Mathematical Modelling of Engineering Problems
Vol. 11, No. 9, September, 2024, pp. 2573-2579
Journal homepage: http://iieta.org/journals/mmep

Maximality Degree Elements of Finite Cyclic Group Z_{p^n}, $Z_{p^m p^n}$

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https://doi.org/10.18280/mmep.110928

Received: 3 July 2023
Revised: 7 January 2024
Accepted: 20 January 2024
Available online: 29 September 2024

Keywords:
maximal sub group, cyclic group, probability, maximality degree

ABSTRACT

In this article, the concept of maximality degree of a finite group G, where G is cyclic group Z_{p^n} or $Z_{p^m q^n}$ is introduced and studied in details. The probability of a random subgroup of G to be maximal is measured by this quantity. For certain special kinds of finite groups, explicit formulas are obtained. We will give a value of one when the probability of $\langle x,y\rangle \leq_{max} G$, and a value of zero when it does not maximal sub group. This will be useful in our research to calculate the degree of probability. Several limits of degrees of maximality are also calculated. We studied three cases, the first is when p is a prime number in Z_p, the second is when p is a prime number raised to a certain degree in Z_{p^n}, and the third case is when p and q are the product of two prime numbers, each of these prime numbers is raised to a certain degree in $Z_{p^n q^m}$. We find an algorithm to compute the probability of maximality degree $P_{max}(G)$. We will use the CAP program to compute the number of maximal subgroups of group G. In this program, we will calculate the max sub groups when p, q is a large number that is difficult to calculate manually.

1. INTRODUCTION

In 2021, many researchers presented some of the studies about computing the cyclicity degree of some of finite groups, If n is greater than or equal to 3, then D_{2n} is defined by $D_{2n} = \{a,b|a^n = b^2 = e, bab = a^{-1}\}$ Suppose that the number of integer n is define by $2^r \prod_{i=1}^{s} p_i^{\alpha_i}$ and odd prime number. Let x be element in G, the cyclicizer of x is denoted by $\boldsymbol{Cyc}(x)$ thus, the cyclicizer of all elements of dihedral group D_{2n} is computed by the following:

$$\boldsymbol{Cyc}(x) = \begin{cases} \{y : \forall\, y \in D_{2n}\} & if\ x = e \\ \{a^j : 1 \leq j \leq n\} & if\ x = a^j \\ \{e, a^i b\} & if\ x = a^i b \end{cases}$$

We get for any group G, $deg\Gamma_G(x) = |Cyc_G(x)| - 1$, where $x \in G$. Let G be a finite group, the cyclicity degree of the group G is define by:

$$P_{Cyc}(G) = \frac{|\{(x,y) \in G \times G | \langle x,y\rangle \leq_{Cy} G, \forall x,y \in G\}|}{|G|^2}$$

The value $P_{Cyc}(G)$ is $0<P_{Cyc}\leq 1$, we recall that for a finite group G we have $P_{Cyc}(G)=1$ if and only if G is an abelian group [1]. It is clear when G is abelian group, then $P_{Cyc}(G)=1$. The cyclicity degree of elements in group G is define by:

$$\eta(x,y) = \begin{cases} 1 & if\ \langle x,y\rangle \leq_{Cy} G\ \exists y \in G \\ 0 & \text{otherwise} \end{cases}$$

and entry in cyclicity degree of elements table is given in reference [2], and its defined by $\eta(x, y)$:

η	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$\eta(x,y)$	
$\vdots$			

Theorem 1.
Taking n be an odd positive integer number, the cyclicity degree elements table of D_{2n}, illustrated by below [3]:

η	$\cdots$ a^j $\cdots$	$\cdots$ $a^j b$ $\cdots$
$\vdots$		
a^i	$\eta(a^i, a^j)$	$\eta(a^i, a^j b)$
$\vdots$		
$a^i b$	$\eta(a^i b, a^j)$	$\eta(a^i b, a^j b)$
$\vdots$		

The entries are defined by:

1. $\eta(a^i, a^j) = 1$ for all $1 \leq i, j \leq n$.
2. $\eta(a^i, a^j b) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$ for all $1 \leq i$.
3. $\eta(a^i b, a^j) = \eta(a^i, a^j b)$.

$$\eta(a^i b, a^j b) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \text{ for all } 1 \leq i, j$$

where, n is a positive integer number, the cyclicity degree of D_{2n} is fixed by:

$$P_{Cyc}(D_{2n}) = \frac{n+3}{4n}$$

One of the older topics in finite group theory that is still regularly investigated is counting the number of subgroups of finite groups. Which was studied by Calhoun and Cavior [4, 5].

A method to ascertain a finite abelian p-group's total number of subgroups was provided by Schmidt [6]. Consider the finite group G, a maximal subgroup of G is a subgroup of G which is not a proper subgroup of any other proper subgroup of G, and the set of all maximal subgroups of G is given by $Max(G)$. In this work, G refers for the cyclic group of order n, referring to $\boldsymbol{Z}_n$. The maximality degree of G is represented by $P_{max}(G)$ and represents a probability that two elements chosen at random x and y that are adjacent in the maximal graph in G. This is how probability is defined:

$$P_{max}(G) = \frac{|\{(x,y)\in G\times G : \langle x,y\rangle \leq_{max} G\}|}{|G|^2}$$

Here, the number of maximality degree elements of group $\boldsymbol{Z}_{p^n}, \boldsymbol{Z}_{p^m q^n}$ is studied and computed.

In 1979, Rusin [7] clarified the probability stated above is obviously not equal to one for any G. It was established that for finite non-abelian groups, the adjacent of two elements commuting is less than or equal to 5/8. Using conjugacy classes, this probability can be calculated. The commutativity degree has been the subject of numerous studies and has been widely generalized.

Note that by calculating the conjugacy classes under some group action on a set lead to obtaining the above probability. For later use, in the following we recall some important concepts concerning graph theory.

Consider the finite group G. The group of $|G|$'s prime divisors are denoted by the symbol (G). If there isn't a suitable subgroup of G that correctly contains a given subgroup H, then that subgroup H is said to be a maximum subgroup. If a group G is non-cyclic yet every appropriate subgroup of G is cyclic, then G is said to be minimally non-cyclic. Suppose that the collection of all maximal subgroups of the group G is denoted as $Max(G)$.

Also, the Frattini subgroup of a group G defines the intersection of all maximal subgroups of G and is denoted by $\Phi(G)$. In this paper we consider for cyclic group of order p^α where p is an odd prime number and $\alpha \geq 1$.

It is commonly known that $|G|$ is only divisible by one prime and that G is cyclic if a finite group G has just one maximum subgroup. In light of this, one would wonder whether the aforementioned conclusion could be expanded if G has exactly two or three maximal subgroups.

If G has precisely three maximal subgroups, neither G must be cyclic nor must $|G|$ be divisible by three prime numbers.

1.1 Basics of number theory [8]

The Euler function, or totient function φ is the number of non-negative integers less than n that are relatively prime to n [8]. Every integer n has a unique prime factor decomposition:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$$

Furthermore, $p_1 < p_2 < \ldots < p_t$ this decomposition is called the canonical prime factor decomposition of n, where p_i is prime number and $\alpha_i \geq 0$ for all i.

Thus, the Euler function φ define by:

$$\varphi(n) = \varphi(p_i^{\alpha_i}) = \Pi(p_i^{\alpha_i} - p_i^{\alpha_i - 1})$$

Let n be a positive integer and let $\tau(n)$ and $\sigma(n)$ be functions defined by:

- $\tau(n)$: the number of divisors of n;
- $\sigma(n)$: the sum of divisors of n.

2. LITERATURE REVIEW

In 1973, Gustafson [9] studied the following question, "What percentage of the time do two group elements commute?" There has been an increase in interest in the application of probability in finite groups over the past 30 years, specifically in the last decade. Erfanian et al. [10] in 2013 introduced some of the probability in finite groups. In 2019, Lazorec [11] presented "Relative cyclic subgroup commutativity degrees of finite groups". In this paper, we will study and compute the number maximality degree elements of group $\boldsymbol{Z}_{p^n}, \boldsymbol{Z}_{p^m q^n}$, and we will find an algorithm to compute the probability of maximality degree $P_{max}(G)$.

3. MAIN RESULTS

In this section we will introduce and compute the probability of the maximal subgroups of the finite group.

3.1 Definition

Let G be a finite group and x is an element of G, the maximalizer of a subset the group G is define as:

$$maxz(x, G) = \{y \in G : \langle x, y\rangle \leq_{max} G\}$$

where, $\langle x, y\rangle \leq_{Max} G$.

3.2 Definition

Let G be a finite group and $x \in G$, the maximality degree of element x in G is defined by:

$$maxd(x, G) = |\{y \in G : \langle x, y\rangle \leq_{max} G\}| = |maxz(x, G)|$$

where, $\langle x, y\rangle \leq_{Max} G$ [12].

3.3 Theorem

Let G be a finite group, the maximality degree of group is given by:

$$maxd(G) = \sum_{\forall g\in G} maxd(g, G).$$

Proof.

Let G be a finite group of order n, that we can write by $G = \{g_1, g_2, \cdots, g_n\}$.

The set of maximalizer is $maxz(g_i, G)$ can be computed by the following:

$$maxz(g_1, G) = \{g_i \in G: \langle g_1, g_i\rangle \leq_{max} G, \exists g_i \in G\}$$
$$maxz(g_2, G) = \{g_i \in G: \langle g_2, g_i\rangle \leq_{max} G, \exists g_i \in G\}$$
$$\vdots$$
$$maxz(g_n, G) = \{g_i \in G: \langle g_n, g_i\rangle \leq_{max} G, \exists g_i \in G\}$$
$$maxd(G) = \sum_{i=1}^{n} |max(g_i, G)|$$

3.4 Theorem

Let G be a finite group, the maximality degree of the group G is given by:

$$P_{Max}(G) = \frac{\sum_{\forall x \in G} |Max_G(x)|}{|G|^2} = \frac{|Max(G)|}{|G|^2}$$

Proof.

Let G be a finite group of order n, say $G=\{x_1, x_2, \cdots, x_n\}$.

The set of maximalizer element x is $Max_G(x_i)$ is computed by the following:

$$Max_G(x_1) = \{(x_1, y) \in G \times G | \langle x_1, y\rangle \leq_{Max} G, y \in G\}$$
$$Max_G(x_2) = \{(x_2, y) \in G \times G | \langle x_2, y\rangle \leq_{Max} G, y \in G\}$$
$$\vdots$$
$$Max_G(x_n) = \{(x_n, y) \in G \times G: \langle x_n, y\rangle \leq_{Max} G, y \in G\}$$
$$Max(G) = Nor_G(x_1) \cup Nor_G(x_2) \cup \cdots \cup Nor_G(x_n)$$
$$|Max(G)| = |Max_G(x_1)| + |Max_G(x_2)| + \cdots |Max_G(x_n)|$$
$$= \sum_{i=1}^{n} |Max_G(x_i)|.$$

Thus, $P_{Max}(G) = \frac{\sum_{\forall x \in G} |Max_G(x)|}{|G|^2} = \frac{|Max(G)|}{|G|^2}$.

3.5 Definition [13]

Let G be a finite group, the portability maximality subgroups of a group G is compute by:

$$P_{max}(G) = \frac{|\{(g_i, g_j) \in G \times G, \langle g_i, g_j\rangle \leq_{max} G\}|}{|G|^2}$$

3.6 Remark

The following are held:

- The parameter $P_{max}(G)$ is $0<P_{max}(G)<1$.
- Usually, the maximalizer max(g, G) of elements is not necessary is a subgroup of G [13].

We can take this example $G=D_{20}$, the set of all elements with a^2 such that maximal subgroup of group $Max(a^2, D_{20}) = \{a^i b_{1\leqslant i \leqslant 10}, a, a^3, a^5, a^7, a^9\}$ is not a subgroup in D_{20}.

Another example, we can see when $G \cong A_4$,

It is clear that the maximalizer $max((1,2)(3,4), A_4) = \{(1,2)(3,4), (1,4)(2,3)\}$ is not subgroup of A_4.

[Computed by GAP] [14].

4. MAXIMALITY DEGREE TABLE

4.1 Definition

The maximality degree element table is a two-dimensional table whose rows and columns are correspond to elements of the group. The entries consist of $\eta(g_i, g_j)$ is defined by:

$$\eta(g_i, g_j) = \begin{cases} 1 & \langle g_i, g_j\rangle \leq_{Max} G \\ 0 & otherwise \end{cases}.$$

In references [15, 16], a max degree has been calculated for the group D_{2n}. We can describe maximal degree for cyclic group $\eta Max(C_n)$ is given by the following table, it is well known the cyclic group is define by:

$$C_n = \langle a,\ a^n = e\rangle$$

	a^j
a^i	$\eta\langle a^i, a^j\rangle$

4.2 Corollary

The following holds:

- $max(g_i, G) = \sum_{\forall j} \eta(g_i, g_j)$;
- $P_{max}(G) = \frac{\sum_{\forall i,j} \eta(g_i, g_j)}{|G|^2}$.

4.3 Lemma

Assume $G=\langle x\rangle$ is a cyclic group of order $n\geq 1$, then a subgroup H of G is ($H \leq_{Max} G$) iff $H=\langle x^p\rangle$ for some prime $p|n$.

Proof. (⇒)

Let H be a maximal subgroup of G. We can write $H=\langle x^s\rangle$ with $s|n$.

Suppose, to reach a contradiction, that s is not prime.

Then we can write $s=qm$ with $1<q<s$.

Since $x^s=x^q m=(x^q)m$, it follows that $H \subseteq \langle x^q\rangle$.

By order considerations, we get strict inclusions $\langle x^s\rangle\langle x^q\rangle\langle G\rangle$ and this contradicts the maximality of H.

Hence, s must be prime.

(⇐)

Suppose $H=\langle x^p\rangle \leq \langle x^s\rangle$ at $p|n$. Lagrange's Theorem provides us with $\frac{n}{p} | \frac{n}{(n,s)} \Rightarrow (n,s)|p$ and thus $|\langle x^s\rangle| \in \{\frac{n}{p} p, n\}$. We deduce that either $\langle x^s\rangle=H$ or $\langle x^s\rangle=G$.

Thus, H is maximal.

4.4 Theorem

For any p-group the probability maximality degree is equal to:

$$P_{max}(G) = \frac{1}{p^2}$$

Proof.

It is obvious that, G is a cyclic p-group when it has one maximum subgroup, and then there exists only one maximum subgroup in the cyclic group C_p, it will be clearer by using the following formula:

If $n=p$, then:

$$Max(C_p, a^i) = \begin{cases} 1 & \text{if } i \text{ is equel to } p \\ 0 & \text{if } i \text{ is not equel to } p. \end{cases}$$

By using Definition 3.4, we get:

$$P_{max}(G) = \frac{|\{(g_i, g_j) \in G \times G, \langle g_i, g_j\rangle \leqslant_{max} G\}|}{|G|^2} = \frac{1}{p^2}$$

From the above results, we can compute the number of maximality degrees table for any elements of p-group then $Max(C_p)$ is given by:

$$\eta(a^i, a^j) = \begin{cases} 1 & if\ \ i = j = p \\ 0 & Otherwise \end{cases}.$$

We can obtain the one and exactly maximal subgroup in C_p its *Group*(()), which is referred to as in previous studies [17-20].

Thus, the identity element is the only element that can generate a maximal subgroup in C_p since there are elements that generate its self-group.

4.5 Theorem

For any element a^i in C_{p^α}, p is a prime number and $\alpha \geq 1$, the following is held:

$$|Max_{C_n}(a^i)| = \begin{cases} p^{\alpha-1} & if\ i = pr, r \geq 1 \\ p^{\varphi(\alpha-1)} & if\ i = p^\alpha r, r \geq 1 \end{cases}.$$

If $i=pr$, $r\geq 1$, then $Max_{C_n}(a^i) = \{a^j | Gcd(i,j) \neq 1\}$, $|Max_{C_n}(a^i)| = p^{\alpha-1}$ and the number of all elements is equal to $\#a^i = \varphi(p^{\alpha-1})$.

If $i=p^\alpha r$ be integer number and $\alpha\geq 2$, then $Max_{C_n}(a^i) = \{a^j | Gcd(i,j) \neq 1 \text{ and } i \neq j\}$, $|Max_{C_n}(a^i)| = \varphi(p^{\alpha-1})$ and $\#a^i=1$.

The maximality degree elements is equal to:

$$P_{Max}(C_p) = \frac{p\varphi(p) + \varphi(p)}{(p^\alpha)^2} = \frac{\varphi(p)(1+p)}{(p^\alpha)^2}$$

Proof.

(1) Since the order elements a^i be equal to p for each i without $i=p$, thus the subgroup $\langle a^i, a^j \rangle$ be isomorphic to C_p and C_p is a Sylow subgroup. $|Max_{C_p}(a^i)| = |\{a^i, 1 \leq i \leq p\}|$ for each i without $i=p$ and $|Max_{C_p}(a^i)| = |\{a^i, 1 \leq i \leq p-1\}|$, when $i=p$.

(2) It is clear that Max_p is isomorphic to C_n.

Thus for any subgroups of type $\langle a^i, a^j \rangle$, if $gcd(i, j)=1$, then it is isomorphic to C_n thus it is Max_p, by other hand, if $gcd(i,j)=t$.

4.6 Theorem

If $n=p^\alpha$, then $P_{max}(C_{p^\alpha}) = \frac{p^2-1}{p^4}$.

Proof.

It is clear that, the maximality degree of elements of group C_{p^α} are generated by the following:

$$Max_{(C_n, a^{pi})} = \{a^{pt}, 1 \leq t \leq p^{\alpha-1}\}$$

where, $i \nmid p$ and the order set $|Max_{(C_n, a^{pi})}| = p^{\alpha-1}$.

By other hand, for the remand elements be equal to $Max_{(C_n, a^{p^r i})}|_{r\geq 2} = \{a^{3t}, 1 \leq t \leq p^{\alpha-1}, \& p \nmid t\}$ where $i \nmid p$ and the order set $|Max_{(C_n, a^{p^r i})}| = \varphi(p^{\alpha-1})$ since:

$$\varphi(\frac{p^\alpha}{p}) = \varphi(p^{\alpha-1}).$$

Now:

$$\#\langle a^{ip} \rangle = \varphi(p^{\alpha-1})$$
$$\#\langle a^{ip^2} \rangle = \varphi(p^{\alpha-2})$$
$$\vdots$$
$$\#\langle a^{ip^{\alpha-1}} \rangle = \varphi(p)$$
$$\#\langle a^{p^\alpha} \rangle = 1$$

Thus, $\varphi(p^{\alpha-1}) |Max_{(C_n, a^{ip})}| = \varphi(p^{\alpha-1}) p^{\alpha-1}$.

So, $\varphi(p^{\alpha-1}) |Max_{(C_n, a^{ip^r})_{r\geqslant 2}}| = \varphi(p^{\alpha-1}).\sum_{i=2}^{\alpha} \varphi(p^{\alpha-i}) = \varphi(p^{\alpha-1})[\varphi(p^{\alpha-2}) + \varphi(p^{\alpha-3}) + \cdots + \varphi(p) + \varphi(1)] = \varphi(p^{\alpha-1})(p^{\alpha-2})$.

Now, we can compute by the following directly by (4.5 Theorem):

$$P_{max}(C_{p^\alpha}) = \frac{p^{\alpha-1}\varphi(p^{\alpha-1}) + p^{\alpha-2}\varphi(p^{\alpha-1})}{(p^\alpha)^2}$$
$$= \frac{(p^{\alpha-1} - p^{\alpha-2})(p^{\alpha-1} + p^{\alpha-2})}{(p^\alpha)^2} = \frac{p^{2\alpha-2} - p^{2\alpha-4}}{(p^\alpha)^2} = \frac{p^2-1}{p^4}.$$

4.7 Theorem

Suppose that $G \cong C_{pq}$ for p, q are prime numbers. The probability maximality degree is equal to:

$$P_{max}(C_{pq}) = \frac{\varphi(p)\sigma(p) + \varphi(q)\sigma(q)}{|pq|^2}$$

Proof.

Form (4.5. Theorem), the maximality degree of element $\max(a^p, C_{pq}) = \{a^p, a^{2p}, \cdots, a^{pq}\} = q$, by similarly for each, $\max(a^{ip}, C_{pq}) = \{a^p, a^{2p}, \cdots, a^{pq}\}$, when $1 \leq i \leq \varphi(q)$.

So, for $\max(a^q, C_{pq}) = \{a^q, a^{2q}, \cdots, a^{pq}\} = p$, by similarly for each $\max(a^{iq}, C_{pq}) = \{a^q, a^{2q}, \cdots, a^{pq}\}$, when $1 \leq i \leq \varphi(p)$, $\max(a^{pq}, C_{pq}) = \{a^p, a^{2p}, \cdots, a^{(p-1)q}\} \cup \{a^q, a^{2q}, \cdots, a^{p(q-1)}\} = \varphi(p) + \varphi(q)$, $\max(C_{pq}) = p\varphi(p) + q\varphi(q) + \varphi(p) + \varphi(q) = \varphi(p)\sigma(p) + \varphi(q)\sigma(q)$.

4.8 Example

In this example we will introduce three cases, for cycle group C_n:

• If $n = 11$, then $C^9=\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}\}$, we have that $|max_{C_{11}}(e)| = 1$, and we have $|max_{C_{11}}(a^i)| = 0\ \forall\ 1 \leq i < 11$, see Table 1.

Thus, we have two ways to find the solution:

Either by using general definition (Definition 3.4):

$$P_{max}(G) = \frac{|\{(g_i, g_j) \in G \times G, \langle g_i, g_j \rangle \leq_{max} G\}|}{|G|^2}$$

$$P_{max}(C_{11}) = \frac{1}{|C_{11}|^2} = \frac{1}{121}$$

Or, by our theorem (Theorem 4.4):

$$P_{max}(C_{11}) = \frac{1}{p^2} = \frac{1}{11^2} = \frac{1}{121}$$

In Table 1, we will show the value is 1 for the elements that satisfy the maximal subgroup, otherwise the value is zero for the group C_{11}.

Table 1. Maximality degree of C_{11}

	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}
e	1	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	0	0
a^2	0	0	0	0	0	0	0	0	0	0	0
a^3	0	0	0	0	0	0	0	0	0	0	0
a^4	0	0	0	0	0	0	0	0	0	0	0
a^5	0	0	0	0	0	0	0	0	0	0	0
a^6	0	0	0	0	0	0	0	0	0	0	0
a^7	0	0	0	0	0	0	0	0	0	0	0
a^8	0	0	0	0	0	0	0	0	0	0	0
a^9	0	0	0	0	0	0	0	0	0	0	0
a^{10}	0	0	0	0	0	0	0	0	0	0	0

Table 2. Maximality degree of C_9

	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
e	0	0	0	1	0	0	1	0	0
a	0	0	0	0	0	0	0	0	0
a^2	0	0	0	0	0	0	0	0	0
a^3	1	0	0	1	0	0	1	0	0
a^4	0	0	0	0	0	0	0	0	0
a^5	0	0	0	0	0	0	0	0	0
a^6	1	0	0	1	0	0	1	0	0
a^7	0	0	0	0	0	0	0	0	0
a^8	0	0	0	0	0	0	0	0	0

•If $n=3^2$, then $C^9=\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$, we have that $|max_{C_9}(a^3)| = |\{e, a^3, a^6\}| = 3$, $|max_{C_9}(a^6)| = |\{e, a^3, a^6\}| = 3$, $|max_{C_9}(e)| = |\{a^3, a^6\}| = 2$, Table 2 shows the elements that satisfy the maximal subgroup, and the elements that do not achieve for the group C_9.

Then, we have two ways to find the solution:

Either by using general definition (Definition 3.4):

$$P_{max}(G) = \frac{|\{(g_i,g_j)\in G\times G, \langle g_i,g_j\rangle \le_{max} G\}|}{|G|^2}$$

$$P_{max}(C_9) = \frac{(1*2)+(2*3)}{|C_9|^2} = \frac{8}{81}$$

Or, by our theorem (Theorem 4.6):

$$P_{max}(C_9) = \frac{p^2-1}{(p^4)} = \frac{3^2-1}{(3)^4} = \frac{8}{81}$$

It is difficult to calculate the maximum degree table when the value of n is large, but it can be easily calculated by (Theory 4.6).

For example, when $n=13^3$, $C_{13^3} = C_{2197}$, then

$$P_{max}(C_{2197}) = \frac{p^2-1}{(p^4)} = \frac{13^2-1}{(13)^4} = \frac{169-1}{28561} = \frac{168}{28561}$$

• If $n=15$, then $C_{15}=\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}\}$, we have that

$$|max_{C_{15}}(e)| = |\{a^3, a^5, a^6, a^9, a^{10}, a^{12}\}| = 6,$$
$$|max_{C_{15}}(a^3)| = |\{e, a^3, a^6, a^9, a^{12}\}| = 5,$$
$$|max_{C_{15}}(a^6)| = |\{e, a^3, a^6, a^9, a^{12}\}| = 5,$$
$$|max_{C_{15}}(a^9)| = |\{e, a^3, a^6, a^9, a^{12}\}| = 5,$$
$$|max_{C_{15}}(a^{12})| = |\{e, a^3, a^6, a^9, a^{12}\}| = 5.$$

Thus 4*5=20.

$$|max_{C_{15}}(a^5)| = |\{e, a^5, a^{10}\}| = 3$$
$$|max_{C_{15}}(a^{10})| = |\{e, a^5, a^{10}\}| = 3$$

Thus 2*3=6, in Table 3, we show the elements that satisfy the maximal subgroup when n equal to the product of two prime numbers.

Then, we have two ways to find the solution:

Either by using general definition (Definition 3.4):

$$P_{max}(G) = \frac{|\{(g_i,g_j)\in G\times G, \langle g_i,g_j\rangle \le_{max} G\}|}{|G|^2}$$

$$P_{max}(C_{15}) = \frac{(1*6)+(4*5)+(2*3)}{|C_{15}|^2} = \frac{32}{225}$$

Or, by our theorem (Theorem 4.7):

$$P_{max}(C_{pq}) = \frac{\varphi(p)\sigma(p)+\varphi(q)\sigma(q)}{|pq|^2} = \frac{4(5+1)+2(3+1)}{(5*3)^2} = \frac{32}{225}$$

We can take another example, when the value of n is large, when $n=91*17$, $C_{91*17}=C_{1547}$, then:

$$P_{max}(C_{91*17}) = \frac{\varphi(p)\sigma(p)+\varphi(q)\sigma(q)}{|pq|^2} = \frac{90(91+1)+16(17+1)}{(1547)^2} = \frac{8280+288}{2393209} = \frac{8568}{2393209}$$

Table 3. Maximality degree of C_{15}

	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}
e	0	0	0	1	0	1	1	0	0	1	1	0	1	0	0
a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^3	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
a^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^5	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
a^6	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
a^7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^9	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
a^{10}	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
a^{11}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^{12}	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
a^{13}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

4.9 Lemma

The following is hold:
If n is prime number, then

- $P_{Max}(C_p) = \frac{1}{9}$ is a largest value, at n=3.
- for $n \to \infty$, then $P_{Max}(C_p) \to 0$

In Figure 1, we can see a curve of the maximum and minimum values of maximality degree of cyclic group C_n when $n =$ prime number.

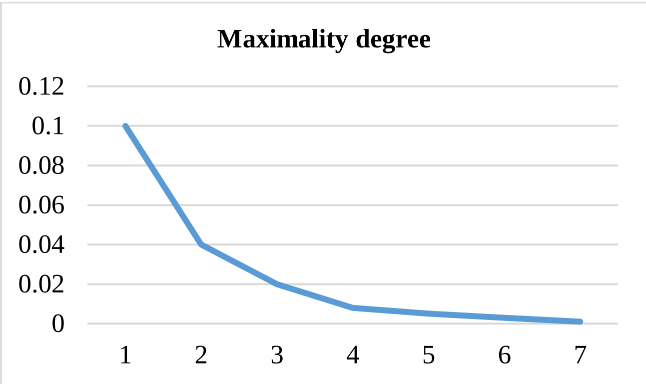

Figure 1. Largest and smallest value of maximality degree of C_p

5. DISCUSSION AND CONCLUSIONS

It is difficult to find the maximum degree by using the general definition when the value of n is large number. Therefore, in this paper, we constructed three algorithms to calculate the probabilities of the maximality degree of finite group Z_n. We indeed consider three cases, the first is when the order group is a prime number in Z_p, second it is when the order group is a prime number raised to a certain degree in Z_{p^m}, and third case is when the order is the product of two prime numbers, each of these primes is raised to a certain degree in $Z_{p^m q^n}$. Those algorithms will be calculated Max_G whatever the value of n.

We hope to develop the research by calculating the maximality degree of another finite group like dihedral group D_{2n}, when n is odd or even number. We will also study in the future the possibility of Decyclic group T_{4n}.

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APPENDIX

GAP program was used to find all the maximal subgroups.
User code of C_9:

```
gap> c:=CyclicGroup(IsPermGroup,9);
Group([ (1,2,3,4,5,6,7,8,9) ])
gap> g:=GeneratorsOfGroup(c);
```

```
[ (1,2,3,4,5,6,7,8,9) ]
gap> a:=g[1];
(1,2,3,4,5,6,7,8,9)
gap> m:=MaximalSubgroups(c);
[ Group([ (1,4,7)(2,5,8)(3,6,9) ]) ]
gap> x:=[]; for t in [2..9] do
[  ]
> h:=Group([a^9,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 6 ]
gap> x:=[]; for t in [2..9] do
[  ]
> h:=Group([a^3,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 6, 9 ]
gap> x:=[]; for t in [2..9] do
[  ]
> h:=Group([a^6,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 6, 9 ]
```

User code of C_{15}:

```
gap> c:=CyclicGroup(IsPermGroup,15);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15) ])
gap> g:=GeneratorsOfGroup(c);
[ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15) ]
gap> a:=g[1];
(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)
gap> m:=MaximalSubgroups(c);
[ Group([ (1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15) ]),
Group([ (1,11,6)(2,12,7)
  (3,13,8)(4,14,9)(5,15,10) ]) ]
gap> x:=[]; for t in [2..15] do
[  ]
> h:=Group([a^15,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 5, 6, 9, 10, 12 ]
gap> x:=[]; for t in [2..15] do
[  ]
> h:=Group([a^3,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 6, 9, 12, 15 ]
gap> x:=[]; for t in [2..15] do
[  ]
> h:=Group([a^6,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 6, 9, 12, 15 ]
gap> x:=[]; for t in [2..15] do
[  ]
> h:=Group([a^9,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 3, 6, 9, 12, 15 ]
gap> x:=[]; for t in [2..15] do
[  ]
> h:=Group([a^5,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 5, 10, 15 ]
gap> x:=[]; for t in [2..15] do
[  ]
> h:=Group([a^10,a^t]); if h in m then; Add(x,t); fi; od; x;
[ 5, 10, 15 ]
gap> h:=Group([a^12,a^t]); if h in m then; Add(x,t); fi; od;
x;
Group([ (1,13,10,7,4)(2,14,11,8,5)(3,15,12,9,6), () ])
Syntax error: expression expected
h:=Group([a^12,a^t]); if h in m then; Add(x,t); fi; od; x;

[ 5, 10, 15, 15 ]
```