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Application of the Homotopy Analysis Method in Solving Fuzzy Nonlinear Integral Equations for Birthrate

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ABSTRACT

In this work, we presented the fuzzy generalized integral equation for population includes future surge in birthrates which is great interesting for future planning throughout the world. So, we formulated the fuzzy problem of birthrates to be investigated any surge in fuzzy birthrates. The fuzzy integral equation (FIE) included the fuzzy birthrates at time t and the fuzzy function of a girl living to age r_2 and unclear role of a girl giving birth to a female kid at this age in an interval time dr_2. This fuzzy integral contributed for fuzzy birthrate from woman in the suitable subinterval of the range of childbearing age r_2. The homotopy analysis method (HAM) for explained the numerical fuzzy solution of periodicity in the surge of birthrate of generalized fuzzy dynamical birthrate of girl born at any as well as all the tables and figures are given in details also the behaviors of the birthrate which described the solution of fuzzy integral.

1. INTRODUCTION

In applications of mathematics there are many types of fuzzy ordinary differential equations and FIEs which interesting of many mathematical models of physics, geographic, medical and biology have been studied by many researchers and the given an interesting result that which made a good role in fuzzy problem of applications and these formulations of integrals depended of some parameters that have a range between [0, 1] which represented as a fuzzy number. The HAM was used to treat the nonlinear FIEs to obtain their solution which described the behaviors of fuzzy mathematical applications. The generalized FIEs contain the fuzzy kernel function and fuzzy non-linear functions which are formulated the birthrate fuzzy integral that are studied and HAM used to obtain the solutions in this paper. The fuzzy integral birthrates applications are a task was the first time introduced to human population by Fuzzy Periodicity in the surge of birthrates and how population growth and explained how to use the numerical method for approximate solution of linear and nonlinear FIEs in 2000 [1].

Several researches [2-7] studied FIEs with some transformation or some numerical methods. Also, Bushnaq et al. [8] in 2020 studied in details the fuzzy integrals with novel hyper method. Esa et al. [9] in 2021 some treatment of numerical solution of Volterra integral equations introduced with some interests steps. In the study conducted by Ezzati et al. [10], the linear and nonlinear FIEs have been studied and their numerical solutions with some applications are given in 2014. Hasan and Abdulqader [11-13] presented many FIEs with different types of numerical methods. The singular kernel involved in some FIEs have been introduced by Samadpour Khalifeh Mahaleh et al. [14]. In 2021, the Bernstein polynomial used to find the numerical solution of FIE is studied by Otadi in 2011 [15], finally the fuzzy Riemann integral introduced with numerical methods for their application. From above literature review there are many authors study the homotopy analytic method as well as used for different types of FIEs, so it is a good method used for explain the numerical solution of fuzzy birthrate of girl to growth the population.

The purpose of this work to generalized the application of the dynamical growth of population by periodicity in the surge of birthrate in fuzzy space and observe the numerical solution of the fuzzy birthrate of all woman.

This paper is organized as follows: there are three different models we applied with our problem formulation and solve it by using two different methods to solve the above models and show the approximate solution and calculate the absolute error and also give the fingers for all models and comparison between the approximate and exact solutions with different α-cut.

2. PRELIMINARIES

Some fundamental ideas for comprehending the birthrate FIE in detail are covered in this section to provide an explanation of the fuzzy numbers in a broader sense.

Definition 2.1 [16] The fuzzy number is a function $u: \mathrm{R} \to [0, 1]$ having the following properties:

(1) G is normal, that is $\exists x_0 \in \mathrm{R}$ such that $G(x_0) = 1$.

(2) G is convex fuzzy set that is $G(\lambda x + (1-\lambda)y) \geq min\,\{G(x), G(y)\}, \forall\, x, y \in \mathrm{R}, 0 \leq \lambda \leq 1$.

(3) G is the upper semi-continuous on R.

(4) The support $[G]_0 = cl\{x \in R: G(x) > 0\}$ is a compact set. The set of all fuzzy numbers is denoted by R_F.

Remark 2.2 [17]

1) Any $a \in$ R which can be described as $\tilde{a} = \chi_{\{a\}}$ and then $R \subset R_F$.
2) The r-level set $[G]_r = \{x \in R: G(x) \geq r\}$ that is a closed interval $[G]_r = [\underline{G}_r, \overline{G}_r]$ for all $0 \leq r \leq 1$.
3) The fuzzy number $(\underline{G}(r), \overline{G}(r))$ where $\underline{G}, \overline{G}$ which let as functions $\underline{G}, \overline{G}: [0,1] \to$ R such that $\underline{G}(r)$ is increasing and $\overline{G}(r)$ is decreasing.

In the following the definitions and remarks of fuzzy number and their operations are introduced as well as constrictions of FIEs.

Definition 2.2 [17] The for $G, v \in R_F$ and $\lambda \in R$, the sum $G +_F v$ defined by $[G +_F v]_r = [G]_r + [v]_r$ which represented the addition of two intervals (as sub sets of R) and the product λG defined by $[\lambda G]_r = \lambda [G]_r$, for all $0 \leq r \leq 1$, which represented the product between a subset of R And a scalar.

The interesting issue is to introduce on properties of fuzzy number operations

Remark 2.3 [18] The following algebraic prosperities for any $G, y, w \in R_F$, hold:

1) $G +_F (v +_F w) = (G +_F v) +_F w$ and $G +_F v = v +_F G$
2) $G +_F \tilde{0} = \tilde{0} +_F G$ for $\tilde{0} = \chi_{\{0\}}$
3) $(a +_F b) ._F G = a ._F G +_F b ._F G$, for all $a, b \in$ R with $ab \geq 0$.
4) $a ._F (G +_F v) = a ._F G +_F a ._F v$ for all $a \in$ R
5) $a ._F (b ._F v) = (a ._F b) ._F G$, for all $a, b \in$ R and $1 ._F G = G$.

The following definition explain the fuzzy Riemann Integral in lower and upper fuzzy number with their membership functions.

Definition 2.3 [19] The fuzzy Riemann integral of $[a, b]$ for $\tilde{f}(x) = [\tilde{f}_\alpha{}^L(x), \tilde{f}_\alpha{}^R(x)]$ which is fuzzy value function and closed and bounded on $[a, b]$. Then the Riemann integral on $[a, b]$ and $\alpha \in [0,1]$ defined as $A_\alpha = [\int_a^b \tilde{f}_\alpha{}^L(x)dx, \int_a^b \tilde{f}_\alpha{}^R(x)\, dx]$ and the membership function of $\int_a^b \tilde{f}(x)\, dx$ is denoted by $M_{\int_a^b \tilde{f}(x)dx}(\alpha) = \sup_{0 \leq \alpha \leq 1} \alpha . 1_{A_\alpha}(r)$.

Definition 2.4 [19] The integral of a fuzzy function was defined using the idea of the Riemann integral.

Let f: $[a, b] \to E^1$. For each p={q0, …, qn} as a partition of the fuzzy function f of:

$$R_P = \sum_{i=1}^{n} f(\xi_i)\Delta(t_i - t_{i-1}) \quad \text{for } \xi_i \in [t_{i-1}, t_i],\ 1 \leq i \leq n \tag{1}$$

$$\Delta(t_i - t_{i-1}) = \max\{|t_i - t_{i-1}|, 1 \leq i \leq n\}.$$

The integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_P \tag{2}$$

If the fuzzy function $f(t)$ is continuous in a suitable metric space, then:

$$\int_a^b \underline{f(t;\alpha)dt}) = \int_a^b \underline{f}(t;\alpha)dt, \quad \int_a^b \overline{f(t;\alpha)}dt) = \int_a^b \overline{f}(t;\alpha)dt \tag{3}$$

We examine that a Lebesgue method can also be used to define the fuzzy integral.

3. FUZZY PERIODICITY IN THE SURGE OF BIRTHRATES AND ITS SOLUTION USING HAM

The birthrate at time t is represented by $\tilde{u}(t)$ and the contribution to the birthrates of girls born at time $r_2 > 0$ when they are at age τ in the range of childbearing age $\alpha \leq r_2 \leq \beta$ must be added. This is similar to the population dynamical model [20], where $n_0\tilde{f}(t)$ represents the woman present at this initial time t=0 and that they will give birth to female child at a rat $\tilde{f}(t)$ per year. Girls born at time $t - r_2$ will at future time belong to the fuzzy birthrate function $\tilde{k}_1(t, r_2) = k\ \tilde{u}(t, r_2)$ where k is a constant. If the probability of girl living to age r_2 is $\tilde{F}_1(r_2, u(r_2))$ and the unclear function of a girl at her age bearing a female child within a certain amount of time.

Δr_2 is a $G(r_2, \int_0^\tau \tilde{F}_2(s, \tilde{u}(s)ds)\Delta\tau$ then the contribution to the birthrate $\tilde{u}(t)$ from woman in the subinterval $\Delta\tau$ of the range of childbearing age r_2,

$$\tilde{k}_1(t, r_2, \tilde{F}_1(r_2, \tilde{u}(r_2)) ._F G(r_2, \int_0^{r_2} \tilde{F}_2(s, \tilde{u}(s)ds) = \tilde{k}_1(t - r_2)\tilde{F}_1(r_2, u(r_2)) ._F G(r_2, \int_0^\tau \tilde{F}_2(s, \tilde{u}(s))ds) \tag{4}$$

[born at $t - \tau$ with birth rate $\tilde{k}_1(t, r_2)$] is the contribution birthrate $k\ \tilde{u}(t, r_2)$ at time t from all woman, except for those present at time t=0 in the childbearing age rang $\alpha \leq r_2 \leq \beta$. If pass to limit and include birthrates due to woman present at time t=0, we have

$$\tilde{u}(t) = n_0\tilde{f}(t) +_F \int_0^t \tilde{k}_1(t - r_2)\tilde{F}_1(r_2, \tilde{u}(r_2)) . G(r_2, \int_0^{r_2} \tilde{F}_2(s, \tilde{u}(s))ds\,)dr_2$$

This is an integral equation representing the birthrate u(t) at time t due girls born after t=0, as in the fuzzy integral, and woman present at t=0, as $n_0\tilde{f}(t)$. The FIE can be written as

$$\tilde{u}(t) = \begin{cases} n_0\tilde{f}(t) +_F \int_0^t \tilde{k}_1(t - r_2)\tilde{F}_1(r_2, \tilde{u}(r_2)) \\ \quad . G(r_2, \int_0^{r_2} \tilde{F}_2(s, \tilde{u}(s))ds)dr_2, & t \leq \beta \\ \int_0^t \tilde{k}_1(t - r_2)\tilde{F}_1(\tau, \tilde{u}(r_2)) . G(\tau, \int_0^{r_2} \tilde{F}_2(s, \tilde{u}(s))ds\,)dr_2, & t > \beta \end{cases} \tag{5}$$

This is a second-order Volterra FIE with a distinct kernel. Now we will representation the FIE in the terms of fuzzy logic, that is as a FIE, letting and $\tilde{u}(t) = [\underline{u}(t), \overline{u}(t)]$ and $n_0\tilde{f}(t) = n_0[\underline{f}(t), \overline{f}(t)]$ and using the Reimman FIE, we obtain and $\tilde{F}_{1\alpha}(t, r_2, u(\tau, \alpha))$ is denoted a fuzzy function of the girl living to age is the fuzzy function equal to $e^{-u(\tau)^\alpha}$ and $G(r_2, \int_a^{r_2} \tilde{F}_{2\alpha}(r_2, s, u(s, \alpha)))$ ds is denoted of fuzzy function the girl is give birth bearing to a female is the fuzzy function equal to $\alpha e^{-\alpha\tau}$ and n_0 is denoted by the number for woman in the time t finally $\tilde{f}(t)$ is the probability to give female per year is a fuzzy function equal to t^α, therefore

$$\overline{u}(t) = n_0 t^{\alpha} + (\int_0^t \alpha e^{-\alpha(t-r_2)}(e^{-\overline{u}(r_2)^{\alpha}}).\int_0^{r_2}(\alpha e^{-\alpha\overline{u}(s)})ds)dr_2$$
$$\underline{u}(t) = n_0 t^{\alpha} + (\int_0^t \alpha e^{-\alpha(t-r_2)}(e^{-\underline{u}(r_2)^{\alpha}}).\int_0^{r_2}(\alpha e^{-\alpha\underline{u}(s)})ds)dr_2 \quad (6)$$

and the range childbearing [0,1]

$$[\underline{u}(t), \overline{u}(t)]_r = n_0[\underline{f}(t), \overline{f}(t)]_r$$
$$+[\int_0^t k_1(t-r_2)F_1(r_2, u(r_2)).G(r_2, \int_0^{r_2} F_2(s, u(s)ds)dr_2$$
$$, \int_0^t k_1(t-r_2)F_1(r_2, u(r_2)).G(r_2, \int_0^{r_2} F_2(s, u(s)ds)dr_2]_r \quad (7)$$

We will use the condition for FIE and using our first problem formula, we obtain:

$$\underline{u}(t,\alpha) = \underline{f}(t,\alpha) +$$
$$\lambda \int_a^t k_1((r_1, r_2, F_{1\alpha}(r_1, r_2, u(r_2,\alpha))).G(\tau, \int_a^{\tau} F_{2\alpha}(r_2, s, u(s,\alpha))ds\, dr_2$$
$$\overline{u}(t,\alpha) = \overline{f}(t,\alpha) +$$
$$\int_a^t k_1((r_1, r_2, F_{1\alpha}(r_1, r_2, u(r_2,\alpha))).G(\tau, \int_a^{\tau} F_{2\alpha}(r_2, s, u(s,\alpha))ds)dr_2 \quad (8)$$

where,

$$k_1(r_1, r_2, F_{1\alpha}(r_1, u(r_1,\alpha))).G(r_1, \textstyle\int_a^{r_1} F_{2\alpha}(r_1, s, u(s,\alpha))ds) =$$
$$\begin{cases} k_1(r_1, r_2, \underline{F}_{1\alpha}(r_1, \underline{u}(r_1,\alpha))).G(r_1, \int_a^{r_1} \underline{F}_{2\alpha}(r_1, s, \underline{u}(s,\alpha))ds), \\ \quad k_1(x, r_1, F_{1\alpha}(r_1, u(t,\alpha))).G(r_1, \int_a^{r_1} F_{2\alpha}(r_1, s, u(s,\alpha))ds) \geq 0 \\ k_1(r_1, \tau, \overline{F}_{1\alpha}(r_1, \overline{u}(r_1,\alpha)).G(r_1, \int_a^{r_1} \overline{F}_{2\alpha}(r_1, s, \overline{u}(s,\alpha))ds), \\ \quad k_1(x, r_1, F_{1\alpha}(r_1, u(r_1,\alpha))).G(r_1, \int_a^{r_1} F_{2\alpha}(r_1, s, u(s,\alpha))ds) < 0 \end{cases}$$
$$k_1(r_1, \tau, F_{1\alpha}(x, r_1, u(r_1,\alpha))).G(r_1, \textstyle\int_a^t F_{2\alpha}(r_1, s, u(s,\alpha))ds) =$$
$$\begin{cases} k_1(r_1, r_2, \overline{F}_{1\alpha}(r_1, \overline{u}(r_1,\alpha))).G(r_1, \int_a^{r_1} \overline{F}_{2\alpha}(r_1, s, \overline{u}(s,\alpha))ds), \\ \quad k_1(r_1, \tau, F_{1\alpha}(x, r_1, u(r_1,\alpha))).G(r_1, \int_a^{r_1} F_{2\alpha}(r_1, s, u(s,\alpha))ds) \geq 0 \\ k_1(r_1, r_2, \underline{F}_{1\alpha}(r_1, \underline{u}(r_1,\alpha))).G(r_1, \int_a^{r_1} \underline{F}_{2\alpha}(r_1, s, \underline{u}(s,\alpha))ds), \\ \quad k_1(r_1, r_2, F_{1\alpha}(r_1, r_2, u(r_1,\alpha))).G(r_1, \int_a^{r_1} F_{2\alpha}(r_1, s, u(s,\alpha))ds) < 0 \end{cases} \quad (9)$$

For each $0 \leq \alpha \leq 1$ and $a \leq x \leq b$.

Now, we employ HAM for solve the system (9), but prior to apply HAM the kernel function will have cases as follows:

$$N[\tilde{u}(x,\alpha)] = 0$$
$$\tilde{u}(t) = \tilde{f}(t)$$
$$+_F \lambda \int_a^t \tilde{k}_1(t, r_2, \tilde{F}_1(t, r_2, \tilde{u}(t))) \cdot_F G(t, \int_a^t \tilde{F}_2(t, s, \tilde{u}(s))ds)dr_2 \quad (10)$$

where,

$$\tilde{u}(t,\alpha) = (\underline{u}(t,\alpha), \overline{u}(t,\alpha))$$
$$\tilde{f}(t,\alpha) = (\underline{f}(t,\alpha), \overline{f}(t,\alpha)) \quad (11)$$

We see that Eq. (10) is transformed into a nonlinear crisp Volterra FIE system

$$\underline{u}(t,\alpha) = \underline{f}(t,\alpha) + \lambda \int_a^t k_1((t, r_2, F_{1\alpha}(t, r_2, u(t,\alpha)))$$
$$.G(t, \int_a^t F_{2\alpha}(t, s, u(s,\alpha))ds)dr_2 \quad (12a)$$

$$\overline{u}(t,\alpha) = \overline{f}(t,\alpha) + \lambda \int_a^x k_1(t, r_2, F_{1\alpha}(t, r_2, u(t,\alpha)))$$
$$.G(t, \textstyle\int_a^t F_{2\alpha}(t, s, u(s,\alpha))ds)dr_2 \quad (12b)$$

We develop the zeroth-order deformation equations to solve system (12) using HAM:

$$(1-p)L[\underline{\emptyset}(t,p;\alpha) - \underline{w}_0(t;\alpha)] = phH[\underline{\emptyset}(t,p;\alpha) - \underline{f}(t,\alpha)$$
$$- \int_a^t k_1(t, r_2, \underline{F}_{1\alpha}(t, r_2, \underline{\emptyset}(t,p;\alpha)))$$
$$.G(t, \int_a^t \underline{F}_{2\alpha}(t, s, \underline{\emptyset}(s,p,\alpha))ds)dt \quad (13)$$

$$(1-p)L[\overline{\emptyset}(t,p;\alpha) - \overline{w}_0(x;\alpha)] = phH[\overline{\emptyset}(t,p;\alpha) - \overline{f}(t,\alpha)$$
$$- \int_a^t k_1(t, r_2, \overline{F}_{1\alpha}(t, r_2, \overline{\emptyset}(t,p;\alpha)))$$
$$.G(t, \int_a^t \overline{F}_{2\alpha}(t, s, \overline{\emptyset}(s,p,\alpha))ds)dr_2 \quad (14)$$

and $p \in [0,1]$ be the embedding parameter, his additional, nonzero parameter, L is a linear auxiliary operator. An auxiliary function is H(x). $(\underline{w}_0(t;\alpha), \overline{w}_0(t;\alpha))$ are initial guesses of $\underline{F}_{1\alpha}(t, \tau, \underline{\emptyset}(t,p;\alpha)), \overline{F}_{1\alpha}(t, s, \overline{\emptyset}(t,p;\alpha)), \overline{F}_{2\alpha}(t, \tau, \overline{\emptyset}(t,p;\alpha))$ and $\underline{F}_{2\alpha}(t, s, \underline{\emptyset}(t,p;\alpha))$ respectively and unknown function $\underline{\emptyset}(t,p;\alpha)$ and $\overline{\emptyset}(t,p;\alpha)$. Let $L(u) = u$ and $H(x) = 1$ used with zeroth-order deformation equations which given above, we get that

$$(1-p)[\underline{\emptyset}(t,p;\alpha) - \underline{w}_0(t;\alpha)] = ph[\underline{\emptyset}(t,p;\alpha) - \underline{f}(t,\alpha)$$
$$- \int_a^t k_1(t, r_2, \underline{F}_{1\alpha}(t, r_2, \underline{\emptyset}(t,p;\alpha)))$$
$$G(t, \int_a^t \underline{F}_{2\alpha}(t, s, \underline{\emptyset}(s,p,\alpha))ds)dr_2 \quad (15)$$

$$(1-p)[\overline{\emptyset}(t,p;\alpha) - \overline{w}_0(t;\alpha)] = ph[\overline{\emptyset}(t,p;\alpha) - \overline{f}(t,\alpha)$$
$$- \int_a^x k_1(t, r_2, \overline{F}_{1\alpha}(t, \tau, \overline{\emptyset}(t,p;\alpha)))$$
$$G(t, \int_a^t \overline{F}_{2\alpha}(t, s, \overline{\emptyset}(s,p,\alpha))ds)dr_2 \quad (16)$$

when $h \neq 0$, $p=0$ and $p=1$ we get that

$$\underline{\emptyset}(t,0;\alpha) = \underline{w}_0(t;\alpha),\ \overline{\emptyset}(t,0;\alpha) = \overline{w}_0(t;\alpha),$$
$$\underline{\emptyset}(t,1;\alpha) = \underline{f}(t,\alpha) + \int_a^t k_1(t, r_2, \underline{F}_{1\alpha}(t, \tau, \underline{\emptyset}(t,1;\alpha)))$$
$$.G(t, \int_a^t \underline{F}_{2\alpha}(t, s, \underline{\emptyset}(s,1,\alpha))ds)dr_2 \quad (17)$$
$$\overline{\emptyset}(t,1;\alpha) = \overline{f}(t,\alpha) + \int_a^t k_1(t, r_2, \overline{F}_{1\alpha}(x, t, \overline{\emptyset}(t,1;\alpha))$$
$$`.G(t, \textstyle\int_a^t \overline{F}_{2\alpha}(t, s, \overline{\emptyset}(s,1,\alpha))ds)dr_2$$

Thus p from 0 to 1 increasing, the function $s\ \underline{\emptyset}(t,p;\alpha)$ $\overline{\emptyset}(t,p;\alpha)$ deforms from the initial guesses $\underline{w}_0(x;\alpha)$, $\overline{w}_0(x;\alpha)$ to the solution of $\underline{F}_{1\alpha}(t, r_2, \underline{\emptyset}(t,p;\alpha)), \overline{F}_{1\alpha}(t, s, \overline{\emptyset}(t,p;\alpha)), \overline{F}_{2\alpha}(t, r_2, \overline{\emptyset}(t,p;\alpha))$ and $\underline{F}_{2\alpha}(t, s, \underline{\emptyset}(t,p;\alpha))$, where $\underline{\emptyset}(t,p;\alpha)$expanded with respect pin Taylors series as follows:

$$\underline{\emptyset}(t,p;\alpha)=\underline{w}_0(t;\alpha)+\sum_{m=1}^{\infty}\underline{u}_m(t,\alpha)p^m \tag{18}$$

$$\overline{\emptyset}(t,p;\alpha)=\overline{w}_0(t;\alpha)+\sum_{m=1}^{\infty}\overline{u}_m(t,\alpha)p^m \tag{19}$$

where, for all $m \geq 1$

$$\begin{aligned}\underline{u}_m(t,r)&=\frac{1}{m!}\frac{\partial^m\underline{\emptyset}(t,p;r)}{\partial p^m}\bigg|_{p=0}\\ \overline{u}_m(t,r)&=\frac{1}{m!}\frac{\partial^m\overline{\emptyset}(t,p;r)}{\partial p^m}\bigg|_{p=0}\end{aligned} \tag{20}$$

It should be noted that $\underline{\emptyset}(t,p;\alpha)=\underline{w}_0(t;\alpha)$, $\overline{\emptyset}(t,p;\alpha)=\overline{w}_0(x;\alpha)$ and differentially the zeroth-order deformation Eq. (17) are differentiated m −times for the embedding parameter p and divided by m! and n!. Finally, if we setting $p=0$, we obtain th mth order deformation equations as follows:

$$\begin{aligned}\underline{u}_m(t;r)&=h[\underline{u}_{m-1}(t;r)-(1-\mathcal{X}_m)\underline{f}(t;r)\\ &-\int_a^x k_1(t,r_2,\underline{F}_{1\alpha}\left(t,r_2,\underline{\emptyset_{m-1}}(t,1;\alpha)\right))\\ &.G(t,\int_a^t\underline{F}_{2\alpha}(t,s,\underline{\emptyset_{m-1}}(s,1,\alpha))ds)dr_2\\ \overline{u}_m(t;r)&=h[\overline{u}_{m-1}(x;r)-(1-\mathcal{X}_m)\bar{f}(x;r)\\ &-\int_a^t k_1((t,r_2,\overline{F}_{1\alpha}\left(t,r_2,\overline{\emptyset}_{m-1}(t,p;\alpha)\right))\\ &G(t,\int_a^t\overline{F}_{2\alpha}(t,s,\overline{\emptyset_{m-1}}(s,p,\alpha))ds)dr_2\end{aligned} \tag{21}$$

where,

$$\mathcal{X}_m=\begin{cases}0, m=1\\ 1, m\neq 1\end{cases} m\geq 1 \tag{22}$$

$$\underline{\emptyset}(t,\alpha)=\underline{w}_0(x;\alpha), \tag{23}$$

$$\underline{R}_{m-1}(t,t;\alpha)=\frac{1}{(m-1)!}\frac{\partial^{m-1}k_1(t,t,\underline{F}_{1\alpha}\left(t,\underline{\emptyset}(x,p;\alpha)\right)).G(t,\int_a^t\underline{F}_{2\alpha}(t,s,\underline{\emptyset}(s,p,\alpha)ds)}{\partial p^{m-1}}\bigg|_{p=0} \tag{24}$$

and

$$\begin{aligned}&\overline{\emptyset}(t,\alpha)=\overline{w}_0(t;\alpha)\\ &\overline{R}_{m-1}(t,t;\alpha)=\frac{1}{(m-1)!}\frac{\partial^{m-1}k_1(t,\tau,\overline{F}_{1\alpha}\left(t,\overline{\emptyset}(x,p;\alpha)\right)).G(t,\int_a^t\overline{F}_{2\alpha}(t,s,\overline{\emptyset}(s,p,\alpha)ds)}{\partial p^{m-1}}\bigg|_{p=0}\end{aligned} \tag{25}$$

From (16) and (17), it may obtain that:

$$\begin{aligned}&\underline{u}_0(t,\alpha)=0\\ &\underline{u}_1(t;\alpha)=h\underline{u}_0(t;\alpha)-h\underline{f}(t;\alpha)\text{-}h[\textstyle\int_0^x\underline{R}_0(t,r_2;\alpha)dt]\\ &\underline{u}_m(t;\alpha)=(1+h)\underline{u}_{m-1}(t;\alpha)-\ h[\int_0^x\underline{R}_{m-1}(t,\tau;\alpha)dt]\end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned}&\overline{u}_0(t,\alpha)=0\\ &\overline{u}_1(t;\alpha)=h\underline{u}_0(t;\alpha)-h\underline{f}(t;\alpha)\text{-}h[\textstyle\int_0^x\underline{R}_0(t,r_2;\alpha)dt],\end{aligned} \tag{27}$$

19 and some general for any $m \geq 1$.

$$\overline{u}_m(t;\alpha)=(1+h)\underline{u}_{m-1}(t;\alpha)\text{-}\ h[\textstyle\int_0^x\underline{R}_{m-1}(t,r_2;\alpha)dt] \tag{28}$$

where,

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$$\begin{aligned}&\underline{R}_0(t,\tau;\alpha)=k_1\left(t,r_2,\underline{F}_{1\alpha}\left(t,\underline{u_0}(t,\alpha)\right)\right).G(t,\int_a^t\underline{F}_{2\alpha}(t,s,\underline{u}_0(s,\alpha))ds)\\ &\overline{R}_0(t,\tau;\alpha)=k_1\left(t,r_2,\overline{F}_{1\alpha}(t,\overline{u_0}(t,\alpha))\right).G(t,\int_a^t\overline{F}_{2\alpha}(t,s,\overline{u}_0(s,\alpha))ds)\end{aligned} \tag{29}$$

As a result, for h if we choose it as an appropriate value, and for p=1 the series (12) is convergent. The solution of system (17) by using homotopy series explained as follows:

$$\begin{aligned}\underline{u}(t,\alpha)&=\underline{u}_0(t;\alpha)+\sum_{m=1}^{\infty}\underline{u}_m(t;\alpha)\\ \overline{u}(t,\alpha)&=\overline{u}_0(t;\alpha)+\sum_{m=1}^{\infty}\overline{u}_m(t;\alpha)\end{aligned} \tag{30}$$

The approximate solution nth order is

21

$$\begin{aligned}\underline{u}(t,r)&=\underline{u}_0(t;r)+\sum_{m=1}^{\infty}\underline{u}_m(t;r)\\ \overline{u}(t,r)&=\overline{u}_0(t;r)+\sum_{m=1}^{\infty}\overline{u}_m(t;r)\end{aligned} \tag{31}$$

Then we have

$$\begin{aligned}\underline{u}(t)&=n_0t^\alpha+\int_0^t(\alpha\exp(-\alpha(t-r_2))*\left(\exp(-\underline{u})\right)\\ &*\int_0^\tau(\alpha\exp\,(-\alpha\underline{u}(s))ds)dr_2)\\ \overline{u}(t)&=n_0t^\alpha+\int_0^t(\alpha\exp(-\alpha(t-r_2))*(\exp(-\overline{u}(\tau)^\alpha))\\ &*\int_0^\tau(\alpha\exp\,(-\alpha\overline{u}(s))ds)dr_2\end{aligned} \tag{32}$$

and the range childbearing [0,1]. Now we will solve our formula by using HAM (Table 1 and Table 2):

$$\begin{aligned}&\underline{u}_0(t,\alpha)=0,\qquad \overline{u}_0(t,\alpha)=0\\ &\underline{u_1}(t,\alpha)=-hn_0\,f(t,\alpha)=-hn_0t^\alpha,\\ &\overline{u}_1(t,\alpha)=-hn_0\,f(t,\alpha)=-hn_0t^\alpha\\ &\underline{u}_2(t,\alpha)=(1+h)\underline{u}_1(t,\alpha)-\\ &\underline{h[\textstyle\int_a^x k_1((t,r_2,F_{1\alpha}(t,r_2,u_1(r_2,\alpha)))}\\ &\underline{.G(\tau,\textstyle\int_a^\tau F_{2\alpha}(r_2,s,u_1(s,\alpha))ds)}dr_2]\end{aligned} \tag{33}$$

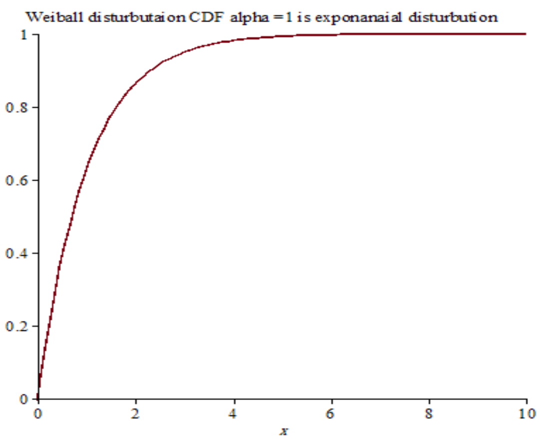
$$\begin{aligned}&\underline{\overline{u}_2(t,\alpha)=(1+h)\overline{u}_1(t,\alpha)}\\ &\underline{-h[\textstyle\int_a^x k_1((t,r_2,F_{1\alpha}(t,r_2,u_1(r_2,\alpha)))}\\ &.G(\tau,\int_a^\tau F_{2\alpha}\left(r_2,s,u_1(s,\alpha)\right)ds)dr_2]\end{aligned}$$

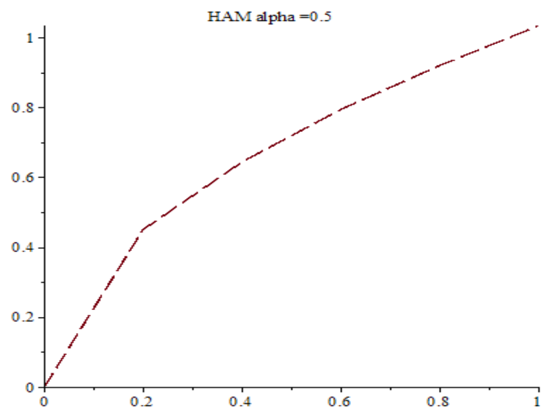
Table 1. Comparison between different value of α–level for lower side

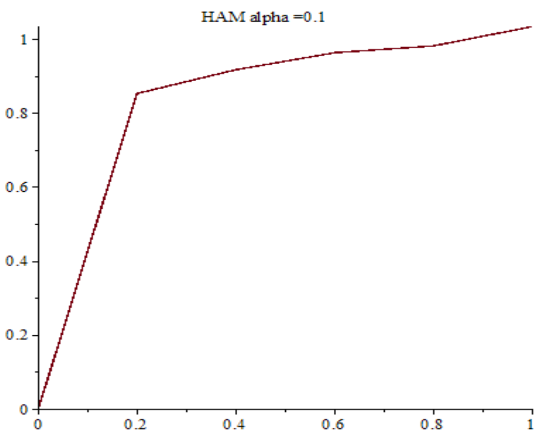
T	HAM α=0.1	HAM α=0.3	HAM α=0.5	HAM α=0.7	HAM α=1
0	0.0000000000	0.000000000	0.000000000	0.0000000000	0.0000000000
0.2	0.8527627004	0.6198952850	0.451302027	0.3297803820	0.2089865791
0.4	0.9179972352	0.7693030324	0.643746107	0.5393728540	0.4161517214
0.6	0.9805087361	0.8777057055	0.793714402	0.7171477769	0.6163292316
0.8	0.9995731871	0.9664994065	0.921171439	0.8258086139	0.8130439053
1	1.0335160020	1.0451791320	1.03383382	1.0212835220	1.0091571819

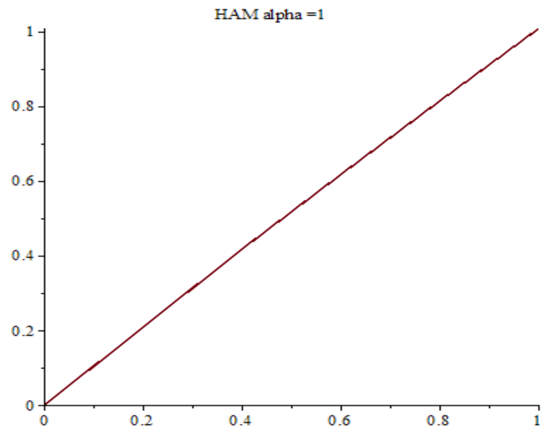
Table 2. Comparison between different value of α–level for upper side

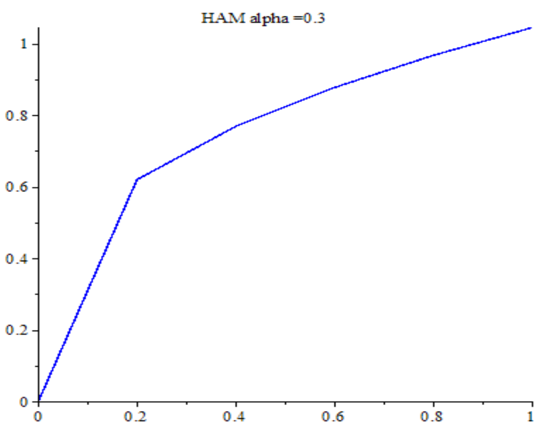
T	HAM α=0.1	HAM α=0.3	HAM α=0.5	HAM α=0.7	HAM α=1
0	0.0000000000	0.0000000000	0.000000000	0.000000000	0.0000000000
0.2	0.8527627004	0.3297803820	0.451302027	0.6198952850	0.2089865791
0.4	0.9179972352	0.5393728540	0.643746107	0.7693030324	0.4161517214
0.6	0.9805087361	0.7171477769	0.793714402	0.8777057055	0.6163292316
0.8	0.9995731871	0.8258086139	0.921171439	0.9664994065	0.8130439053
1	1.0335160020	1.0212835220	1.033833821	1.0451791320	1.0091571819

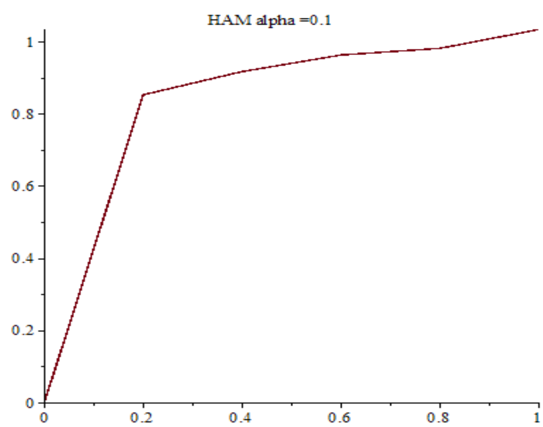


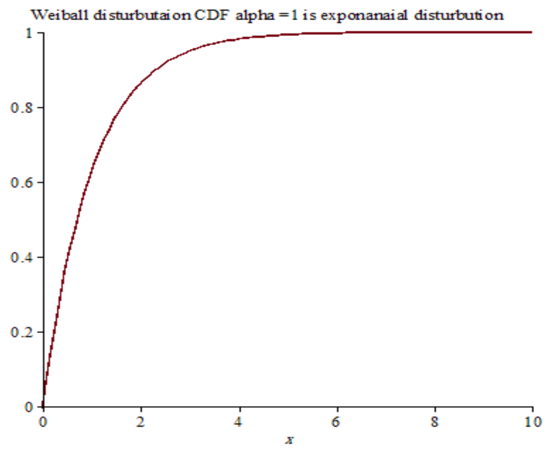


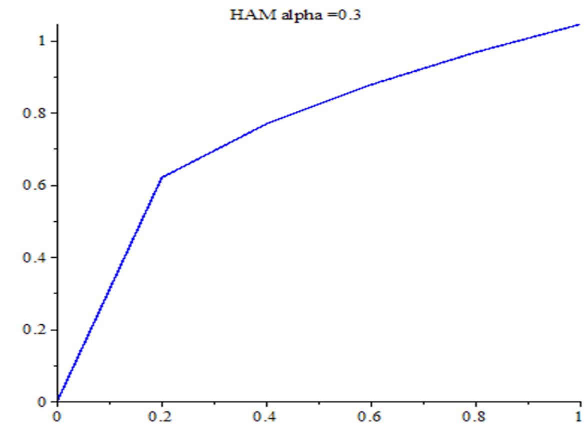


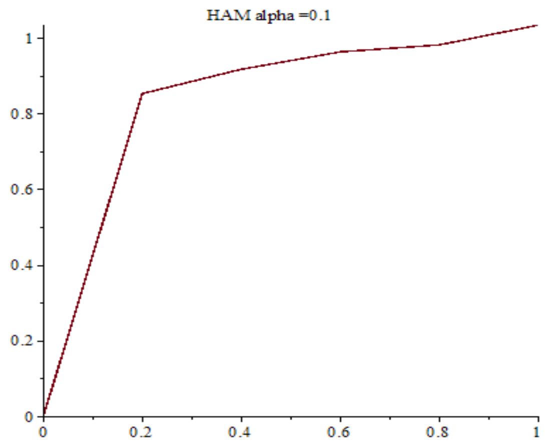


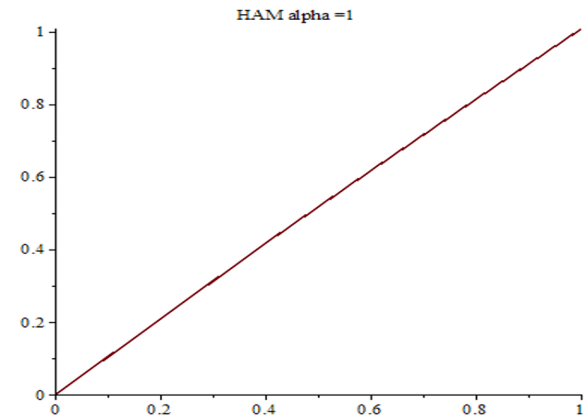


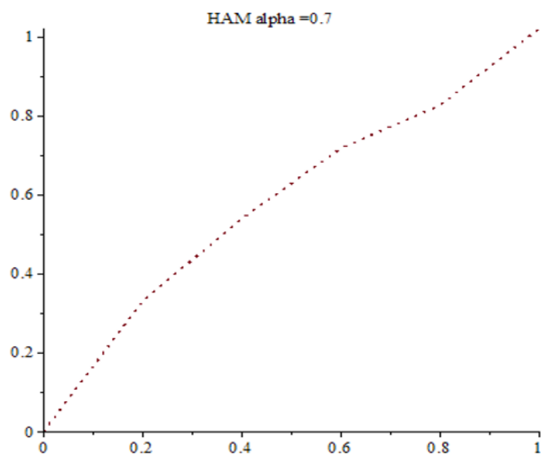












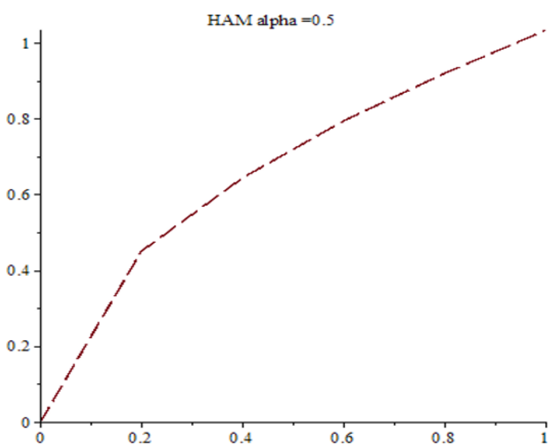


Figure 1. Periodicity in the surge of birthrates for Weibull distribution on CDF for different values of alpha-cuts

As shown in Figure 1, the periodicity in the surge of birthrates for the Weibull distribution on the cumulative distribution function (CDF) is analyzed for different values of alpha-cuts. Additionally, a comparison of the results across these different alpha-cuts has been performed to evaluate the impact of parameter variations on the periodicity

4. SECOND PROGRESS FUZZY PERIODICITY IN THE SURGE OF BIRTHRATES

The nonlinear Volterra FIE of the second class is the nonlinear FIE with integral kernel that is examined in this paper. It looks like this:

$$\tilde{u}(x) = \tilde{f}(t) +_{F} \lambda \int_a^x k((x,t,\tilde{F}_1(x,t,\tilde{u}(t)) \cdot_{F} G(t, \int_a^t \tilde{F}_2(t,s,\tilde{u}(s))ds))dt \tag{34}$$

We have,

$$\underline{\underline{u}(t,\alpha) = \underline{f}(t,\alpha) + \int_a^t k_1((t,r_2,F_{1\alpha}(t,r_2,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))ds))dr_2} \tag{35}$$

$$\overline{\overline{u}(t,\alpha) = \overline{f}(t,\alpha) +\lambda \int_a^t k_1((t,r_2,F_{1\alpha}(t,r_2,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))\,ds))dr_2} \tag{36}$$

let $r_2, t, s \in [a,b]$

$$\underline{k_1((t,r_2,F_{1\alpha}(t,r_2,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))ds)} = \begin{cases} k_1(t,r_2,F_{1\alpha}\left(t,\underline{u}(t,\alpha)\right), k_1(t,r_2) \ \geq 0 \\ k_1(t,r_2,G(t,\int_a^t F_{2\alpha}(t,s,\overline{u}(s,\alpha))ds)) \ , k_1(t,r_2) < 0 \end{cases} \quad (37)$$

$$\overline{k_1((t,r_2,F_{1\alpha}(t,r_2,u(t,\alpha)),G(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))}$$
$$= \begin{cases} k_1\left(t,r_2,G(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{u}(s,\alpha))ds)\right), k(t,r_2) \geq 0 \\ k_1\left(t,r_2,F_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right), k(t,r_2) < 0 \end{cases} \quad (38)$$

Then

$$\underline{u}(t,\alpha) = \underline{f}(t,\alpha) + \int_c^t k_1(t,r_2,G\left(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{u}(s,\alpha))ds\right)dr_2 \quad (39)$$

$$\overline{u}(t,\alpha) = \overline{f}(t,\alpha) + \int_c^t k_1\left(t,r_2,\underline{F}_{1\alpha}\left(t,\underline{u}(t,\alpha)\right)\right)dr_2 \quad (40)$$

$$\underline{f}(t,\alpha) = \min(n_0\underline{f}(t,\alpha)) \quad (41)$$

$$\underline{u}(t) = n_0 t^{\alpha} + \int_0^c (\alpha exp - \alpha(t - r_2) * (exp - \underline{u}(r_2)^{\alpha})d\tau + \int_c^t m(r_2)\int_0^{\tau}(\alpha exp - \alpha\overline{u}(s))ds)dr_2) \quad (42)$$

$$\overline{u}(t) = n_0 t^{\alpha} + \int_0^c m(r_2)\int_0^{\tau}(\alpha exp - \alpha\overline{u}(s))ds)d\tau + \int_c^t (\alpha exp - \alpha(t - r_2) * (exp - \underline{u}(\tau)^{\alpha})dr_2 \quad (43)$$

where, $0 \leq r_2 \leq c$, $c \leq r_2 \leq t$.

Now we will use the HAM to solve our application (Table 3 and Table 4), let c=0.5, we have

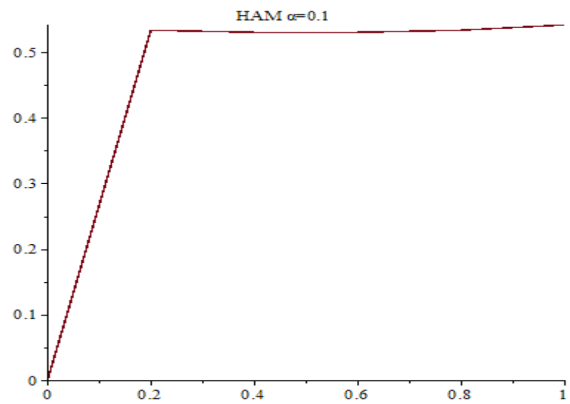
$$\underline{u}_0(t,\alpha) = 0;\ \overline{u}_0(t,\alpha) = 0$$
$$\underline{u}_1(t,\alpha) = -hn_0;\ \underline{f}(t,\alpha) = -hn_0 t^{\alpha} \quad (44)$$

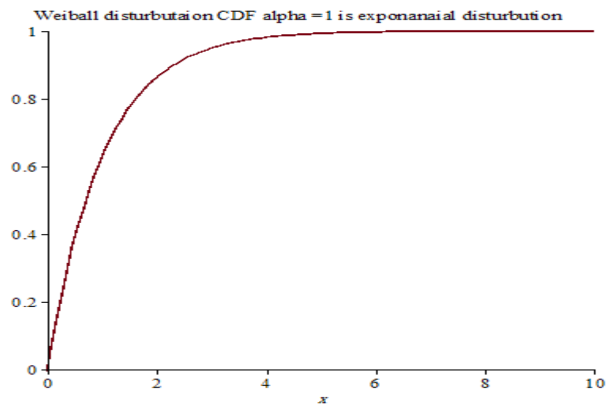
Table 3. Comparison between different value of α–level for lower side

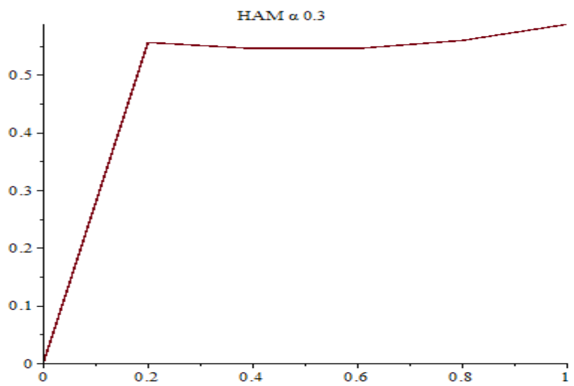
T	HAM $\alpha = 0.1$	HAM $\alpha = 0.3$	HAM $\alpha = 0.5$	HAM $\alpha = 0.7$	HAM $\alpha = 0.9$	HAM $\alpha = 1$
0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.2	0.533581612	0.556517993	0.545070296	0.530375309	0.531281389	0.539681580
0.4	0.529600159	0.544695215	0.525571084	0.503367931	0.496937563	0.501734995
0.6	0.529684978	0.545486955	0.527816261	0.507851025	0.504483149	0511139185
0.8	0.533918338	0.559676475	0.554108799	0.548604439	0.562288758	0.578517345
1	0.542418197	0.588442720	0.608088329	0.633562942	0.684955990	0.735728888

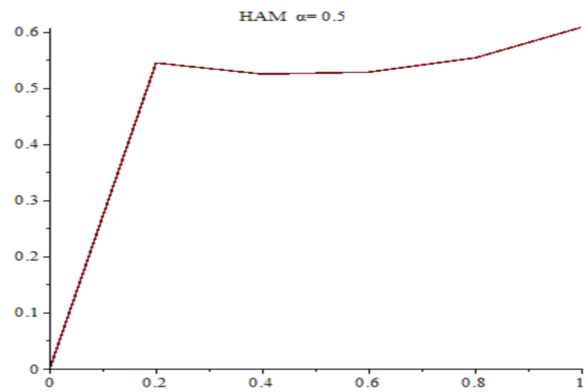
Table 4. Comparison between different value of α–level for upper side

T	HAM $\alpha = 0.1$	HAM $\alpha = 0.3$	HAM $\alpha = 0.5$	HAM $\alpha = 0.7$	HAM $\alpha = 0.9$	HAM $\alpha = 1$
0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.2	0.145968540	0.287826332	0.429662441	0.571476271	0.713367705	0.784153892
0.4	0.297656546	0.343497400	0.389102469	0.434692309	0.480259255	0.503033947
0.6	0.451162768	0.403409045	0.355633620	0.307859210	0.260015327	0.236096250
0.8	0.680324212	0.459647078	0.316030490	0.247826287	0.248729441	0.451102957
1	0.752389773	0.507089998	0.261768550	0.172391342	0.228947970	0.351633890









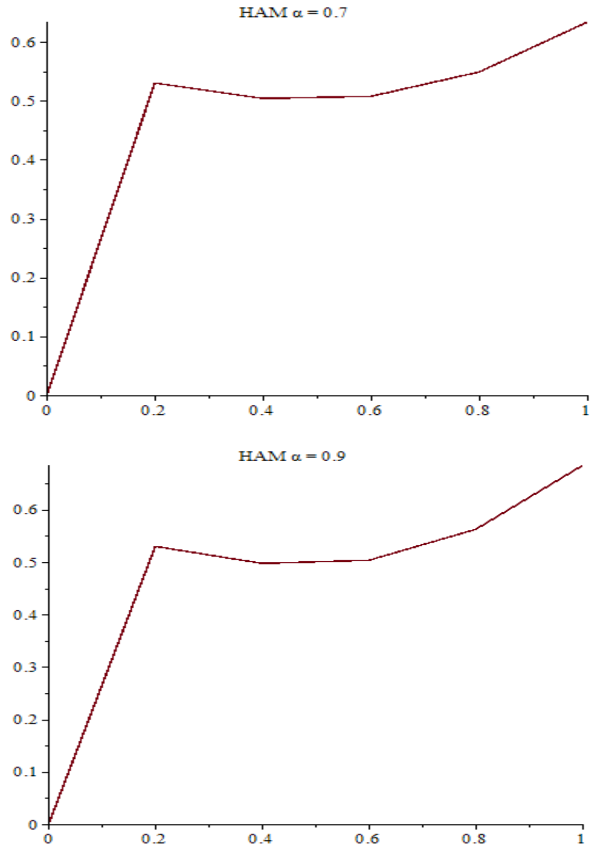


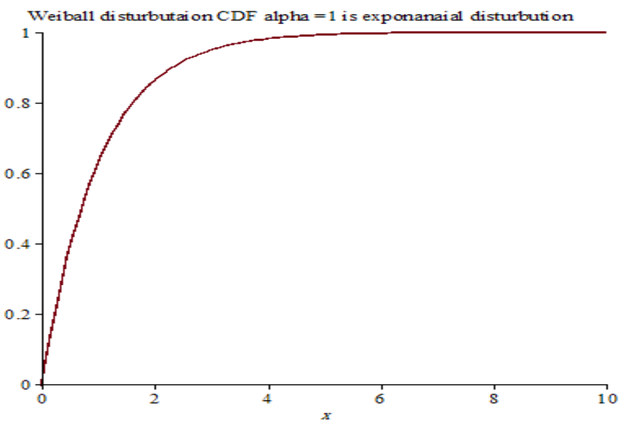
Figure 2. The second progress fuzzy periodicity in the surge of birthrates for Weibull distribution on CDF for α–level for lower side

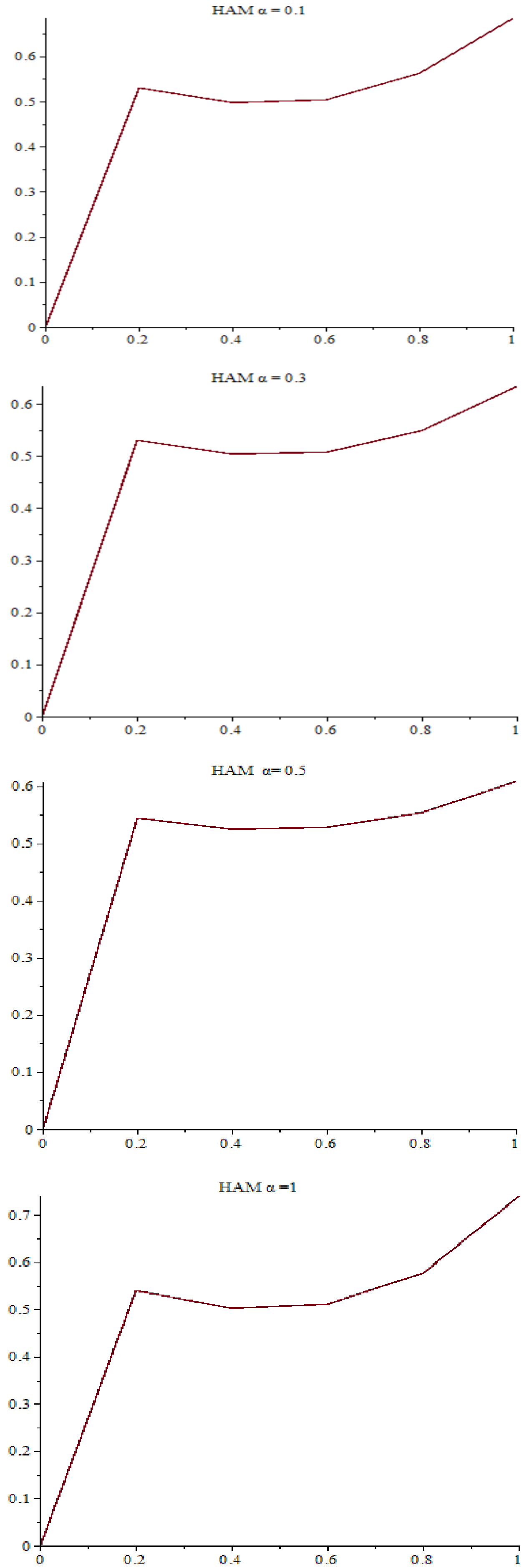
Figure 2 explains the periodicity in the surge of birthrates for Weibull distribution on CDF for difference values of alpha-cuts and compression between the results, α–level for lower side.

$$\overline{u}_1(t,\alpha) = -hn_0\,;\overline{f}(t,\alpha) = -hn_0 t^{\alpha}$$

$$\begin{aligned} &\overline{u}_2(t,\alpha) = (1+h)\underline{u}_1(t,\alpha) \\ &-h\Big(\int_0^c (\alpha exp - \alpha(t-\mathrm{r}_2) * \big(exp - \underline{u}(\mathrm{r}_2)^{\alpha}\big) d\tau \\ &+ \int_c^t m(\mathrm{r}_2)\int_0^{\tau} (\alpha exp - \alpha\overline{u}(s))ds)d\mathrm{r}_2\Big) \end{aligned} \tag{45}$$

$$\begin{aligned} &\overline{u}_2(t,\alpha) = (1+h)\overline{u}_1(t,\alpha) \\ &-h\lambda\Big(\int_0^c m(\mathrm{r}_2)\int_0^{\tau} (\alpha exp - \alpha\overline{u}(s))ds)d\mathrm{r}_2 \\ &+ \int_c^t (\alpha exp - \alpha(t-\mathrm{r}_2) * \big(exp - \underline{u}(\mathrm{r}_2)^{\alpha}\big) d\tau\Big) \end{aligned} \tag{46}$$





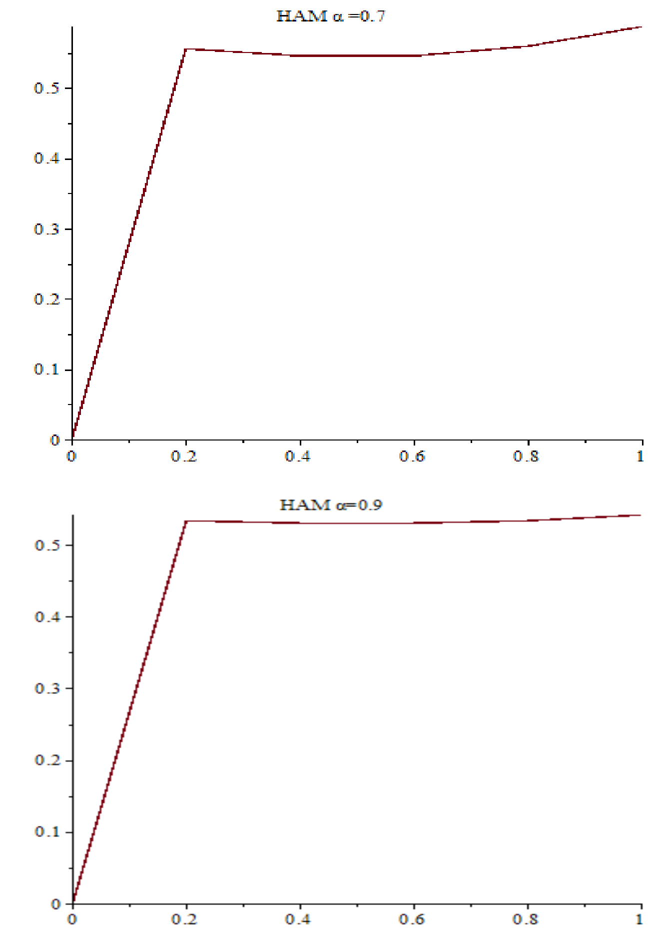


Figure 3. The second progress fuzzy periodicity in the surge of birthrates for Weibull distribution on CDF for α–level for upper side

Figure 3 explains the periodicity in the surge of birthrates for Weibull distribution on CDF for difference values of alpha-cuts and compression between the results, α–level for upper side.

5. CONCLUSIONS

The proposal FIE for application of fuzzy periodicity in the surge of fuzzy birthrates have been extended of classical modeling for population dynamical such that all the components are extended to fuzzy nonlinear functions such as birth to a female child which is birthrate, a girl bearing a female child and girl living to age between choosing interval to be more practical where the solution has membership between zero and one, not as classical solution only zero or one and the HAM numerical method was used confirmed that membership of solution more generalized and more that a best accuracy and efficient for study the FIE.

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