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A Fuzzy Fractional Initial Value Problem with Applications Under New Conformable Fractional Granular Differentiability

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ABSTRACT

This paper introduces a novel definition of the fractional order derivative for fuzzy set-valued functions (FSVF), utilizing the concept of granular difference, which we refer to as the new conformable fractional granular derivative (NCFGD). It also presents the corresponding integral, termed the new conformable fractional granular integral (NCFGI). In terms of results, the basic properties of both the NCFGD and NCFGI are rigorously established and proven, providing a strong mathematical foundation for these new operators. Several illustrative examples demonstrate the effectiveness and applicability of the proposed definitions. Additionally, the paper explores methods for solving the new conformable fractional granular initial value problem (NCFG IVP), which is integral to understanding the behavior of fuzzy set-valued functions under fractional calculus. We apply these methods to solve new conformable fractional granular differential equations (NCFG DEqs) related to growth and decay models, offering a more flexible approach to modeling dynamic systems with fuzzy uncertainty. This approach offers a more flexible and adaptive framework for modeling dynamic systems characterized by fuzzy uncertainty, as it allows for fractional order differentiation. The incorporation of these concepts into the solution of fuzzy fractional differential equations ensures more precise and adaptable models. Ultimately, these findings contribute significantly to advancing both fuzzy calculus and fractional calculus, enhancing the understanding and modeling of complex systems with uncertainty.

1. INTRODUCTION

Since 19th century, the theory of fuzzy sets has found numerous applications in science and engineering domains to address uncertainty in real-world processes. Recently, the fuzzy calculus has emerged to handle the challenges posed by vagueness and imprecision inherent in various situations, with applications of fuzzy derivatives being extensively analyzed across different research areas. In 1965, Zadeh [1] first developed the theory of fuzzy sets. Several authors have since introduced derivatives for fuzzy functions [2-6]. However, fuzzy DEqs involving these derivatives had some drawbacks, such as the absence of a unique solution, unbounded solution diameters, monotonicity, Unnatural Behavior in Modelling (UBM). Mazandarani et al. [7] later introduced a granular derivative using the horizontal membership function (*H.M.F.*), which is more efficient than the aforementioned derivatives. *H.M.F.* for FSVFs was introduced by Piegat and Pluciński [8, 9], who recently distinguished between the *H.M.F.* and the inverse membership function. Recently, Nagalakshmi and Suresh Kumar presented a solution of initial and boundary value problems for FSVF under granular differentiability [10, 11].

Fractional derivatives have seen significant development over the past few decades, dating back to the introduction of fractional calculus in 1695. Fractional calculus is used in mathematical models of real-world phenomena, with the fractional Riemann-Liouville and fractional Caputo derivatives [12] being the most frequently used. The fractional R-L and Caputo gH-derivatives [13, 14], as well as SGH fuzzy fractional derivatives [6], have been introduced and discussed. Fuzzy fractional DEqs under generalized differentiability and granular Caputo and R-L fractional derivatives for fuzzy non-integer order linear dynamic systems have also been explored [15, 16].

Khalil et al. [17] proposed the new derivative called the conformable fractional derivative, proving fundamental characteristics that differentiate it from existing formulations. Anderson and Ulness [18] later presented a more precise conformable derivative motivated by a proportional derivative controller. This new derivative's main advantage is that the derivative of order zero acts as the identity operator, a property not satisfied by earlier fractional derivatives. The study of conformable fractional calculus in time scales was introduced by Segi Rahmat [19].

The fuzzy fractional DEqs may lack similar analytical properties, limiting theoretical analyses and modeling studies. The motivation for developing NCFGD is to overcome limitations associated with classical fuzzy fractional derivatives and improve the computational convenience of the methods used for solving practical applications of fractional order DEqs in various scientific and engineering disciplines. This paper aims to addresses a fuzzy fractional calculus (FFC) under NCFGD using the concept of *H.M.F.* Our approach is natural and shares advantages with crisp functions. We obtain the solution to some example problems and illustrate them by graphically.

The paper is grouped into sections: Section 2 provides the key idea of FFC. Moving forward, Section 3 introduces the novel concept of the NCFGD and discuss its fundamental principles. Section 4 presents the definition of the NCFGI. Section 5 derives NCFG DEqs and demonstrates the examples on the earlier discussed theoretical results.

2. PRELIMINARIES

Below are the key notions of FFC that will be employed throughout the paper.

$\mathfrak{R}$ - denotes the set of real number set.

K_1 - denotes the set of fuzzy number set on $\mathfrak{R}$.

A proportional derivative controller for controller y with respect to time t has the algorithm

$$y(t) = k_p\, e(t) + k_d \frac{d}{dt} e(t)$$

where, k_p is the proportional gain, k_d is the derivative gain, and e be the error between the state and process variables. Based on this, a new conformable derivative was introduced below.

Definition 2.1 [18]

The continuous functions $k_0, k_1: [0,1] \times \mathfrak{R} \rightarrow [0, \infty)$ that

$$\lim_{p\to 0^+} k_1(p,u) = 1,\ \lim_{p\to 0^+} k_0(p,u) = 0$$
$$\lim_{p\to 1^-} k_1(p,u) = 0,\ \lim_{p\to 1^-} k_0(p,u) = 1, \forall\, p \in \mathfrak{R}$$
$$k_1(\mathrm{p}, u) \neq 0, \mathrm{p} \in [0,1)\ \ k_0(\mathrm{p}, u) \neq 0, \mathrm{p} \in (0,1]$$

Now we define the new conformable fractional derivative as,

$$D^{\mathrm{p}}\mathrm{F}(u) = k_1(\mathrm{p}, u)\mathrm{F}(u) + k_0(\mathrm{p}, u)\, \frac{d\mathrm{F}}{du}$$

where, k_1 is called as proportional gain and k_0 is called as derivative gain.

A fuzzy number (FN) $\tilde{n}: \mathfrak{R} \rightarrow [0, 1]$ is defined as a normal, fuzzy convex, upper semi-continuous, and compactly supported sets of real numbers.

The function $\tilde{\mathrm{F}}(u): [c, d] \subseteq \mathfrak{R} \rightarrow K_1$ is FSVF and $[\tilde{\mathrm{F}}(u)]^\delta = [\mathrm{F}_L^\delta(u), \mathrm{F}_R^\delta(u)]$, where $\mathrm{F}_L^\delta(u), \mathrm{F}_R^\delta(u)$ are left and right end points of $\delta -$ level set $[\tilde{\mathrm{F}}]^\delta$.

Definition 2.2 [7]

The FN $\tilde{n}$ whose *H.M.F.* $n^{gr}: [0,1] \times [0,1] \rightarrow [c, d]$ is defined as $n^{gr}(\delta, r_n) = u$ and $H(\tilde{n}) = n^{gr}(\delta, r_n)$. Moreover, $n^{gr}(\delta, r_n) = n_L^\delta + (n_R^\delta - n_L^\delta)r_n$, where δ, r_n in [0, 1] and r_n is the relative distance measure (RDM) variable.

Note 2.1 The inverse *H.M.F.* is defined as

$$[\tilde{n}(u)]^\delta = \left[\inf_{\delta\le\gamma} \min_{r_n} n^{gr}(\gamma, r_n), \sup_{\delta\le\gamma} \max_{r_n} n^{gr}(\gamma, r_n)\right] \qquad (1)$$

where, gr refer to the information granule contained in $u \in [c, d], \delta \in [0,1]$ is the membership degree of u in $\tilde{n}(u)$ and $r_n \in [0,1]$ is the relative-distance-measure (RDM) variable which make possible to find points lying between the end points of $\delta -$ level set of FN $\tilde{n}$. For $r_n = 0$, we obtain the left and for $r_n = 1$, we get the right point.

Note 2.2 The *H.M.F.* of triangular FN $\tilde{n} = (\omega_1, \omega_2, \omega_3)$ is defined as $H(\tilde{n}) = [\omega_1 + (\omega_2 - \omega_1)\delta] + [(1-\delta)(\omega_3 - \omega_1)]r_n$.

Remark. [20] The following properties hold for any two FNs $\tilde{m}_1, \tilde{m}_2$ are:

1) Linearity: $H(\tilde{m}_1 + \tilde{m}_2) = H(\tilde{m}_1) + H(\tilde{m}_2)$.

2) $H(a\, \tilde{m}_1) = a\, H(\tilde{m}_1)$, a is real valued constant.

Definition 2.3 [7] Let $\tilde{u}_1, \tilde{u}_2$are two FNs and $*$ represents one of the arithmetic operators $+, -, \div$ and $\times$. Then $\tilde{u}_1 * \tilde{u}_2$ is a FN $\tilde{n}$ such that $H(\tilde{n}) = H(\tilde{\mathrm{u}}_1) * H(\tilde{\mathrm{u}}_2)$ provided $0 \notin H(\tilde{\mathrm{u}}_2)$ when * denotes division operator.

Note 2.2 [7] If $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3 \in K_1$ then the following relations holds:

1. $\tilde{m}_1 - \tilde{m}_2 = -[\tilde{m}_2 - \tilde{m}_1]$
2. $\tilde{m}_1 - \tilde{m}_1 = 0$
3. $\tilde{m}_1 \div \tilde{m}_1 = 1$
4. $[\tilde{m}_1 + \tilde{m}_2]\tilde{m}_3 = \tilde{m}_1\tilde{m}_3 + \tilde{m}_2\tilde{m}_3$

Note 2.3 [7] The fuzzy numbers $\tilde{m}_1$and $\tilde{m}_2$ are equal if and only if their *H.M.F.*s are equal i.e., $\tilde{m}_1 = \tilde{m}_2 \Leftrightarrow H(\tilde{m}_1) = H(\tilde{m}_2)$ and whenever $H(\tilde{m}_1) \geq H(\tilde{m}_2)$ then $\tilde{m}_1 \geq \tilde{m}_2$for all $\mathrm{r}_{m_1} = \mathrm{r}_{m_2} \in [0,1]$.

Note 2.4 [16] Let $\tilde{\mathrm{G}}$ is FSVF, then *H.M.F.* of $\tilde{F}\left(\tilde{\mathrm{G}}(u)\right)$ is defined as $\mathrm{H}\left[\tilde{\mathrm{F}}\left(\mathrm{H}\left(\tilde{\mathrm{G}}(\mathrm{u})\right)\right)\right]$.

Definition 2.4 [7] Let $\tilde{m}_1, \tilde{m}_2$be two FNs, then $D^{gr}: K_1 \times K_1 \rightarrow R^+ \cup \{0\}$ is defined as

$$D^{gr}[\tilde{\mathrm{m}}_1, \tilde{\mathrm{m}}_2] = \sup_{\delta\in[0,1]} \max_{\mathrm{r}_{m_1}\mathrm{r}_{m_2}\in[0,1]} \left|m_1^{gr}(\delta, \mathrm{r}_{m_1}) - m_2^{gr}(\delta, \mathrm{r}_{m_2})\right|$$

Definition 2.5 [7] Suppose that $\tilde{\mathrm{F}}: [c, d] \subseteq \mathfrak{R} \rightarrow K_1$ be the FSVF, if there exists an FN $\frac{d_{gr}\tilde{\mathrm{F}}(u)}{du} \in K_1$ such that

$$\lim_{h\to 0} \frac{\tilde{\mathrm{F}}[u+h] - \tilde{\mathrm{F}}[u]}{h} = \frac{d_{gr}\tilde{\mathrm{F}}(u)}{du}$$

The limit exists in the (D^{gr}, K_1) metric space is called as the granular differentiable (gr-differentiable) of $\tilde{F}$ at $u \in [c, d]$.

Theorem 2.1 [7] Let $\tilde{F}: [c, d] \subseteq \mathfrak{R} \rightarrow K_1$ be FSVF is gr-differentiable at the point $u \in [c, d]$ if and only if its *H.M.F.* is differentiable with respect to u at that point.

Moreover,

$$H\left(\frac{d_{gr}\tilde{\mathrm{F}}(u)}{du}\right) = \frac{\partial}{\partial u} H\left(\tilde{\mathrm{F}}(u)\right)$$

Definition 2.6 [7] Suppose the continuous FSVF $\tilde{F}: [c,d] \subseteq \mathfrak{R} \to K_1$ with *H.M.F.* $F^{gr}(u,\delta,r_F)$ is integrable on $u \in [c,d]$ and $\int_c^d \tilde{F}(u)du$ is the integral of $\tilde{F}$ on $[c,d]$. The FSVF $\tilde{F}$ is said to be granularly integrable in $[c,d]$ if there exists a FN $\tilde{n} = \int_c^d \tilde{F}(u)du$ such that $H(\tilde{n}) = \int_c^d H[\tilde{F}(u)]du$.

Theorem 2.2 [7] If the FSVF $\tilde{F}: [c,d] \subseteq \mathfrak{R} \to K_1$ is gr-differentiable, $\frac{d_{gr}\tilde{F}(u)}{du}$ be a continuous FSVF in interval $[c,d]$, then

$$\int_c^d \left(\frac{d_{gr}\tilde{F}(u)}{du}\right) d\,u = \tilde{F}(d) - \tilde{F}(c)$$

3. NEW CONFORMABLE FRACTIONAL GRANULAR DERIVATIVE

Definition 3.1 Let FSVF $\tilde{F}: [c,d] \subseteq \mathfrak{R} \to K_1$ is said to be new conformable fractional granular derivative if $\forall \epsilon > 0, \exists \vartheta > 0, |h| < \vartheta$ such that

$$D_{gr}(k_1\tilde{F}(u) + k_0\tilde{F}'(u), D_{gr}^{p}\,\tilde{F}(u)) < \epsilon$$

where, $\tilde{F}'(u)$ is granular derivative of FSVF $\tilde{F}(u)$.

On a limit basis, we write above as,

$$\begin{aligned} D_{gr}^{p}\,\tilde{F}(u) &= [k_1\,\tilde{F}(u) + k_0\,\tilde{F}'(u)\,] \\ &= \lim_{h\to 0}\left[k_1\,\tilde{F}(u) + k_0\frac{\tilde{F}(u+h) - \tilde{F}(u)}{h}\right] \end{aligned}$$

exists, we say that $\tilde{F}(u)$ is new conformable fractional granular differentiable.

Theorem 3.1 The FSVF $\tilde{F}(u)$ is a new conformable fractional granular derivative at point $u \in [c,d]$ if its *H.M.F.* is differentiable with respect to the point *u*. Furthermore,

$$H\left[D_{gr}^{p}\tilde{F}(u)\right] = k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)$$

Proof. We know that,

$$H[\tilde{F}'(u)] = \frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)$$

Consider,

$$\begin{aligned} H[D_{gr}^{p}\tilde{F}(u)] &= H[k_1\tilde{F}(u) + k_0\tilde{F}'(u)] \\ &= k_1 H[\tilde{F}(u)] + k_0 H\left[\tilde{F}'(u)\right] \\ &= k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F). \end{aligned}$$

Example 3.1 Suppose that $\tilde{F}(u) = e^{-\tilde{5}u}$ where $\tilde{5} = (3,5,7)$ for $u \in [0,1]$. Since $H[\tilde{5}] = [3 + 2\delta + 4(1-\delta)r_5]$ then from Theorem 3.1, we have

$$H\left[D_{gr}^{p}\tilde{F}(u)\right] = [(1-p)u^p - p\,u^{1-p}\,(3 + 2\delta + 4(1-\delta)\,r_5)]e^{-[3+2\delta+4(1-\delta)r_5]u}.$$

Therefore

$$\left[D_{gr}^{p}\tilde{F}(u)\right]^{\delta} = H^{-1}\big([(1-p)u^p - p\,u^{1-p}\,(3 + 2\delta + 4(1-\delta)\,r_5)]e^{-[3+2\delta+4(1-\delta)r_5]u}\big).$$

Theorem 3.2 The FSVFs $\tilde{F}(u)$, $\tilde{G}(u)$ and the continuous functions defined as k_0, $k_1: [0,1] \times \mathfrak{R} \to [0,\infty)$, then the following properties hold:

1. $D_{gr}^{p}[\tilde{F}(u) + \tilde{G}(u)] = D_{gr}^{p}\tilde{F}(u) + D_{gr}^{p}\tilde{G}(u)$
2. $D_{gr}^{p}[\tilde{c}] = k_1\tilde{c}, \forall$ constants $\tilde{c} \in K_1$
3. $D_{gr}^{p}[\tilde{F}(u)\,\tilde{G}(u)] = \tilde{F}(u)D_{gr}^{p}\tilde{G}(u) + \tilde{G}(u)D_{gr}^{p}\tilde{F}(u) - k_1\tilde{F}(u)\tilde{G}(u)$
4. $D_{gr}^{p}\left[\frac{\tilde{F}(u)}{\tilde{G}(u)}\right] = k_1\frac{\tilde{F}(u)}{\tilde{G}(u)} + \frac{\tilde{G}(u)D_{gr}^{p}\tilde{F}(u) - \tilde{F}(u)D_{gr}^{p}\tilde{G}(u)}{[\tilde{G}(u)]^2}$

Proof.

(1) From Theorem 3.1, we have $H\left[D_{gr}^{p}\tilde{F}(u)\right] = k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)$.

Consider,

$$\begin{aligned} &H\left[D_{gr}^{p}\left(\tilde{F}(u) + \tilde{G}(u)\right)\right] \\ &= k_1\big(F^{gr}(u,\delta,r_F) + G^{gr}(u,\delta,r_G)\big) \\ &+k_0\frac{\partial}{\partial u}\big(F^{gr}(u,\delta,r_F) + G^{gr}(u,\delta,r_G)\big) \\ &= k_1 F^{gr}(u,\delta,r_F) + k_1 G^{gr}(u,\delta,r_G) \\ &+k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}G^{gr}(u,\delta,r_G) \\ &=\left(k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)\right) \\ &+\left(k_1 G^{gr}(u,\delta,r_G) + k_0\frac{\partial}{\partial u}G^{gr}(u,\delta,r_G)\right) \\ &= H\left(D_{gr}^{p}\tilde{F}(u)\right) + H\left(D_{gr}^{p}\tilde{G}(u)\right) \\ &= H\left[D_{gr}^{p}\tilde{F}(u) + D_{gr}^{p}\tilde{G}(u)\right] \end{aligned}$$

From Note 2.3, we have $D_{gr}^{p}\big(\tilde{F}(u) + \tilde{G}(u)\big) = D_{gr}^{p}\tilde{F}(u) + D_{gr}^{p}\tilde{G}(u)$.

(2) From Theorem 3.1, we have $H\left[D_{gr}^{p}\tilde{F}(u)\right] = k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)$.

Consider, $H\left[D_{gr}^{p}\tilde{c}\right] = k_1 c^{gr}(\delta,r_c) + k_0\,\frac{\partial}{\partial u}c^{gr}(\delta,r_c) = k_1 c^{gr}(\delta,r_c)$.

From Note 2.3, we have $D_{gr}^{p}[\tilde{c}] = k_1\tilde{c}, \forall$constants $\tilde{c} \in K_1$.

(3) From Theorem 3.1, we have $H\left[D_{gr}^{p}\tilde{F}(u)\right] = k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)$.

Consider,

$$\begin{aligned} &H\left[D^{\alpha}\big(\tilde{F}(u)\tilde{G}(u)\,\big)\right] \\ &=k_1 F^{gr}(u,\delta,r_F)G^{gr}(u,\delta,r_G) \\ &+k_0\frac{\partial}{\partial u}\big(F^{gr}(u,\delta,r_F)G^{gr}(u,\delta,r_G)\big) \\ &= F^{gr}(u,\delta,r_F)\left(k_1 G^{gr}(u,\delta,r_G) + k_0\frac{\partial}{\partial u}G^{gr}(u,\delta,r_G)\right) \\ &+k_0 G^{gr}(u,\delta,r_G)\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F) \\ &+k_1 F^{gr}(u,\delta,r_F)G^{gr}(u,\delta,r_G) - k_1 F^{gr}(u,\delta,r_F)G^{gr}(u,\delta,r_G) \\ &=F^{gr}(u,\delta,r_F)\left(k_1 G^{gr}(u,\delta,r_G) + k_0\frac{\partial}{\partial u}G^{gr}(u,\delta,r_G)\right) \\ &+G^{gr}(u,\delta,r_G)\left(k_1 F^{gr}(u,\delta,r_F) + k_0\frac{\partial}{\partial u}F^{gr}(u,\delta,r_F)\right) \\ &-k_1 F^{gr}(u,\delta,r_F)G^{gr}(u,\delta,r_G) \\ &=H[\tilde{F}(u)]H\left[D_{gr}^{p}\tilde{G}(u)\right] + H\left[\tilde{G}(u)\right]H\left[D_{gr}^{p}\tilde{F}(u)\right] \\ &-k_1 H[\tilde{F}(u)]H\left[\tilde{G}(u)\right]. \end{aligned}$$

From Note 2.3, we have $D^{p}\left[\tilde{F}(u)\tilde{G}(u)\right] = \tilde{F}(u)D_{gr}^{p}\tilde{G}(u) + \tilde{G}(u)D_{gr}^{p}\tilde{F}(u) - k_1\left(\tilde{F}(u)\tilde{G}(u)\right)$.

(4) From Theorem 3.1, we have $H\left[D_{gr}^{p}\tilde{F}(u)\right] = k_1 \mathrm{F}^{gr}(u,\delta,\mathrm{r_F}) + k_0 \frac{\partial}{\partial u}\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})$.

Consider,

$$H\left[D_{gr}^{\mathrm{p}}\left(\frac{\tilde{\mathrm{F}}(u)}{\tilde{\mathrm{G}}(u)}\right)\right] = k_1 \frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})}{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})} + k_0 \frac{\partial}{\partial u}\left(\frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})}{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})}\right)$$
$$= k_1 \frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})}{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})}$$
$$+ k_0 \frac{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\frac{\partial}{\partial u}\mathrm{F}^{gr}(u,\delta,\mathrm{r_F}) - \mathrm{F}^{gr}(u,\delta,\beta_\psi)\frac{\partial}{\partial u}\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})}{[\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})]^2}$$
$$= k_1 \frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})}{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})}$$
$$+ \frac{k_0 \mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\frac{\partial}{\partial u}\mathrm{F}^{gr}(u,\delta,\mathrm{r_F}) - k_0 \mathrm{F}^{gr}(u,\delta,\mathrm{r_F})\frac{\partial}{\partial u}\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})}{[\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})]^2}$$
$$+ \frac{k_1 \mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\,\mathrm{F}^{gr}(u,\delta,\mathrm{r_F}) - k_1 \mathrm{F}^{gr}(u,\delta,\mathrm{r_F})\,\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})}{[\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})]^2}$$
$$= k_1 \frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})}{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})} + \frac{\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\left(k_1 \mathrm{F}^{gr}(u,\delta,\mathrm{r_F}) + k_0 \frac{\partial}{\partial u}\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})\right)}{\left(\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\right)^2}$$
$$- \frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})\left(k_1 \mathrm{G}^{gr}(u,\delta,\mathrm{r_G}) + k_0 \frac{\partial}{\partial u}\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\right)}{\left(\mathrm{G}^{gr}(u,\delta,\mathrm{r_G})\right)^2}$$
$$= k_1 \frac{H\left(\tilde{\mathrm{F}}(u)\right)}{H\left(\tilde{\mathrm{G}}(u)\right)} + \frac{H\left(\tilde{\mathrm{G}}(u)\right) H\left(D_{gr}^{\mathrm{p}}\tilde{\mathrm{F}}(u)\right) - H\left(\tilde{\mathrm{F}}(u)\right) H\left(D_{gr}^{\alpha}\tilde{\mathrm{G}}(u)\right)}{H\left[\left(\tilde{\mathrm{G}}(u)\right)^2\right]}$$

From Note 2.3, we have $D_{gr}^{\mathrm{p}}\left(\frac{\tilde{\mathrm{F}}(u)}{\tilde{\mathrm{G}}(u)}\right) = k_1 \frac{\tilde{\mathrm{F}}(u)}{\tilde{\mathrm{G}}(u)} + \frac{\tilde{\mathrm{G}}(u) D_{gr}^{\mathrm{p}}\tilde{\mathrm{F}}(u) - \tilde{\mathrm{F}}(u) D_{gr}^{\mathrm{p}}\tilde{\mathrm{G}}(u)}{\left(\tilde{\mathrm{G}}(u)\right)^2}$.

Example 3.2 Suppose that $\tilde{F}(u) = \tilde{5}\cos u$ and $\tilde{G}(u) = \tilde{5}\sin u$, where $\tilde{5} = (3, 5, 7)$ for $u \in [0,1]$ then we have

$$H[\tilde{F}(u)] = [3 + 2\delta + 4(1-\delta)r_5]\, cos\, u$$
$$H[\tilde{G}(u)] = [3 + 2\delta + 4(1-\delta)r_5]\, sin\, u$$

Using the Theorem 3.2, we get

(1)
$$H\left[D_{gr}^{p}(\tilde{F} + \tilde{G})\right] = [\,3 + 2\delta + 4(1-\delta)r_5\,]$$
$$[(1-p)u^p(cos\,u + sin\,u) + pu^{1-p}(cos\,u - sin\,u)]$$

Therefore
$$\left[D_{gr}^{\mathrm{p}}(\tilde{\mathrm{F}} + \tilde{\mathrm{G}})\right]^{\delta} =$$
$$H^{-\mathbb{1}}([3 + 2\delta + 4(1-\delta)\mathrm{r}_5][(1-\mathrm{p})u^{\mathrm{p}}(\cos u + \sin u) + \mathrm{p}u^{1-\mathrm{p}}(\cos u - \sin u)])$$

(2)
$$H\left[D_{gr}^{\mathrm{p}}(\tilde{\mathrm{F}} - \tilde{\mathrm{G}})\right] =$$
$$[3 + 2\delta + 4(1-\delta)]r_5[(1-p)u^p(\cos\ u - \sin\ u) - \mathrm{p}u^{1-\mathrm{p}}(\cos u + \sin u)\ ]$$

Therefore
$$\left[D_{gr}^{\mathrm{p}}(\tilde{\mathrm{F}} - \tilde{\mathrm{G}})\right]^{\delta} =$$
$$H^{-1}([3 + 2\delta + 4(1-\delta)]r_5[(1-p)u^p(\cos u - \sin u) - pu^{1-p}(\cos u + \sin u)])$$

(3)
$$H[D_{gr}^{\mathrm{p}}(\tilde{\mathrm{F}}\tilde{\mathrm{G}})] =$$
$$[3 + 2\delta + 4(1-\delta)\mathrm{r}_5]^2$$
$$[(1-\mathrm{p})\, u^{\mathrm{p}} \sin u \cos u + \mathrm{p}u^{1-\mathrm{p}}(\cos^2 u - \sin^2 u)]$$

Therefore
$$\left[D_{gr}^{\mathrm{p}}(\tilde{\mathrm{F}}\tilde{\mathrm{G}})\right]^{\delta} =$$
$$H^{-\mathbb{1}}([3 + 2\delta + 4(1-\ \delta)\mathrm{r}_5]^2[(1-\mathrm{p})u^{\mathrm{p}}\, sin\, u \cos u + \mathrm{p}u^{1-\mathrm{p}}(\cos^2 u - \sin^2 u)])$$

4. NEW CONFORMABLE FRACTIONAL GRANULAR INTEGRAL

Definition 4.1 Let $\tilde{\mathrm{F}}(u)$ be a continuous function whose *H.M.F.* $\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})$ is integrable on $u \in [c,d]$. Let $\int_c^u \frac{\tilde{\mathrm{F}}(s)e_0(u,s)}{k_0(\mathrm{p},s)}ds$ is new conformable fractional integral on [c, d]. Then the FSVF $\tilde{\mathrm{F}}$ is called as a new conformable fractional granular integrable on [c, d] if there exists a FN $\tilde{m} = \int_c^u \frac{\tilde{\mathrm{F}}(s)e_0(u,s)}{k_0(\mathrm{p},s)}ds$ such that $H(\tilde{m}) = \int_c^u \frac{H[\tilde{\mathrm{F}}(s)]e_0(u,s)}{k_0(\mathrm{p},s)}ds$ recall that $e_0(u,s) = e^{-\int_s^u \frac{k_1(\mathrm{p},\tau)}{k_0(\mathrm{p},\tau)}d\tau}$.

Theorem 4.1 Let the FSVFs $\tilde{F}(u), \tilde{G}(u): \mathfrak{R} \to K_1$ then

$$I_{gr}^{\mathrm{p}}[\tilde{\mathrm{F}}(u) + \tilde{\mathrm{G}}(u)] = I_{gr}^{\mathrm{p}}\tilde{\mathrm{F}}(u) + I_{gr}^{\mathrm{p}}\tilde{\mathrm{G}}(u)$$

Proof. From the definition of NCFGI,

$$H\left[I_{gr}^{\mathrm{p}}\left(\tilde{\mathrm{F}}(u) + \tilde{\mathrm{G}}(u)\right)\right]$$
$$= \int_c^u \frac{H[\tilde{\mathrm{F}}(s) + \tilde{\mathrm{G}}(s)]e_0(u,s)}{k_0(\mathrm{p},s)}ds$$
$$= \int_c^u \frac{H[\tilde{\mathrm{F}}(s)]e_0(u,s)}{k_0(\mathrm{p},s)}ds + \int_c^u \frac{H[\tilde{G}(s)]e_0(u,s)}{k_0(\mathrm{p},s)}ds$$
$$= H\left[I_{gr}^{\mathrm{p}}\left(\tilde{\mathrm{F}}(u)\right)\right] + H\left[I_{gr}^{\mathrm{p}}\left(\tilde{\mathrm{G}}(u)\right)\right].$$

From Note 2.3, we get

$$I_{gr}^{\alpha}[\tilde{\mathrm{F}}(u) + \tilde{\mathrm{G}}(u)] = I_{gr}^{\mathrm{p}}\tilde{\mathrm{F}}(u) + I_{gr}^{\mathrm{p}}\tilde{\mathrm{G}}(u).$$

Theorem 4.2 Suppose FSVFs $\tilde{F}(u)$, $\tilde{G}(u)$ be new conformable fractional granular differentiable as needed and let the functions k_0, k_1 be continuous and satisfy Definition 2.1, then the derivative of the definite integral of FSVF $\tilde{F}$ is given by $D_{gr}^{\mathrm{p}}\left(\int_c^u \frac{\tilde{\mathrm{F}}(s)e_0(u,s)}{k_0(\mathrm{p},s)}ds\right) = \tilde{\mathrm{F}}(u)$.

Proof.

$$H\left[D_{gr}^{\mathrm{p}}\left(\int_c^u \frac{\tilde{\mathrm{F}}(s)e_0(u,s)}{k_0(\mathrm{p},s)}ds\right)\right]$$
$$= k_0(\mathrm{p},u)\frac{d}{du}\left(\int_c^u \frac{\mathrm{F}^{gr}(s,\delta,\mathrm{r_F})e_0(u,s)}{k_0(\mathrm{p},s)}ds\right)$$
$$+ k_1(\mathrm{p},u)\int_c^u \frac{\mathrm{F}^{gr}(s,\delta,\mathrm{r_F})e_0(u,s)}{k_0(\mathrm{p},s)}ds$$
$$= k_0(\mathrm{p},u)\left[\frac{-k_1(\mathrm{p},u)}{k_0(\mathrm{p},u)}\int_c^u \frac{\mathrm{F}^{gr}(s,\delta,\mathrm{r_F})e_0(u,s)}{k_0(\mathrm{p},s)}ds + \frac{\mathrm{F}^{gr}(u,\delta,\mathrm{r_F})e_0(u,u)}{k_0(\mathrm{p},u)}\right]$$
$$+ k_1(p,u)\int_c^u \frac{\mathrm{F}^{gr}(s,\delta,\mathrm{r_F})e_0(u,s)}{k_0(\mathrm{p},s)}ds$$
$$= -\mathrm{k}_1(\mathrm{p},\mathrm{u})\int_{\mathrm{c}}^{\mathrm{u}} \frac{\mathrm{F}^{\mathrm{gr}}(\mathrm{s},\delta,\mathrm{r_F})\mathrm{e}_0(\mathrm{u},\mathrm{s})}{\mathrm{k}_0(\mathrm{p},\mathrm{s})}\mathrm{ds} + \mathrm{F}^{\mathrm{gr}}(\mathrm{u},\delta,\mathrm{r_F})$$
$$+ \mathrm{k}_1(\mathrm{p},\mathrm{u})\int_c^u \frac{F^{gr}(s,\delta,r_F)\, e_0(u,s)}{k_0(p,s)}\, ds$$
$$= \mathrm{F}^{gr}(u,\delta,\mathrm{r_F}) = H[\tilde{\mathrm{F}}(u)].$$

From the Note 2.3, we get

$$D_{gr}^{P}\left(\int_c^u \frac{\tilde{F}(s)e_0(u,s)}{k_0(p,s)}ds\right) = \tilde{F}(u)$$

5. NEW CONFORMABLE FRACTIONAL GRANULAR IVP AND APPLICATIONS

Consider NCFG IVP,

$$D_{gr}^{p}\,\tilde{G}(u) = \tilde{F}\left(u, \tilde{G}(u)\right) \quad (2)$$

$$\tilde{G}(u_0) = \tilde{G}_0 \quad (3)$$

where, $\tilde{G}: [c,d] \subseteq \mathfrak{R} \to K_1$, $\tilde{F}: [c,d] \to K_1 \times K_1$ is called fuzzy mapping and initial condition as $\tilde{G}_0 \in K_1$.

To obtain the solution of NCFG IVP Eq. (2), we follow the method given below:

Apply *H.M.F.* on Eq. (2), we get

$$k_1 G^{gr}(u,\delta,r_G) + k_0 \frac{\partial}{\partial u} G^{gr}(u,\delta,r_G) = F^{gr}(u, G^{gr}(u,\delta,r_G), r_F) \quad (4)$$

$$G^{gr}(u_0,\delta,r_G) = G_0^{gr}\left(\delta, r_{G_0}\right) \quad (5)$$

where, $r_F, r_{G_0}, r_G \in [0,1]$. Hence, Eq. (2) is converted as a partial differential Eq. (4) in single independent variable u. Therefore, this equation is considered as conformable fractional DEq.

The Solution corresponding to the Eq. (4) is taken as,

$$H\left(\tilde{G}(u)\right) = G^{gr}(u,\delta,r_G) \quad (6)$$

Take inverse *H.M.F.* on Eq. (6), we get the δ-cut solution corresponding to the NCGF IVP (2) as

$$\left[\tilde{G}(u)\right]^{\delta} = \left[\inf_{\delta\le\gamma\le 1} \min_{r_G} G^{gr}(u,\gamma,r_G), \sup_{\delta\le\gamma\le 1} \max_{r_G} G^{gr}(u,\gamma,r_G)\right]$$

Example 5.1 Consider the growth model NCFG IVP with the initial condition as triangular FN.

$$D_{gr}^{p}\tilde{G}(u) = 0.5\tilde{G}(u),\ u \in [0,5] \quad (7)$$

$$\text{Subject to, } \tilde{G}(0) = [0.4, 0.6, 0.9] \quad (8)$$

Now, $\left[\tilde{G}(0)\right]^{\delta} = [0.4 + 0.2\delta, 0.9 - 0.3\delta]$ where $\delta \in [0,1]$.

Apply *H.M.F.* on Eq. (7) and Eq. (8), we get

$$pu^{1-p}\frac{\partial}{\partial u} G^{gr}(u,\delta,r_G) + (1-p)u^{p} G^{gr}(u,\delta,r_G) = 0.5\, G^{gr}(u,\delta,r_G) \quad (9)$$

$$H\left[\tilde{G}(0)\right] = 0.4 + 0.2\delta + 0.5(1-\delta)r_G \quad (10)$$

where, $r_G \in [0,1]$.

Solving Eq. (9) and Eq. (10), we get

$$G^{gr}(u,\delta,r_G) = [0.4 + 0.2\delta + 0.5(1-\delta)r_G]\, e^{\left[\frac{0.5u^{p}}{p^2} - \frac{(1-p)u^{(2p)-1}}{2p^2}\right]} \quad (11)$$

which gives the solution corresponding to Eq. (9).

Take inverse *H.M.F.* on Eq. (11), we get the $\delta -$cuts solution corresponding to the IVP (7) as

$$\left[\tilde{G}(u)\right]^{\delta} = \left[\inf_{\delta\le\gamma\le 1} \min_{r_G} G^{gr}(u,\gamma,r_G), \sup_{\delta\le\gamma\le 1} \max_{r_G} G^{gr}(u,\gamma,r_G)\right]$$

Using MATLAB draws $\delta -$level sets and presented in Figure 1.

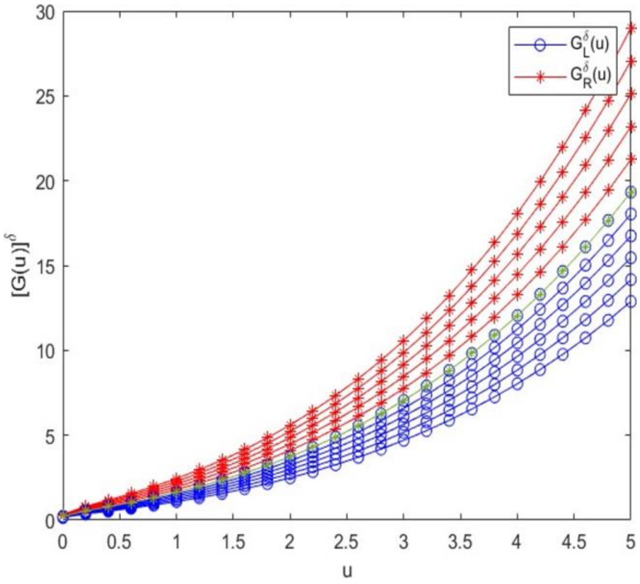

Figure 1. The solution span in granular form corresponding to the IVP (7) with $p = 0.5$

Example 5.2 Consider the decay model NCFG IVP with the initial condition as triangular FN.

$$D_{gr}^{p}\tilde{G}(u) = -0.3\,\tilde{G}(u), u \in [0,1] \quad (12)$$

$$\text{Subject to } \tilde{G}(0) = (3.97, 4.3, 5.1) \quad (13)$$

Now $\left[\tilde{G}(0)\right]^{\delta} = [3.97 + 0.33\delta, 5.1 - 0.8\delta]$, where $\delta \in [0,1]$.

Apply *H.M.F.* on Eq. (12) and Eq. (13), we get

$$pu^{1-p}\frac{\partial}{\partial u} G^{gr}(u,\delta,r_G) + (1-p)u^{p} G^{gr}(u,\delta,r_G) = -0.3G^{gr}(u,\delta,r_G) \quad (14)$$

$$\text{with } \left[\tilde{G}(0)\right]^{gr}(\delta,r_G) = 3.97 + 0.33\delta + 1.13\,(1-\delta)r_G \quad (15)$$

where, $r_G \in [0,1]$.

Solving Eq. (14) and Eq. (15), we get

$$G^{gr}(u,\delta,r_G) = [3.97 + 0.33\delta + 1.13\,(1-\delta)r_G]\, e^{\left[\frac{-0.3u^{p}}{p^2} - \frac{(1-p)u^{(2p)-1}}{2p^2}\right]} \quad (16)$$

which gives the solution corresponding to the Eq. (14).

Using inverse *H.M.F.* on Eq. (16), we get

$$\left[\tilde{G}(u)\right]^{\delta} = \left[\inf_{\delta\le\gamma\le 1} \min_{r_G} G^{gr}(u,\gamma,r_G), \sup_{\delta\le\gamma\le 1} \max_{r_G} G^{gr}(u,\gamma,r_G)\right]$$

which is the $\delta -$cuts solution corresponding to the IVP (13), using MATLAB draws $\delta -$level sets and presented in Figure 2.

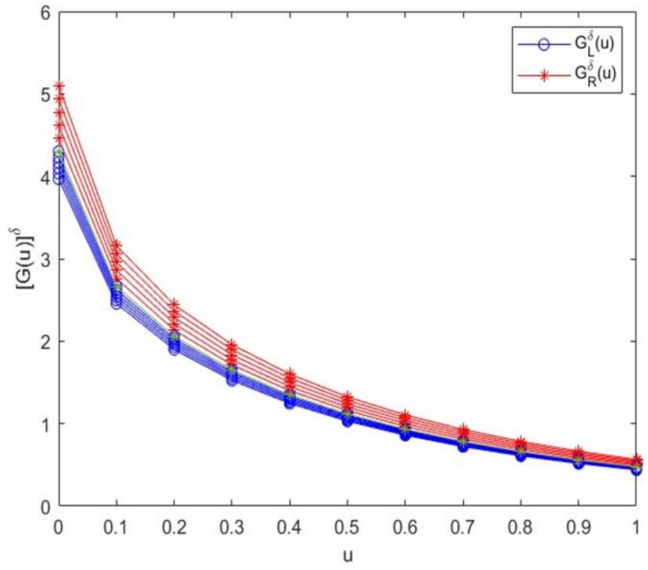

Figure 2. The solution span in granular form corresponding to the IVP (12) with p=0.5

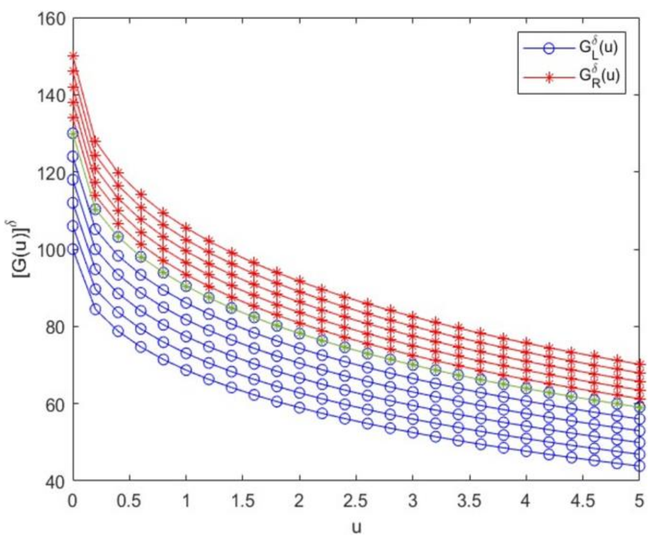

Figure 3. The solution span in granular form corresponding to the IVP (17) with p=0.5

Example 5.3 A body with temperature $(100, 130, 150)^0F$ kept in an environment of $\tilde{L} = (5, 10, 15)^0F$. Then NCFG IVP that describes the temperature inside the body $\tilde{G}(u)$ is modeled by

$$D_{gr}^{p}\tilde{G}(u) = -k[\tilde{G}(u) - (5,10,15)],\ u \in [0,5] \tag{17}$$

$$\text{Subject to, } \tilde{G}(0) = (100, 130, 150) \tag{18}$$

Now, taking $= \frac{1}{10}$, then $\left[\tilde{G}(0)\right]^\delta = [100 + 30\delta, 150 - 20\delta]$ where $\delta \in [0,1]$.

Apply *H.M.F.* on Eq. (17) and Eq. (18) and in particular, we relax the non-zero condition on k_1 i.e., we take $k_1 = 0\ and\ k_0 = pu^{1-p}$, we have

$$pu^{1-p}\frac{\partial}{\partial u}G^{gr}(u,\delta,r_G) = -0.1\,[G^{gr}(u,\delta,r_G) - (5 + 5\delta + 10(1-\delta)r_L] \tag{19}$$

$$H\left[\tilde{G}(0)\right] = 100 + 30\delta + 50(1-\delta)r_{G_0} \tag{20}$$

where $\beta_{G_0}, \beta_L \in [0,1]$.

Solving Eq. (19) and Eq. (20), we get

$$G^{gr}(u,\delta,r_G) = [5 + 5\delta + 10(1-\delta)r_L] + [95 + 25\delta + (1-\delta)(50r_{G_0} - 10r_L)]\, e^{\left[\frac{-0.1u^p}{p^2}\right]} \tag{21}$$

which gives the solution corresponding to the Eq. (19).

Using inverse *H.M.F.* on Eq. (21), we get

$$\left[\tilde{G}(u)\right]^\delta = \left[\inf_{\delta\le\gamma\le 1}\min_{r_G} G^{gr}(u,\gamma,r_G),\ \sup_{\delta\le\gamma\le 1}\max_{r_G} G^{gr}(u,\gamma,r_G)\right]$$

which is the δ −cuts solution corresponding to the IVP (17), using MATLAB draws δ −level sets and presented in Figure 3.

6. CONCLUSION

We have constructed a theoretical framework for NCFGD and its corresponding integral NCFGI. Our approach utilizes the *H.M.F.* to define and analyze these concepts. We demonstrate the application of NCFGD and NCFGI to solve IVPs associated with fuzzy fractional differential equations involving growth, decay and colling processes. By incorporating the relative-distance-measure variable, our method ensures unique and accurate solutions, which are comparable to those obtained for crisp functions. We provide detailed examples and graphical illustrations to showcase the effectiveness of our approach. Future work will focus on extending the NCFGD and NCFGI framework to more complex systems, including BVPs and systems of DEqs. Additionally, we intend to investigate the potential applications of our approach in various fields, including control theory, signal processing, and biological systems with appropriate data sets.

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