3.2 Bottom boundary conditions at thermocline regime

In deep water, the thermocline regime as indicated in Figure 2 is associated with the distribution of temperature vertically with depth. So, thermocline is the depth interval where there is a significant change in water temperature, salinity, and other deep water properties. Hence, bottom boundary condition plays very significant role in determining the thermocline regime in a stratified deep water under modified gravity, where, η is the free surface elevation and ϕ is the surface flux for the flow.

In Figure 2, h is the depth of the water column. Then the substantive derivative or material derivative at the boundary can be expressed as:

$$y = h(x, t), \frac{D}{Dt}\left(y - \eta(x, t)\right) = 0$$
$$\Longrightarrow -\phi_y = h_x\,\phi_n + h_t, h(x, t) = const$$

$\phi_y = 0$ on $y = -h$, this is the bottom boundary conditions at thermocline regime where gravitational force is different from that on earth giving rise to modified gravity.

The impact of this modified gravity is what our model has sufficiently captured.

Assume that η is small, the quantity associated with ϕ is equally small

$$\phi_x/y = \eta = \phi_x/y = 0 + \eta\frac{\partial \phi_x}{\partial x}/y = 0 + \frac{\eta^2}{2!}\frac{\partial^2 \phi_x}{\partial_x^2} + \cdots \tag{5}$$

Similarly,

$$\phi_y/y = \eta = \phi_y/y = 0 + \eta.\frac{\partial \phi_y}{\partial y} + \frac{\eta^2}{2!}\frac{\partial^2}{\partial_y^2}(\phi_y) + \cdots \tag{6}$$

Substitute for ϕ_x, ϕ_y at $y = \eta$ as in Eq. (4) in both dynamic and kinematic conditions and neglect the product/higher power terms, which yields:

$$\frac{\phi_x^2}{y} = 0, \frac{\phi_y^2}{y} = 0, \eta.\frac{\phi_x}{y} = 0, \eta^2\phi_x \ldots \tag{7}$$

If the higher power terms are negligible, then the result from the dynamic condition and kinematic condition Eq. (7) become linearized boundary conditions

$$Dynamic \longrightarrow \phi_t + g\eta = 0 \text{ on } y = 0 \tag{8}$$

Condition

$$Kinematic \longrightarrow \eta_t = \phi_y \text{ on } y = 0 \tag{9}$$

Now from the above kinematic conditions Eqs. (8) and (9), we obtain

$$\phi_{tt} + g\eta_t = 0 \text{ on } y = 0 \tag{10}$$

$$\phi_{tt} + g\phi_y = 0 \text{ on } y = 0 \tag{11}$$

Free surface boundary condition which is the combination of dynamic condition and kinematic condition.

Eq. (3) necessitate the flow and Eq. (1) which is Laplacian equation is the equation that govern the body of stratified deep water at the thermocline regime.

3.3 Vertical and horizontal amplitude of stratified deep water

At the surface, the vertical displacement δz must correspond to a traveling wave

$$\delta z = A cos(kx\text{- }\omega t)$$

Because vertical displacement of deep water corresponds to the traveling wave due to the principles of wave propagation. In deep water, the wave's motion is primarily vertical, and this shows that water particles move up and down as this wave passes through. The amount of vertical displacement depends on the wave's amplitude, which is the maximum distance the water particles move from their resting position.

$$\delta z = Acos(kx - \omega t) \tag{12}$$

$$\delta z = -A_x \sin(kx - \omega t) \tag{13}$$

$$\omega = \frac{d}{dt}(\delta z) = A_z \omega \sin(kx - \omega t) \tag{14}$$

$$u = \frac{d}{dt}(\delta x) = A_x \omega \cos(kx - \omega t) \tag{15}$$

But the condition that the velocity field is non-divergent and Irrotational arises from conservation of mass in absence of viscosity; hence, Eqs. (16) and (17):

$$\frac{du}{dx} + \frac{d\omega}{dz} = 0 \tag{16}$$

$$\frac{d\omega}{dx} - \frac{du}{dz} = 0 \tag{17}$$

Applying Eq. (14) into Eq. (16) gives

$$-A_z k\omega sin(kx - \omega t) + \frac{d}{dz} A_z \omega \sin(kx - \omega t) = 0 \tag{18}$$

$$-kA_z + \frac{d}{dz} A_z = 0 \tag{19}$$

Using Eq. (15) in Eq. (17), we have

$$A_z k\omega \cos(kx - \omega t) - \frac{d}{dz} A_x \omega \cos(kx - \omega t) = 0 \tag{20}$$

Differentiating Eq. (19) further gives

$$\frac{d^2 A_z}{dz^2} - k^2 A_z \tag{21}$$

Eq. (21) is a second order differential equation and the solution becomes

$$A_z = \alpha e^{kz} + \beta e^{-kz} \tag{22}$$

where, α and β are unknown arbitrary constants and when $z = 0$,

$$\Rightarrow A_z = \alpha + \beta = A \tag{23}$$

When, $z = h$, the bottom condition which is at the thermocline regime where there is no vertical displacement, therefore

$$\delta z(-h) = 0 = \alpha e^{-kh} + \beta e^{kh} \tag{24}$$

$$A_Z = \alpha - \frac{\alpha e^{-kh}}{e^{kh}} = \alpha \left(\frac{e^{kh} - e^{-kh}}{e^{kh}} \right)$$

Likewise

$$A_Z = \beta - \frac{\beta e^{kh}}{e^{-kh}} = \beta \left(\frac{e^{-kh} - e^{kh}}{e^{-kh}} \right) \tag{25}$$

$$\text{or } \alpha = A \frac{e^{kh}}{e^{kh} - e^{-kh}}, \beta = \frac{-A\, e^{-kh}}{e^{kh} - e^{-kh}} \tag{26}$$

So,

$$A_Z = \alpha e^{kz} + \beta e^{kz} = \frac{A}{e^{kh} - e^{-kh}} (e^{kh} e^{kz} - e^{-kh} e^{-kz}) = \frac{A e^{k(h+z)} - e^{-k(h+z)}}{e^{kh} - e^{-kh}} \tag{27}$$

$$A_Z = A \frac{\sin h\big(k(h+z)\big)}{\sin h(kh)} \tag{28}$$

Eq. (28) is the vertical amplitude which stands for the vertical movement or oscillation of water particles in deep water. It helps to determine the wave energy and potential impact on coastal structures and marine environment.

Applying Eq. (28) to obtain the horizontal amplitude A_x, by substituting solution for A_Z, back into Eq. (27), $-KA + \frac{d}{dz} A_Z = 0$, gives

$$-KA_x + \frac{d}{dz}\left(\frac{A e^{k(h+z)} - e^{-k(h+z)}}{e^{kh} - e^{-kh}} \right) = 0,$$

$$\text{then, } KA_x = AK \left(\frac{A e^{k(h+z)} + e^{-k(h+z)}}{e^{kh} - e^{-kh}} \right)$$

So if we take vertical derivative, it yields

$$A_x = A \frac{\cosh\,(k(z+h))}{\sin\,(kh)} \tag{29}$$

Eq. (29) is the horizontal amplitude of the moving wave which is influenced by the interaction between the waves and stratified water layers. When *z* becomes negative then $\sinh\,(k(z+h))$ disappears and the amplitude decreases with depth and when the depth is significantly greater than the wavelength then we have deep water.

Importantly, when Eqs. (27) and (28) are equal then equation of stratified deep water is established at the thermocline regime.

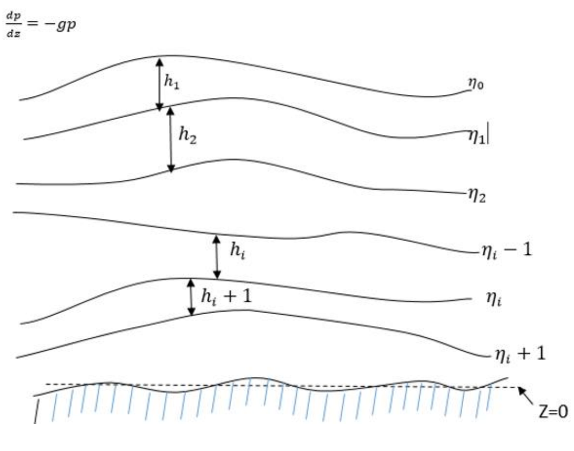

Figure 3. Stratification effects in different fluid layers under modified gravity conditions

3.4 Variation of pressure in stratified deep water

In stratified deep water, where multiple layers are stacked atop one another, the pressure within each layer is determined by the hydrostatic approximation. This pressure variation is influenced by several factors, including depth, temperature, salinity, and density gradients. As depicted in Figure 3, these layers possess distinct densities. Consequently, the pressure at any given depth is affected by the density of the layer above it. This results in a layered effect where pressure changes are not uniform across the deep water column.

For the Bosques equation, the hydrostatic approximation is given as

$$\frac{dp}{dz} = -gp \tag{30}$$

The layers in Figure 3 are numbered from top down. The coordinates of the interfaces are denoted by η, and the layer thickness h_i. So integrating from η_0 to η_z

$$\int dp = \int_{\eta_0}^{\eta_z} (-g\rho dz) \\ p_1 = -\rho_1 g(z - \eta_0), p_1 = \rho_1 g(\eta_0 - z) \tag{31}$$

For the second layer, integrating from η_0 to η_1 and η_1 to z, we have

$$p_2 = -\rho g \int_{\eta_0}^{\eta_1} dz - \rho g \int_{z}^{\eta_1} dz$$
$$p_2 = \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z)$$
$$= \rho_1 g\eta_0 - \rho_1 g'_1\eta_1 + \rho_2 g'_1\eta_1 - \rho_2 gz$$
$$p_2 = \rho_1 g\eta_0 + (\rho_2 - \rho_1) g'\eta_1 - \rho_2 gz$$

At $z = 0$, $p_2 gz = 0$, hence

$$p_2 = \rho_1 g\eta_0 + (\rho_2 - \rho_1) g'\eta_1 \tag{32}$$

From Eq. (32), $p_1 = \rho_1 g\eta_0$ and

$$\rho_1 g'_1\eta_1 = g(\rho_2 - \rho_1)\eta_1 \tag{33}$$

From Eq. (33), we get

$$g'_1 = g\frac{(\rho_2 - \rho_1)}{\rho_1} = g\frac{\Delta\rho}{\bar{\rho}} \tag{34}$$

Eq. (34) is the modified gravity, where $\frac{\Delta\rho}{\bar{\rho}} = \frac{\rho_{i+1}}{\rho_i}$, ρ is density.

The term involving z is irrelevant for the dynamics, because only the horizontal motion is considered; hence Eqs. (32) and (33) become

$$p_1 = \rho_1 g\eta_0, p_2 = \rho_1 g\eta_0 + \rho_1 g'_1\eta_1 \tag{35}$$

Summing from the top down, we have

$$\eta_0 = h_1 + h_2 + \eta_b, \eta_1 = h_2 + \eta_b \tag{36}$$

Therefore, putting Eqs. (32) and (33) the pressure in the two layers' system can be expressed as:

$$p_1 = \rho_1 g\eta_0 = \rho_1 g(h_1 + h_2 + \eta_b) \tag{37}$$

$$p_2 = \rho_1 g\eta_0 + \rho_1 g'_1\eta_1 \\ = \rho_1 g(h_1 + h_2 + \eta_b) + \rho_1 g'_1(h_2 + \eta_b) \tag{38}$$

Now for the conservation equation of mass, each layer has the same form as the single-layer case and it is expressed as

$$\frac{Dh_n}{Dt} + h_n\nabla.u_n \tag{39}$$

Considering the three-layer model, pressure can be given as

$$p_1 = \rho_1 g\eta_0 = \rho_1 g(h_1 + h_2 + h_3 + \eta_b) \tag{40}$$

$$p_2 = \rho_1 g\eta_0 + \rho_1 g_1'\eta_1 \\ = \rho_1 g(h_1 + h_2 + h_3 + \eta_b) + \rho_1 g_1'(h_2 + h_3 + \eta_b) \tag{41}$$

$$p_3 = \rho_1 g\eta_0 + \rho_1 g_1'\eta_1 + \rho_1 g'_2\eta_2 \\ = \rho_1 g(h_1 + h_2 + h_3 + \eta_b) + \rho_1 g_1'(h_2 + h_3 + \eta_b) \\ + \rho_1 g'_2(h_3 + \eta_b) \tag{42}$$

where, η_b is the nth layer of the strata.

Hence,

$$\eta_1 = \eta_b + \sum_{i=n+1}^{i=N} h_i \tag{43}$$

Therefore, for the nth layer model, the dynamical pressure is given as:

$$p_n = \rho_i \sum_{i=0}^{n-1} g_i\eta_i \tag{44}$$

Eq. (39) is a continuity equation in stratified deep water and Eq. (44) is the pressure variation for nth stratification of deep water.

$$g'_i = g\frac{(\rho_{i+1} - \rho_i)}{\rho_i} = g\frac{\Delta\rho}{\bar{\rho}} \tag{45}$$

Eqs. (34) and (45) show the effect of modified gravity on deep water stratification.

It leads to changes in the density and temperature gradients within stratified deep water column.

Taking $\rho_0 = 0$.

Therefore, incorporating modified gravity into the stratified deep water equations at the thermocline regime become:

$$\frac{\partial(h_1u_1)}{\partial t} + \frac{\partial(h_1u_1^2 + g\frac{\Delta\rho}{\bar{\rho}}h_1^2/2)}{\partial x} + \frac{\partial(h_1u_1v_1)}{\partial y} \\ = -g\frac{\Delta\rho}{\bar{\rho}}h_1\frac{\partial h_2}{\partial x} - g\frac{\Delta\rho}{\bar{\rho}}h_1\frac{\partial(\xi)}{\partial x} + fv_1 \tag{46}$$

$$\frac{\partial(h_1v_1)}{\partial t} + \frac{\partial(h_1u_1v_1)}{\partial x} + \frac{\partial(h_1v_1^2 + h_1^2/2)}{\partial y} \\ = -g\frac{\Delta\rho}{\bar{\rho}}h_1\frac{\partial h_2}{\partial y} - g\frac{\Delta\rho}{\bar{\rho}}h_1\frac{\partial(\xi)}{\partial y} + fu_1 \tag{47}$$

$$\frac{\partial h_1}{\partial t} + \frac{\partial(h_1u_1)}{\partial x} + \frac{\partial(h_1v_1)}{\partial y} = 0 \tag{48}$$

$$\frac{\partial(h_2u_2)}{\partial t}+\frac{\partial(h_2u_2^2+g\frac{\Delta\rho}{\bar{\rho}}h_2^2/2+g\frac{\Delta\rho}{\bar{\rho}}h_2h_1)}{\partial x}+\frac{\partial(h_2u_2v_2)}{\partial y}=-g\frac{\Delta\rho}{\bar{\rho}}h_1\frac{\partial h_2}{\partial x}-g\frac{\Delta\rho}{\bar{\rho}}h_2\frac{\partial(\xi)}{\partial x}+fv_2 \quad (49)$$

$$\frac{\partial(h_2v_2)}{\partial t}+\frac{\partial(h_2u_2v_2)}{\partial x}+\frac{\partial(h_2v_2^2+g\frac{\Delta\rho}{\bar{\rho}}h_2^2/2+g\frac{\Delta\rho}{\bar{\rho}}h_2h_1)}{\partial y}=-g\frac{\Delta\rho}{\bar{\rho}}h_1\frac{\partial h_2}{\partial y}-g\frac{\Delta\rho}{\bar{\rho}}h_2\frac{\partial(\xi)}{\partial y}+fu_2 \quad (50)$$

$$\frac{\partial h_2}{\partial t}+\frac{\partial(h_2u_2)}{\partial x}+\frac{\partial(h_2v_2)}{\partial y}=0 \quad (51)$$

We can now provide solution to the model equation in Eqs. (46)-(51) by considering the characteristic equations in matrix form.

In conservation form,

$$\frac{\partial q}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\psi \quad (52)$$

where,

$$q=\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix}=\begin{bmatrix}h\\hu\\hv\end{bmatrix},$$
$$F=\begin{bmatrix}q_2\\ \frac{q_2^2}{q_1}+\frac{g\frac{\Delta\rho}{\bar{\rho}}q_1^2}{2}\\ \frac{q_2q_3}{q_1}\end{bmatrix}=\begin{bmatrix}hu\\ hu^2+\frac{g\frac{\Delta\rho}{\bar{\rho}}h^2}{2}\\ huv\end{bmatrix},$$
$$G=\begin{bmatrix}q_3\\ \frac{q_2q_3}{q_1}\\ \frac{q_3}{q_1}+\frac{g\frac{\Delta\rho}{\bar{\rho}}q_1^2}{2}\end{bmatrix}=\begin{bmatrix}hv\\ huv\\ hv^2+\frac{g\frac{\Delta\rho}{\bar{\rho}}h^2}{2}\end{bmatrix}, \quad (53)$$
$$\psi=\begin{bmatrix}0\\ -g\frac{\Delta\rho}{\bar{\rho}}h\xi_x+fv\\ -g\frac{\Delta\rho}{\bar{\rho}}h\xi_y-fu\end{bmatrix}$$

The subscripts x,y denote differentiation with respect to that variable. In vector form, the unknown is $q=[h;hu;hv]^T$ is the vector of conserved variables.

The right-hand side vector of source terms, ψ, includes the effects of density stratification at the transition zone (thermocline) and the effects of the Coriolis force. To calculate the eigenvalues of the deep water equations, define the Jacobian matrix with differentiated coefficients of $F(q)=[f_1;f_2;f_3]^T$ as $F'(q)=\frac{\partial f_1}{\partial q_1}$ for $i,j=1,2,3$.

$$F'(q)=\begin{bmatrix}\frac{\partial f_1}{\partial q_1}&\frac{\partial f_1}{\partial q_2}&\frac{\partial f_1}{\partial q_3}\\ \frac{\partial f_2}{\partial q_1}&\frac{\partial f_2}{\partial q_2}&\frac{\partial f_2}{\partial q_3}\\ \frac{\partial f_3}{\partial q_1}&\frac{\partial f_3}{\partial q_2}&\frac{\partial f_3}{\partial q_3}\end{bmatrix}$$

And an equivalent expression for

$$G'(q)q_1+F'(q)\frac{\partial q}{\partial x}+G'(q)\frac{\partial q}{\partial y}=\psi \quad (54)$$

where,

$$F'(q)=\begin{bmatrix}0&1&0\\ g\frac{\Delta\rho}{\bar{\rho}}h-u^2&2u&0\\ -uv&v&u\end{bmatrix} \quad (55)$$

$$G'(q)=\begin{bmatrix}0&0&1\\ -uv&v&u\\ g\frac{\Delta\rho}{\bar{\rho}}h-v^2&0&2v\end{bmatrix}$$ and $F'(q).x=\lambda x$

That is

$$(F'(q)-\lambda I)x=0 \quad (56)$$

The characteristics determinant:

$$\left|F'(q)-\lambda I\right|=\begin{vmatrix}-\lambda&1&0\\ g\frac{\Delta\rho}{\bar{\rho}}h-u^2&2u-\lambda&0\\ -uv&v&u-\lambda\end{vmatrix} \quad (57)$$

Will lead to the characteristic equation:

$$\left|F'(q)-\lambda I\right|=\begin{vmatrix}-\lambda&1&0\\ g\frac{\Delta\rho}{\bar{\rho}}h-u^2&2u-\lambda&0\\ -uv&v&u-\lambda\end{vmatrix}=0 \quad (58)$$

Solving the 3 by 3 matrix in Eq. (45),

$$-\lambda\{(2u-\lambda)(u-\lambda)\}-1\left\{\left(g\frac{\Delta\rho}{\bar{\rho}}h-u^2\right)(u-\lambda)\right\}=0$$
$$(u-\lambda)\left(\lambda^2-2u\lambda-\left(g\frac{\Delta\rho}{\bar{\rho}}h-u^2\right)\right)=0$$
$$(u-\lambda)=0,\lambda=u$$
$$\text{or }\left(\lambda^2-2u\lambda-\left(g\frac{\Delta\rho}{\bar{\rho}}h-u^2\right)\right)=0$$

Solving using the quadratic formula: $\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where, $a=1,b=-2u$ and $c=-\left(g\frac{\Delta\rho}{\bar{\rho}}h-u^2\right)$, substituting these values in the formula; we obtain:

$$\lambda=u\pm\sqrt{g\frac{\Delta\rho}{\bar{\rho}}h} \quad (59)$$

This leads us to the eigenvalues,

$$\lambda_1(q)=u-\sqrt{g\frac{\Delta\rho}{\bar{\rho}}h},\lambda_2(q)=u+\sqrt{g\frac{\Delta\rho}{\bar{\rho}}h},\lambda_3(q)=u \quad (60)$$

The following eigenvectors are obtained using Maple Software;

For $\lambda_1(q)$ we have:

$$\lambda_1(q)=\begin{bmatrix}\dfrac{\sqrt{g\frac{\Delta\rho}{\bar\rho}h}\left(g\frac{\Delta\rho}{\bar\rho}h-u^2\right)}{uv(-u-\sqrt{g\frac{\Delta\rho}{\bar\rho}h})(u-\sqrt{g\frac{\Delta\rho}{\bar\rho}h})}\\ \dfrac{\sqrt{g\frac{\Delta\rho}{\bar\rho}h}\left(g\frac{\Delta\rho}{\bar\rho}h-u^2\right)}{uv(-u-\sqrt{g\frac{\Delta\rho}{\bar\rho}h})}\\ 1\end{bmatrix}$$

$$\lambda_2(q)=\begin{bmatrix}\dfrac{-\sqrt{g\frac{\Delta\rho}{\bar\rho}h}(g\frac{\Delta\rho}{\bar\rho}h-u^2)}{uv(-u+\sqrt{g\frac{\Delta\rho}{\bar\rho}h})(u+\sqrt{g\frac{\Delta\rho}{\bar\rho}h})}\\ \dfrac{-\sqrt{g\frac{\Delta\rho}{\bar\rho}h}\left(g\frac{\Delta\rho}{\bar\rho}h-u^2\right)}{uv(-u-\sqrt{g\frac{\Delta\rho}{\bar\rho}h})}\\ 1\end{bmatrix} \tag{61}$$

$$\lambda_3(q)=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Similarly, for $G'(q)$

$$\lambda_1(q)=v-\sqrt{g\frac{\Delta\rho}{\bar\rho}h},\lambda_2(q)=v+\sqrt{g\frac{\Delta\rho}{\bar\rho}h},\lambda_3(q)=v \tag{62}$$

With the corresponding eigenvectors,

$$\chi_1(q)=\begin{bmatrix}\dfrac{1}{v-\sqrt{g\frac{\Delta\rho}{\bar\rho}h}}\\ \dfrac{-ug\frac{\Delta\rho}{\bar\rho}h+uv^2+uv\left(v-\sqrt{g\frac{\Delta\rho}{\bar\rho}h}\right)-2uv^2}{(g\frac{\Delta\rho}{\bar\rho}h-v^2)\sqrt{g\frac{\Delta\rho}{\bar\rho}h}}\\ 1\end{bmatrix} \tag{63}$$

$$\chi_2(q)=\begin{bmatrix}\dfrac{1}{v+\sqrt{g\frac{\Delta\rho}{\bar\rho}h}}\\ \dfrac{-ug\frac{\Delta\rho}{\bar\rho}h+uv^2+uv\left(v+\sqrt{g\frac{\Delta\rho}{\bar\rho}h}\right)-2uv^2}{(g\frac{\Delta\rho}{\bar\rho}h-v^2)\sqrt{g\frac{\Delta\rho}{\bar\rho}h}}\\ 1\end{bmatrix} \tag{64}$$

$$\chi_3(q)=\begin{bmatrix}0\\1\\0\end{bmatrix} \tag{65}$$

Since this system has two spatial dimensions for Eqs. (59) and (60), the waves (solutions) move in horizontal direction establishing the vertical amplitude of the wave and the horizontal amplitude which is given by the unit normal vector n where $n(n_x;n_y)$.

$$\left|n_xF'(q)+n_yG'(q)-\lambda_iI\right|=0 \text{ for } i=1,2,3 \tag{66}$$

that yield

$$\lambda_1=\frac{n_xq_2}{q_1}+\frac{n_yq_3}{q_1}-\sqrt{g\frac{\Delta\rho}{\bar\rho}q_1},$$
$$\lambda_2=\frac{n_xq_2}{q_1}+\frac{n_yq_3}{q_1},\lambda_3=\frac{n_xq_2}{q_1}+\frac{n_yq_3}{q_1}+\sqrt{g\frac{\Delta\rho}{\bar\rho}q_1} \tag{67}$$

$$\lambda_1(q)=u+v-\sqrt{g\frac{\Delta\rho}{\bar\rho}h},\lambda_2(q)=u+v,$$
$$\lambda_3(q)=u+v+\sqrt{g\frac{\Delta\rho}{\bar\rho}h} \tag{68}$$

where

$$c=\sqrt{g\frac{\Delta\rho}{\bar\rho}h} \tag{69}$$

Eq. (69) provides the speed of stratified deep water at equilibrium position which is when the vertical amplitude is equal to the horizontal amplitude.

The characteristic equations/eigenvalues obtained in Eq. (68) are very useful in deep water analysis as it tells us the behavior of the deep water waves. The eigenvalues help us to predict the growth or decay rates of wave modes in deep water as demonstrated in Figure 1.

The Froude Number is a dimensionless quantity mainly used in fluid dynamics in defining the nature of boundary layer. Fundamentally, Froude Number is defined as the ratio of the inertial force to gravitational force acting on a fluid.

The eigenvalues take the form of a convective velocity minus/plus a phase velocity. Dividing convective by phase yields a Froude number for each x- and y- direction, therefore, F_r is Froude number which can be expressed as:

$$F_{(r)i}\equiv\frac{U_x}{\sqrt{\left(g\frac{\Delta\rho}{\bar\rho}\right)h}}=\frac{V_y}{\sqrt{g\frac{\Delta\rho}{\bar\rho}}} \tag{70}$$

Eq. (70) is the Froude number expressed in terms of vertical and horizontal velocities respectively.

Now in x-direction, the Froude number between two stratified layers can be expressed as:

$$F_r=\frac{u}{\sqrt{g\frac{(\rho_2-\rho_1)}{\rho_1}h}}=\frac{u}{c} \tag{71}$$

Eq. (71) is the dimensionless quantity which helps to determine the stability and propagation of internal waves which occurs at the interface between the two layers.

$$q=\begin{bmatrix}h_1\\h_1u_1\\h_1v_1\\h_2\\h_2u_2\\h_2v_2\end{bmatrix}$$

$$S(q)=\begin{bmatrix}0\\ -\frac{g\Delta\rho}{\bar\rho}h_1\frac{\partial h_2}{\partial x}-\frac{g\Delta\rho}{\bar\rho}h_1\frac{\partial\xi}{\partial x}+fv_1\\ -\frac{g\Delta\rho}{\bar\rho}h_1\frac{\partial h_2}{\partial y}-\frac{g\Delta\rho}{\bar\rho}h_1\frac{\partial\xi}{\partial y}-fu_1\\ 0\\ -\frac{g\Delta\rho}{\bar\rho}h_1\frac{\partial h_2}{\partial x}-\frac{g\Delta\rho}{\bar\rho}h_2\frac{\partial\xi}{\partial x}+fv_2\\ -\frac{g\Delta\rho}{\bar\rho}h_1\frac{\partial h_2}{\partial y}-\frac{g\Delta\rho}{\bar\rho}h_2\frac{\partial\xi}{\partial y}-fu_2\end{bmatrix}$$

$$f(q)=\begin{bmatrix} h_1u_1 \\ h_1u_1^2+\frac{1}{2}\frac{g\Delta\rho}{\bar{\rho}}h_1^2 \\ h_1u_1v_1 \\ h_2u_2 \\ h_2u_2^2+\frac{1}{2}\frac{g\Delta\rho}{\bar{\rho}}h_2^2+\frac{g\Delta\rho}{\bar{\rho}}h_2h_1 \\ h_2u_2v_2 \end{bmatrix}\qquad g(q)=\begin{bmatrix} h_1v_1 \\ h_1u_1v_1 \\ h_1v_1^2+\frac{1}{2}\frac{g\Delta\rho}{\bar{\rho}}h_1^2 \\ h_2v_2 \\ h_2u_2v_2 \\ h_2v_2^2+\frac{1}{2}\frac{g\Delta\rho}{\bar{\rho}}h_2^2+\frac{g\Delta\rho}{\bar{\rho}}h_2h_1 \end{bmatrix}$$

We can then calculate, the flux Jacobians of $f(q)$ as:

$$f'(q)=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -u_1^2+\frac{g\Delta\rho}{\bar{\rho}}h_1 & 2u_1 & 0 & 0 & 0 & 0 \\ -u_1v_1 & v_1 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{g\Delta\rho}{\bar{\rho}}h_2 & 0 & 0 & -u_2^2+\frac{g\Delta\rho}{\bar{\rho}}h_2+\frac{g\Delta\rho}{\bar{\rho}}h_1 & 2u_2 & 0 \\ 0 & 0 & 0 & -u_2v_2 & v_2 & u_2 \end{bmatrix} \tag{72}$$

$$g'(q)=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -u_1v_1 & v_1 & u_1 & 0 & 0 & 0 \\ -v_1^2+\frac{g\Delta\rho}{\bar{\rho}}h_1 & 0 & 2v_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u_2v_2 & v_2 & u_2 \\ \frac{g\Delta\rho}{\bar{\rho}}h_2 & 0 & 0 & -u_2^2+\frac{g\Delta\rho}{\bar{\rho}}h_2+\frac{g\Delta\rho}{\bar{\rho}}h_1 & 0 & 2v_2 \end{bmatrix} \tag{73}$$

3.5 Calculating the eigenspace

A key step in solving multi-layer deep water equations is calculating the Eigen space of the system, which consists of two sets: eigenvalues and eigenvectors, one set corresponds to slightly perturb classical deep water gravity waves, while the other set corresponds to internal waves traveling at a much lower speed. By the use of the Maple Software, we have the following eigenvalues and eigenvectors, for $A(q)$:

$$\lambda_1(q)=u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1},\lambda_2(q)=u_1+\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1},\lambda_3(q)=u_2,\lambda_5(q)=u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}+\frac{g\Delta\rho}{\bar{\rho}}h_2,\lambda_6(q)=u_1 \tag{74}$$

And the corresponding eigenvectors

$$\chi_1=\begin{bmatrix} \dfrac{\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\left(-u_1^2+\frac{g\Delta\rho}{\bar{\rho}}h_1\right)^2\left(u_2^2-\frac{g\Delta\rho}{\bar{\rho}}h_2-\frac{g\Delta\rho}{\bar{\rho}}h_1+\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)^2-2u_2\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\right)}{v_2\left(-u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)^2\frac{g\Delta\rho}{\bar{\rho}}h_2\left(-u_1^2-\frac{g\Delta\rho}{\bar{\rho}}h_1+u_1\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\right)} \\ \dfrac{\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\left(-u_1^2+\frac{g\Delta\rho}{\bar{\rho}}h_1\right)^2\left(u_2^2-\frac{g\Delta\rho}{\bar{\rho}}h_2-\frac{g\Delta\rho}{\bar{\rho}}h_1+\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)^2-2u_2\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\right)}{v_2\left(-u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\frac{g\Delta\rho}{\bar{\rho}}h_2\left(-u_1^2-\frac{g\Delta\rho}{\bar{\rho}}h_1+u_1\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\right)} \\ \dfrac{\left(-u_1^2+\frac{g\Delta\rho}{\bar{\rho}}h_1\right)^2\left(u_2^2-\frac{g\Delta\rho}{\bar{\rho}}h_2-\frac{g\Delta\rho}{\bar{\rho}}h_1+\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)^2-2u_2\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\right)v_1}{v_2\left(-u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\frac{g\Delta\rho}{\bar{\rho}}h_2} \\ \dfrac{-u_1^2+\frac{g\Delta\rho}{\bar{\rho}}h}{v_2\left(-u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)\left(u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)} \\ \dfrac{-u_1^2+\frac{g\Delta\rho}{\bar{\rho}}h}{v_2\left(-u_1-\sqrt{\frac{g\Delta\rho}{\bar{\rho}}h_1}\right)} \\ 1 \end{bmatrix} \tag{75}$$

Eq. (75) presents the eigenvector equations that are crucial for studying the behavior of waves, deep water currents, and many other oceanic phenomena. These equations provide information about both the direction and magnitude of wave motion, which are essential for predicting wave patterns and assessing their impact on offshore structures. They are also valuable for seismic analysis in deep water.

4. RESULT AND DISCUSSION

The investigation shows that the stratification of the deep water column influences the behavior of water particles, currents, and waves under the influence of modified gravity. This leads to changes in deep water dynamics and stability.

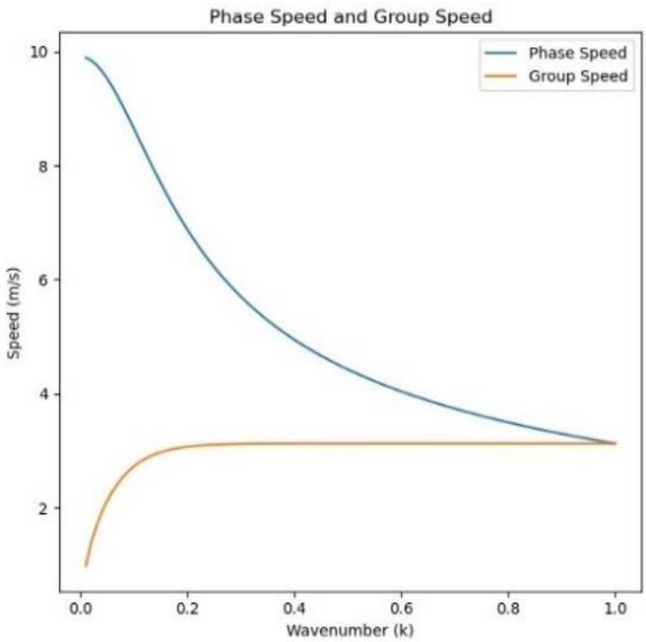

Figure 4. Speed vs. wavenumber

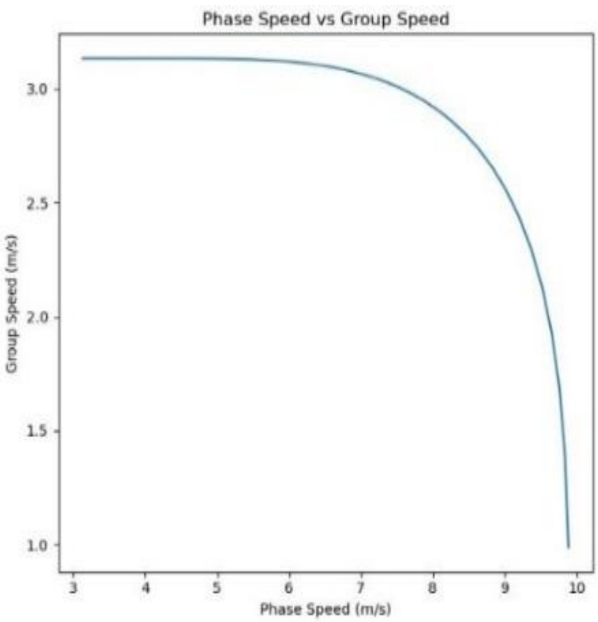

Figure 5. Group speed vs. phase speed

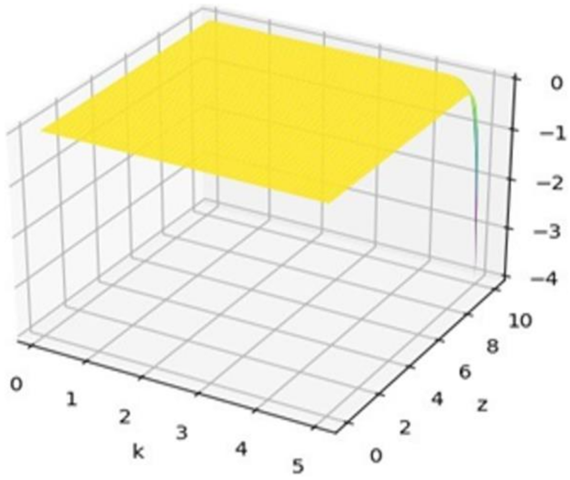

Figure 6. 3D mesh plot of A vs. wave number (k) and z

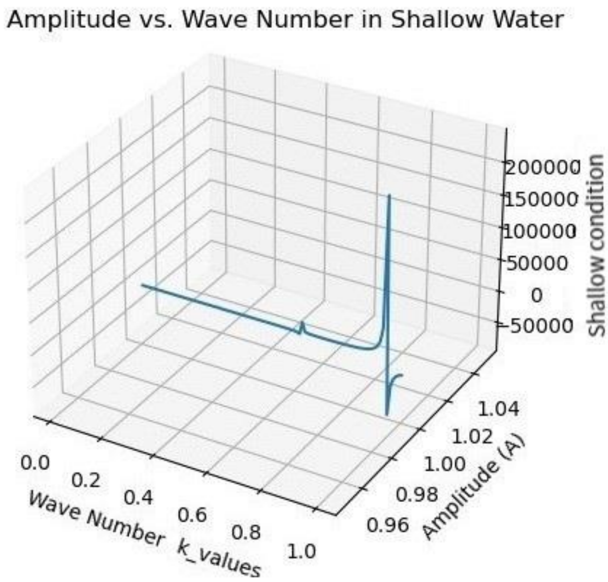

Figure 7. Amplitude vs. wave number in shallow water

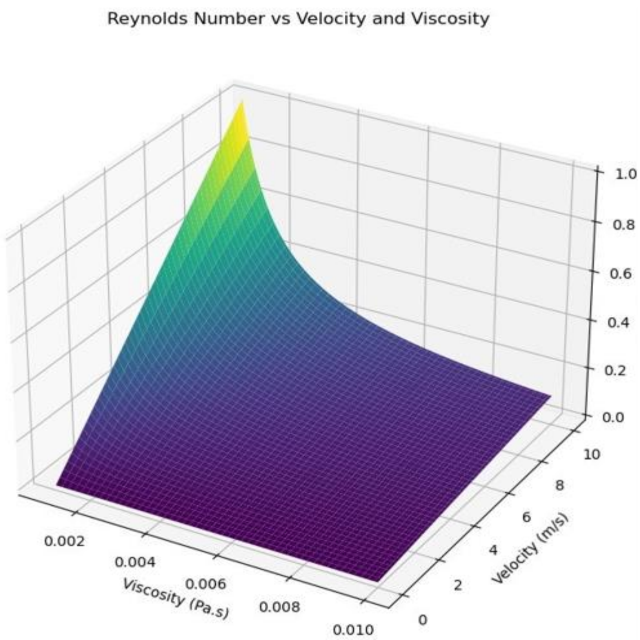

Figure 8. Reynolds number vs. velocity and viscosity

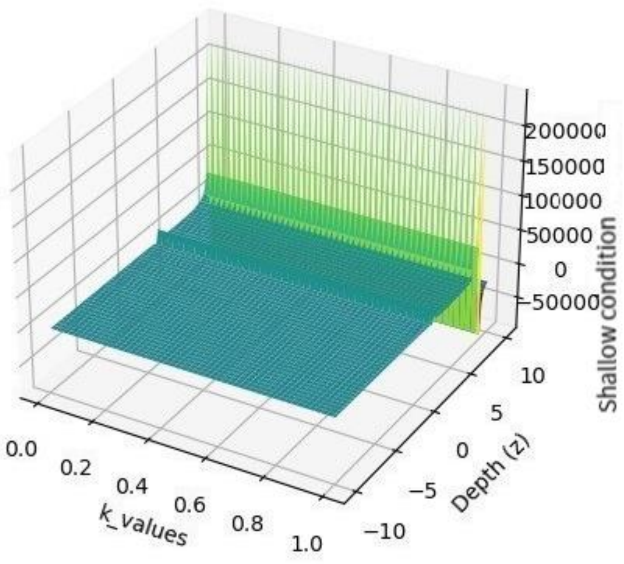

Figure 9. Depth vs. wavenumber (k)

Figure 4 displays the dispersion for wave propagation with modified gravity in deep water as density changes abruptly at the interface for stratified deep water flow within established thermocline regime. Figure 5 is the effect of phase speed with modified gravity in stratified deep water when density is greater near the bottom at the transition zone (thermocline). Figure 6 shows how gravity and friction can restore the bottom boundary condition with gravity modification, $g\frac{\Delta\rho}{\bar{\rho}}$. Figure 7 shows that the density increases with height and the fluid is unstable when the bottom is perturbed due to inequalities resulting from the wave amplitudes. Figure 8 shows that when a flow is relative to obstacle, the approach flow is uniform with height and this underscores the significant of vertical and horizontal amplitude of the surface wave as demonstrated in Eqs. (48) and (54). Figure 9 shows amplitudes of surface waves which depict the condition when the vertical amplitude is equal to the horizontal amplitude, hence the sufficient condition for deep water stratification.

5. CONCLUSION

We have sufficiently established that the condition for deep water stratification is when the vertical amplitude of the surface waves is equal to the horizontal amplitude of the wave as demonstrated in our simulation in Figure 8 and the amplitude of vertical and horizontal surface waves when equal establishes the stratified deep water waves. The effects of amplitude on deep water waves varies and depends on the specific stratification of water column, the layers with different densities as we demonstrated with our numerical simulations.

While the effect of gravity modification, vertical and horizontal amplitudes, Coriolis force and friction was considered and incorporated into our work unlike in previous works. The model will make significant contributions to the understanding of wave dynamics, coastal engineering and oceanography. The research improves understanding of wave forecasting models; understanding of wave-induced currents and in designing structures that can withstand load stress.

The limitation of the research is the problem of not obtaining accurate measurements of amplitude of stratified deep water waves and using estimated data values for simulations, the complexity of wave and current interaction in stratified deep water poses a significant challenge to the understanding of the amplitude of deep water and must be factored in subsequent model assumptions and formulation.

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NOMENCLATURE

$u = (u, v, w)$	The three-dimensional velocity vector
ρ	The density
p	The pressure
g	The gravity constant
f	Coriolis parameter
u	Velocity in the horizontal x direction
v	Velocity in the horizontal y direction
L	Length scale
ρ_0, T_0, p_0	Are reference values of density, temperature and Salinity respectively
h	Vertical length scale
ς	Free surface elevation
x	Horizontal, x direction
y	Horizontal, y direction
z	Vertical z direction
t	Time
$\frac{D}{Dt}$	Material derivative
$h(x, y, t)$	The height of water surface from the same reference height
H	Depth of stratified deep water
Fr	Froude number
$\xi(x, y)$	Denotes the thermocline regime
H	Undisturbed free surface level
h^*	The water height above the thermocline regime
δx	Width in the x −direction
δy	Width in the y −direction
u_1	Velocity in the first layer in the x −direction
v_1	Velocity in the first layer in the y −direction
u_2	Velocity in the second layer in the x − direction
v_2	Velocity in the second layer in the y − direction
F	Sum of all forces
m	Mass
a	Acceleration of the block of water
λ	Eigenvalues
χ	Eigenvector
A_Z	Horizontal amplitude
A_X	Vertical amplitude
A	Amplitude
K	Wave number
$\Delta\rho$	Change in density
$\bar{\rho}$	Density bar
g'	Modified gravity