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Operational Matrices of Genocchi Polynomials for Solving High-Order Linear Fredholm Integro-Differential Equations

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ABSTRACT

This work presents a numerical method based on the Genocchi polynomials to solve linear Fredholm Integro-Differential Equations (LFIDEs). The process of the method is to transform the (LFIDE) into a matrix equation. This is done by approximating the unknown function, its derivatives, and integral kernel using Genocchi polynomials. After using the equidistant collocation points we solve the corresponding linear system with unknown Genocchi coefficients. To prove the accuracy and efficiency of the current method we mentioned some numerical examples. Comparing the obtained results with the exact solutions and some existing methods, it turns out that the current method gives a better approximation.

1. INTRODUCTION

One of the most important branches of modern mathematics is Integro-Differential Equations (IDEs). It arises naturally in many areas including engineering and science, such as in financial mathematics and control theory [1]. It is also found in describing physical phenomena like wind ripple in the desert, the glass-forming process, fluid mechanics, and the theory of elasticity physics [2].

These equations have an essential role in formulating optimal control problems, fluid dynamics, electrostatics, boundary value problems of gravitation [3-6].

IDEs appear in different forms, the most founded are equations in which the derivative of the unknown function is just outside the integral symbol.

Recently, researchers have increased interest in finding numerical methods to solve this type of equations, because finding analytical solutions is often difficult. Some of these methods can be mentioned like B-spline method [7], the Adomian decomposition method [8], CAS wavelet [9], modified Adomian decomposition method [10], Homotopy perturbation method [11], Hermite wavelet method [12], Modified variational iteration method [13], Haar wavelet bases [14], exponential spline method [15], Schauder bases [16], differential transformation [17], Bernoulli matrix method [18], Bessel collocation method [19], Legendre Galerkin method [20], Taylor collocation method [21], shifted Chebyshev polynomials [22], Euler matrix method [23].

It is known that Genocchi polynomials can be used in series expansions to approximate functions, offering advantages of fewer terms and smaller coefficients, which lead to efficient and stable numerical computations.

This research provides an approximate solution for high-order LFIDEs using an operational matrix based on Genocchi polynomials.

We consider the LFIDE of order m in its general form:

$$\sum_{k=0}^{m} R_k(t)x^{(k)}(t) = g(t) + \lambda\int_a^b K(t,z)x(z)dz, \quad 0 \le a \le t, z \le b \tag{1}$$

Under the given initial-boundary conditions:

$$\sum_{k=0}^{m-1}(\alpha_{pk}x^{(k)}(a) + \beta_{pk}x^{(k)}(b)) = \mu_p \tag{2}$$

for all p=0, 1, ..., m-1, where, $x(t)$ is an unknown function; $R_k(t)$, $g(t)$ are functions defined on $[a, b]$ and the kernel function $k(t, z)$ is defined and continuous on $[a, b]\times[a, b]$; α_{pk}, β_{pk} and μ_p are appropriate constants.

Then, we express the approximate solution x_N of the problem (1) in the form of Genocchi polynomials: $x(t) \cong x_N(t) = \sum_{n=1}^{N} a_n H_n(t)$, $(0 \le t \le 1)$, a_n, (n=1, 2, ..., N), are the coefficients to be determined, whereas $H_n(t)$ represent Genocchi polynomials.

2. GENOCCHI POLYNOMIALS AND FUNCTIONS APPROXIMATION

In this section, we present the definition of Genocchi polynomials along with some of their key properties, as well as Genocchi approximation formulas for functions of one and two variables.

Definition

The Genocchi polynomials $H_n(t)$ are defined by references [24, 25]:

$$\sum_{n=0}^{\infty} H_n(t)\frac{z^n}{n!} = \frac{2ze^{zt}}{e^z+1}$$

where, $H_n(t) = \sum_{k=1}^{n}\binom{n}{k-1}H_{n-k+1}t^{k-1}, n = 1,2,\dots,N,$ and H_n represents Genocchi numbers, which are computed by the Bernoulli numbers according to the following relationship:

$$H_n = 2(1-2^n)B_n$$

Here are the first four Genocchi polynomials:

$$H_0(t)=0;\ H_1(t)=1;\ H_2(t)=2t-1;\ H_3(t)=3t^2-3t$$

At t=0, the Genocchi polynomials yield the Genocchi numbers: H_n=H_n(0).

These polynomials have some important properties such as:

$$H_n(0)+H_n(1)=0,\quad n>1.$$
$$\frac{dH_n(t)}{dt} = nH_{n-1}(t),\quad n\geq 1.$$
$$\int_0^1 H_p(z)H_q(z)dz = \frac{2(-1)^p p!q!}{(p+q)!}H_{p+q},\quad p,q\geq 1.$$

The following theorem presents the approximation formula for a function of one variable S stands for the set Span{$H_1(x)$, $H_2(x)$, ..., $H_N(x)$}.

Theorem [26]

Let $x \in L^2[0,1]$ an arbitrary function then x has a unique best approximation x_N in the finite dimensional vector space S by Genocchi polynomials:

$$x(t) \cong x_N(t) = \sum_{n=1}^{N} a_n H_n(t) \tag{3}$$

The Genocchi coefficients a_n given in Eq. (3) can be computed by applying the formula below:

$$a_n = \frac{1}{2n!}(x^{(n-1)}(0)+x^{(n-1)}(1)), n=1,\dots,N \tag{4}$$

Proof. See the study conducted by Hashemizadeh et al. [26].

For two variables, we mention the following theorem.

Theorem [27]

The approximation of a continuous function $k(t, z)$ in terms of Genocchi polynomials is defined by:

$$K(t,z) \simeq \sum_{i=1}^{N}\sum_{j=1}^{N} k_{pq}H_p(t)H_q(z) = H(t)K_H H^T(z) \tag{5}$$

the matrix $K_H = [k_{pq}]_{N\times N'}$, where $k_{pq} = \frac{1}{4(p!q!)}(K^{(p-1,q-1)}(0,0) + K^{(p-1,q-1)}(0,1) + K^{(p-1,q-1)}(1,0) + K^{(p-1,q-1)}(1,1))$.

For all i, j=1, 2, …, N.

Proof. See the study conducted by Loh and Phang [27].

3. SOLUTION STEPS

To find the approximate solution to Eq. (1) using Eq. (3), we follow these steps:

Step 1: We write the matrix forms of the truncated Genocchi series as well as its derivatives:

$$x(t) \cong x_N(t) = \sum_{n=1}^{N} a_n H_n(t) = T(t)FA \tag{6}$$

and

$$x^{(k)}(t) \cong x_N^{(k)}(t) = T(t)D^k FA, (k=0,1,\dots,m) \tag{7}$$

Indeed,

$$T^{(1)}(t) = T(t)D$$
$$T^{(2)}(t) = T^{(1)}(t)D = T(t)D^2$$
$$\vdots$$
$$T^{(k)}(t) = T^{(k-1)}(t)D = T(t)D^k$$

Therefore,

$$x^{(k)}(t) \cong x_N^{(k)}(t) = T(t)D^k FA, (k=0,1,\dots,m)\ .$$

where, $T(t) = [1, t, t^2, \dots, t^{N-1}]$; $A = [a_1, a_2, \dots, a_N]^T$,

$$F = \begin{bmatrix} \binom{1}{0}H_1 & \binom{2}{0}H_2 & \binom{3}{0}H_3 & \dots & \binom{N}{0}H_N \\ 0 & \binom{2}{1}H_1 & \binom{3}{1}H_2 & \dots & \binom{N}{1}H_{N-1} \\ 0 & 0 & \binom{3}{2}H_1 & \dots & \binom{N}{2}H_{N-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \binom{N}{N-1}H_1 \end{bmatrix};$$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & N-1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the above, we get the matrix relation for the first side of Eq. (1):

$$\sum_{k=0}^{m} R_k(t)x^{(k)}(t) = \sum_{k=0}^{m} R_k(t)T(t)D^k FA \tag{8}$$

Step 2: To obtain the matrix relation of the integral part in the problem (1) we use the formula (5) and (6). Thus:

$$I(t)=\int_0^1 K(t,z)x(z)dz=T(t)FK_HHA \tag{9}$$

where,

$$H=[H_{pq}]_{N\times N},(p,q=1,2,\dots,N);$$
$$H_{pq}=\frac{2(-1)^p p!\,q!}{(p+q)!}H_{p+q}.$$

Therefore, the second side of Eq. (1) is:

$$g(t)+\lambda\int_0^1 K(t,z)x(z)dz=G+\lambda T(t)FK_HHA \tag{10}$$

Next, we substitute (8) and (10) into Eq. (1):

$$\sum_{k=0}^{m}R_kTD^kFA=G+\lambda TFK_HHA \tag{11}$$

Step 3: We write the matrix corresponding to conditions (2), which we obtain by Eq. (7) as:

$$\sum_{k=0}^{m-1}(\alpha_{pk}T(a)+\beta_{pk}T(b))D^kFA=\mu_p\,,\quad p=0,1,\dots,m-1 \tag{12}$$

Step 4: We define the collocation equidistant points $t_i=\frac{1}{N}i, i=0,1,\dots,N-1$.

Then, we substitute these points in (11). So, we obtain:

$$\left(\sum_{k=0}^{m}R_kTD^kF-\lambda TFK_HH\right)A=G \tag{13}$$

where,

$$R_k=\begin{bmatrix} r_k(t_0) & 0 & 0 & \dots & 0\\ 0 & r_k(t_1) & 0 & \dots & \\ 0 & 0 & r_k(t_2) & \dots & 0\\ . & . & . & & .\\ . & . & . & & .\\ 0 & 0 & 0 & \dots & r_k(t_{N-1})\end{bmatrix}$$

$$T=\begin{bmatrix} 1 & t_0^1 & t_0^2 & \dots & t_0^{N-1}\\ 1 & t_1^1 & t_1^2 & \dots & t_1^{N-1}\\ . & . & . & \dots & .\\ . & . & . & \dots & .\\ 1 & t_{N-1}^1 & t_{N-1}^2 & \dots & t_{N-1}^{N-1}\end{bmatrix};G=\begin{bmatrix} g(t_0)\\ g(t_1)\\ .\\ .\\ g(t_{N-1})\end{bmatrix}$$

The above matrix relation can be expressed as $\Psi A=G$ or $[\Psi;G]$, where, $\Psi=[\psi_{pq}]=\sum_{k=0}^{m}R_k\widehat{T}D^kF-\lambda\widehat{T}FK_HH,\ p,q=0,1,\dots,N-1$.

$$[\Psi;G]=\begin{bmatrix} \psi_{00} & \psi_{01} & \dots & \psi_{0(N-1)} & ; & g(t_0)\\ \psi_{10} & \psi_{11} & \dots & \psi_{1(N-1)} & ; & g(t_1)\\ \dots & \dots & \dots & \dots & ; & \dots\\ \psi_{(N-1)0} & \psi_{(N-1)1} & \dots & \psi_{(N-1)(N-1)} & ; & g(t_{N-1})\end{bmatrix} \tag{14}$$

Besides, the form of the matrix (12) for the conditions (2) becomes:

$$C_pA=\left[\mu_p\right] \text{ or } \left[C_p;\mu_p\right]=\left[c_{p0}\ \ c_{p1}\ \dots\ c_{p(N-1)}\ ;\ \mu_p\right] \tag{15}$$

where, $C_p=\sum_{k=0}^{m-1}(\alpha_{pk}T(a)+\beta_{pk}T(b))D^kF, p=0,1,\dots,m-1$.

For m initial conditions, the augmented matrix becomes:

$$\left[\Psi;G\right]=\begin{bmatrix} \psi_{00} & \psi_{01} & \psi_{02} & \dots & \psi_{0(N-1)} & ; & g(t_0)\\ \psi_{10} & \psi_{11} & \psi_{12} & \dots & \psi_{1(N-1)} & ; & g(t_1)\\ . & . & . & \dots & . & . & .\\ . & . & . & \dots & . & . & .\\ \psi_{(N-1-m)0} & \psi_{(N-1-m)1} & \psi_{(N-1-m)2} & \dots & \psi_{(N-1-m)(N-1)} & ; & g(t_{N-1-m})\\ c_{00} & c_{01} & c_{02} & \dots & c_{0(N-1)} & ; & \mu_0\\ c_{10} & c_{11} & c_{12} & \dots & c_{1(N-1)} & ; & \mu_1\\ . & . & . & \dots & . & . & .\\ . & . & . & \dots & . & . & .\\ c_{(m-1)0} & c_{(m-1)1} & c_{(m-1)2} & \dots & c_{(m-1)(N-1)} & ; & \mu_{m-1}\end{bmatrix} \tag{16}$$

Finally, if $rank\,\widehat{\Psi}=rank\left[\widehat{\Psi}\,;\widehat{G}\right]=N$, we get the unique solution of (1) by Genocchi series solution (3).

4. ERROR ANALYSIS

The following theorem provides the error estimate for the function approximation used to solve problem (1).

Theorem [28]

Consider $x\in L^2[0,1]$ an arbitrary function with $\left|x^{(n-1)}(t)\right|\le\rho$ (ρ is finite), if $x_N(t)$ is an approximation of $x(t)$ by truncated Genocchi series. $x_N(t)=\sum_{n=1}^{N}a_nH_n(t)$ then the error $E_N=\|x(t)-x_N(t)\|_2$ is bounded above by:

$$E_N\le\left(\sum_{n=N+1}^{\infty}\sum_{k=0}^{n}\frac{\rho^2}{2\times(n!)^2}\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1}\right)^{\frac{1}{2}}.$$

where, $E_N=(\int_0^1|x(t)-x_N(t)|^2dt)^{\frac{1}{2}}$

Proof. The study conducted by Loh et al. [28].

$$\begin{aligned}E_N^2 &= \int_0^1\left|x(t)-x_N(t)\right|^2dt = \int_0^1\left|x(t)-\sum_{n=1}^{N}a_nH_n(t)\right|^2dt\\ &= \int_0^1\left|\sum_{n=N+1}^{\infty}a_nH_n(t)\right|^2dt\\ &\le \sum_{n=N+1}^{\infty}\left|a_n\right|^2\int_0^1\left|H_n(t)\right|^2dt,\end{aligned}$$

We have, $H_n(t)=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}t^k$. Hence

$$\begin{aligned}E_N^2 &\le \sum_{n=N+1}^{\infty}\left|a_n\right|^2\sum_{k=0}^{n}\binom{n}{k}^2H_{n-k}^2\int_0^1t^{2k}dt\\ &= \sum_{n=N+1}^{\infty}\left|a_n\right|^2\sum_{k=0}^{n}\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1}\end{aligned}.$$

Using the formula (4) we have:

$$\begin{aligned} E_N^2 &\le \sum_{n=N+1}^{\infty}\left|\frac{1}{2n!}\left(x^{(n-1)}(0)+x^{(n-1)}(1)\right)\right|^2\sum_{k=0}^{n}\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1} \\ &\le \sum_{n=N+1}^{\infty}\sum_{k=0}^{n}\frac{1}{4(n!)^2}\left(\left|x^{(n-1)}(0)\right|^2+\left|x^{(n-1)}(1)\right|^2\right)\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1} \\ &\le \sum_{n=N+1}^{\infty}\sum_{k=0}^{n}\frac{1}{4(n!)^2}(2\rho^2)\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1} \\ &= \sum_{n=N+1}^{\infty}\sum_{k=0}^{n}\frac{\rho^2}{2(n)^2}\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1}. \end{aligned}$$

Therefore,

$$E_N \le \left(\sum_{n=N+1}^{\infty}\sum_{k=0}^{n}\frac{\rho^2}{2(n!)^2}\binom{n}{k}^2\frac{H_{n-k}^2}{2k+1}\right)^{\frac{1}{2}}.$$

If the exact solution is unknown. The error estimation is given by the following theorem.

Theorem [28]

Let $x(t)$ be the unknown solution and $x_N(t)$, $x_{N+1}(t)$ be the approximate solutions of $x(t)$. The error estimation given by $e_N = \|x_N(t) - x_{N+1}(t)\|_2$ is convergent.

Proof.

$$\begin{aligned} e_N &= \left\|x_{N+1}(t)-x_N(t)\right\|_2 = \left\|x_{N+1}(t)-x(t)+x(t)-x_N(t)\right\|_2 \\ &\le \left\|x_{N+1}(t)-x(t)\right\| + \left\|x(t)-x_N(t)\right\| \\ &\le E_{N+1}+E_N \end{aligned}$$

e_N is convergent because E_{N+1} and E_N are also convergent.

5. NUMERICAL EXAMPLES

The aim of this section is to prove the efficiency and effectiveness of the method used in this research for solving the higher-order LFIDE through six different examples, using MATLAB program. The results obtained are shown in the tables and figures below.

Example 1. [29]
Consider the following equation:

$$\begin{cases} x'(t)-2tx(t) = -2t^3-2t^2+\dfrac{23}{6}t+1+2\displaystyle\int_0^1 tzx(z)dz, \ 0\le t,z\le 1 \\ x(0)=-1 \end{cases}$$

where, $k(t, z)=tz$, $g(t) = -2t^3-2t^2+\frac{23}{6}t+1$, $\lambda=2$ and $R_0(t)=-2t$, $R_1(t)=1$.
The approximate solution $x(t)$ of this equation by the Genocchi polynomials is:

$$x(t)\cong x_3(t)=\sum_{n=1}^{3}a_nH_n(t)$$

where, $0\le t\le 1$.
The collocation points for N=2 are: $\left\{t_0=0, t_1=\frac{1}{2}, t_2=1\right\}$, hence $\left(\sum_{k=0}^{1} R_k\,\widehat{T}\,D^kF - \lambda\,\widehat{T}\,FK_HH\right)A = G$, where,

$$R_0=\begin{bmatrix}0&0&0\\0&-1&0\\0&0&-2\end{bmatrix},\ R_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ D=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix},$$

$$F=\begin{bmatrix}1&-1&0\\0&2&-3\\0&0&3\end{bmatrix},\ G=\begin{bmatrix}1\\ \frac{13}{6}\\ \frac{5}{6}\end{bmatrix}\ K_H=\begin{bmatrix}\frac{1}{4}&\frac{1}{4}&0\\ \frac{1}{4}&\frac{1}{4}&0\\0&0&0\end{bmatrix},$$

$$H=\begin{bmatrix}1&0&-\frac{1}{2}\\0&\frac{1}{3}&0\\-\frac{1}{2}&0&\frac{3}{10}\end{bmatrix},\ T=\begin{bmatrix}1&0&0\\1&\frac{1}{2}&\frac{1}{4}\\1&1&1\end{bmatrix},$$

Altogether, we find that:

$$[\Psi;G]=\begin{bmatrix}0&2&-3&;&1\\-\frac{3}{2}&\frac{11}{6}&1&;&\frac{13}{6}\\-3&-\frac{1}{3}&\frac{7}{2}&;&\frac{5}{6}\end{bmatrix}$$

The matrix representation of the initial condition for N=3 is $[C_0; \mu_0]$=[1 -1 0].
We get the new augmented matrix related to the condition:

$$\left[\Psi;G\right]=\begin{bmatrix}0&2&-3&;&1\\-\frac{3}{2}&\frac{11}{6}&1&;&\frac{13}{6}\\1&-1&0&;&-1\end{bmatrix}$$

After solving the above linear system, we obtain the unknown Genocchi coefficients $A=\left[0\ 1\ \frac{1}{3}\right]^T$.
Hence, the numerical solution for the first example is expressed as:

$$x_3(t)=\sum_{n=1}^{3}a_nH_n(t)=a_1H_1(t)+a_2H_2(t)+a_3H_3(t)=t^2+t-1.$$

We note that the current method gave us the exact solution.

Example 2. [11-13, 16]
Consider the first linear LFIDE

$$\begin{cases} x'(t)-\displaystyle\int_0^1 tx(z)dz = te^t+e^t-t, \quad x(0)=0 \\ x(t)=te^t \end{cases}$$

Table 1 and Figure 1 display the Exact Solution (Ex.S) and the Approximate Solutions (App.S) obtained by the current method for Example 2, with N=8 and N=10. These results are compared with some existing methods [11-13, 16] in Table 2.

Figure 2 presents the Absolute Errors (Ab.E) obtained by the current method for Example 2 (when $4 \le N \le 10$). These results show that the errors decrease when N increases.

Table 1. Solutions of Example 2

t	Ex.S	App.S (N=8)	App.S (N=10)
0.1	0.11051709180	0.11051707001	0.11051709175
0.2	0.24428055163	0.24428052896	0.24428055158
0.3	0.40495764227	0.40495762191	0.40495764222
0.4	0.59672987905	0.59672985385	0.59672987901
0.5	0.82436063535	0.82436060931	0.82436063531
0.6	1.09327128023	1.09327125290	1.09327128019
0.7	1.40962689522	1.40962685865	1.40962689519
0.8	1.78043274279	1.78043271555	1.78043274275
0.9	2.21364280004	2.21364275749	2.21364280003

Table 2. Absolute errors corresponding to Example 2 for N=10 compared with existing methods

t	HP.M [11]	S-B.M [16]	MVI.M [13]	Her-W.M [12]	Ab.E (N=*10*)
0.1	0.231*e*–05	1.017*e*–07	3.000*e*–09	4.938*e*–10	5.714*e*–11
0.2	0.925*e*–05	4.827*e*–07	1.000*e*–09	4.192*e*–10	4.858*e*–11
0.3	0.208*e*–04	1.017*e*–07	2.800*e*–09	4.552*e*–10	4.662*e*–11
0.4	0.370*e*–05	1.619*e*–06	7.300*e*–09	4.658*e*–10	4.186*e*–11
0.5	0.578*e*–04	2.308*e*–06	9.100*e*–09	4.905*e*–10	3.821*e*–11
0.6	0.833*e*–04	3.093*e*–06	1.000*e*–08	5.205*e*–10	3.530*e*–11
0.7	0.113*e*–03	3.978*e*–06	1.060*e*–07	5.494*e*–10	3.302*e*–11
0.8	0.148*e*–03	4.995*e*–06	1.700*e*–07	5.971*e*–10	3.830*e*–11
0.9	0.187*e*–03	6.135*e*–06	1.590*e*–07	6.216*e*–10	7.990*e*–12

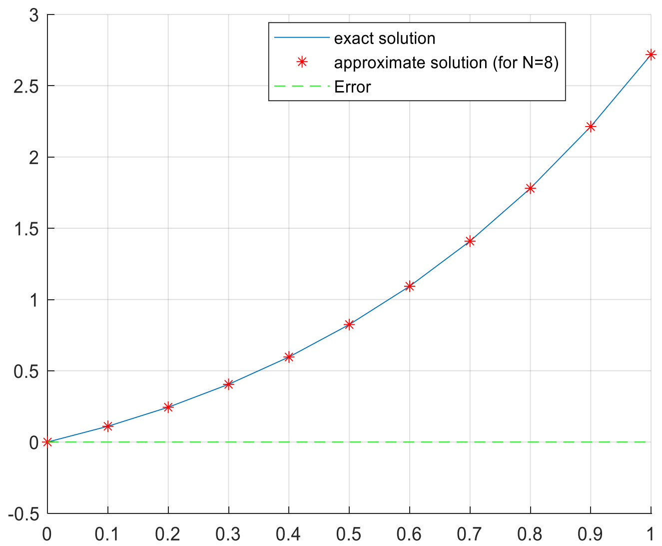

Figure 1. Solutions and absolute errors corresponding to Example 2

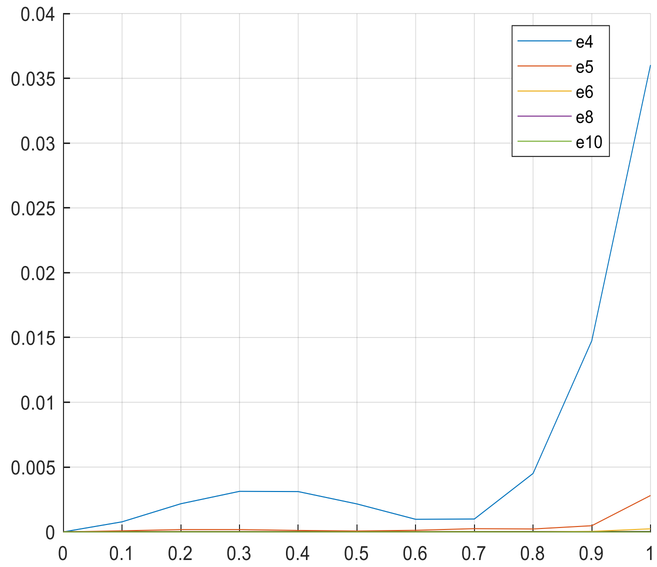

Figure 2. The absolute errors of Example 2 for $4 \le N \le 10$

Table 3. The absolute errors of the current method for Example 3

t	Ab.E (N=10)	Ab.E (N=12)	Ab.E (N=14)
0.125	8.350*e* – 05	9.890*e* – 07	2.085*e* – 07
0.250	1.545*e* – 04	1.831*e* – 06	1.528*e* – 07
0.375	2.019*e* – 04	2.392*e* – 06	7.390*e* – 08
0.500	2.192*e* – 04	2.597*e* – 06	1.630*e* – 08
0.625	2.053*e* – 04	2.432*e* – 06	1.042*e* – 07
0.750	1.649*e* – 04	1.954*e* – 06	1.768*e* – 07
0.825	1.061*e* – 04	1.266*e* – 06	2.232*e* – 07

Example 3. [15]
Consider the following LFIDE:

$$\begin{cases} x''(t)+t\,x'(t)+\pi^2 x(t)-\int_0^1 (t+z)x(z)dz = \pi t\cos(\pi t)-\dfrac{2t+1}{\pi} \\ x(0)=x(1)=0;\ x(t)=\sin(\pi t) \end{cases}$$

Figure 3 displays the Exact Solution (Ex.S) and the Approximate Solutions (App.S) obtained by the current method for Example 3, with N=10. Table 3 and Figure 4 present the Absolute Errors (Ab.E) obtained by the current method for Example 3. These results show that the errors diminish as N increases.

Table 4 provides a comparison of the Ab.E achieved by the current method (for N=14) with those obtained using the exponential spline method [15].

Table 4. Comparison between current method and ES.M for Example 3

t	Ab.E (N=14)	ES.M [15]
0.125	$2.09e-07$	$1.41e-06$
0.250	$1.53e-07$	$2.53e-06$
0.375	$7.39e-08$	$3.31e-06$
0.500	$1.63e-08$	$3.68e-06$
0.625	$1.04e-07$	$3.59e-06$
0.750	$1.77e-07$	$2.99e-06$
0.825	$2.23e-07$	$1.81e-06$

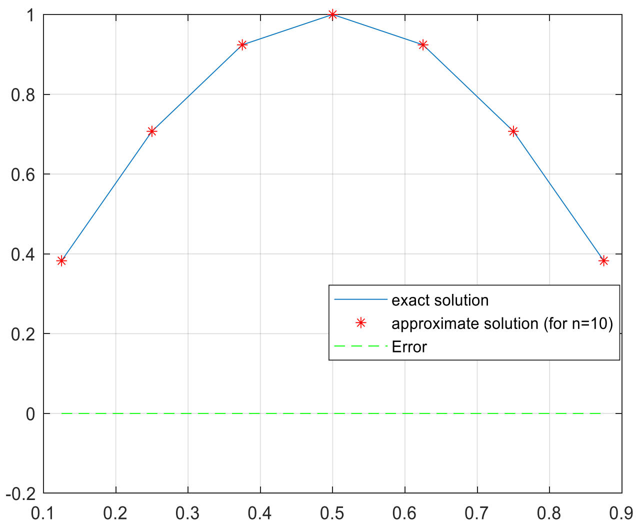

Figure 3. Solutions and absolute errors corresponding to Example 3

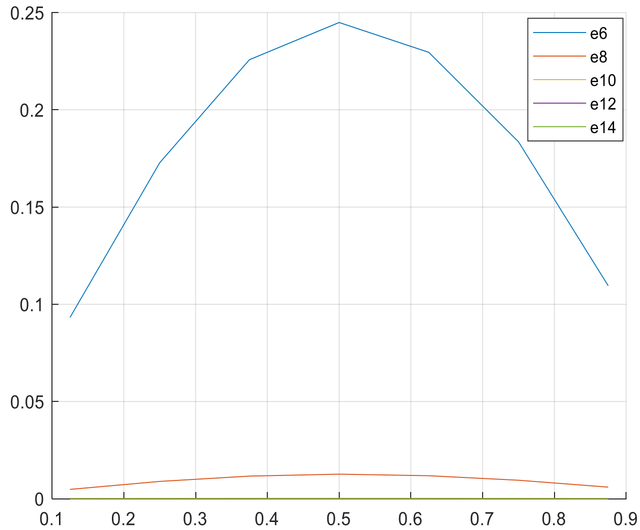

Figure 4. The absolute errors of Example 3

Example 4.
Consider the following example:

$$\begin{cases} x'(t)-\int_0^1 tzx(z)dz = 1-\dfrac{1}{3}t, \quad x(0)=0 \\ x(t)=t \end{cases}$$

Table 5 represents a comparison of the Absolute Errors (Ab.E) obtained by the current method with existing methods [9, 14, 16, 17] corresponding to Example 4. It is observed that the results obtained using Genocchi polynomials are more accurate than [9, 14, 16, 17]. Solutions and Absolute Errors for N=6 are shown in Figure 5.

Table 5. Absolute errors of the current method and [9, 14, 16, 17] for Example 4

t	DT.M [17]	CAS-W.M [9]	S-B.M [16]	Haar-W.M [14]	Current.M (N=2)
0.1	1.60e–03	2.17e–04	3.79e–06	1.60e–06	0
0.2	6.09e–03	6.32e–04	1.51e–05	2.36e–06	0
0.3	1.32e–02	7.91e–04	3.41e–05	2.26e–06	0
0.4	2.29e–02	2.15e–02	6.06e–05	1.31e–06	0
0.5	3.51e–02	4.99e–02	9.47e–05	4.85e–07	0
0.6	6.69e–02	2.21e–02	1.36e–05	9.28e–07	0
0.7	7.12e–02	1.05e–04	1.85e–05	1.48e–06	0
0.8	8.63e–02	1.43e–03	2.42e–04	1.91e–06	0
0.9	1.08e–01	2.07e–02	3.06e–04	5.40e–05	0

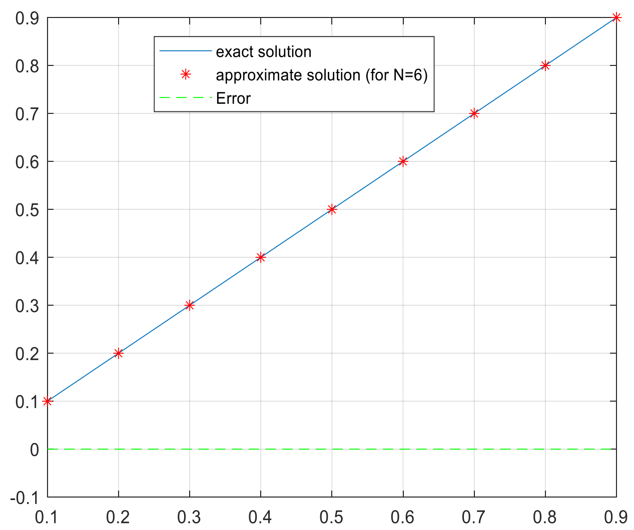

Figure 5. Solutions and absolute errors corresponding to Example 4

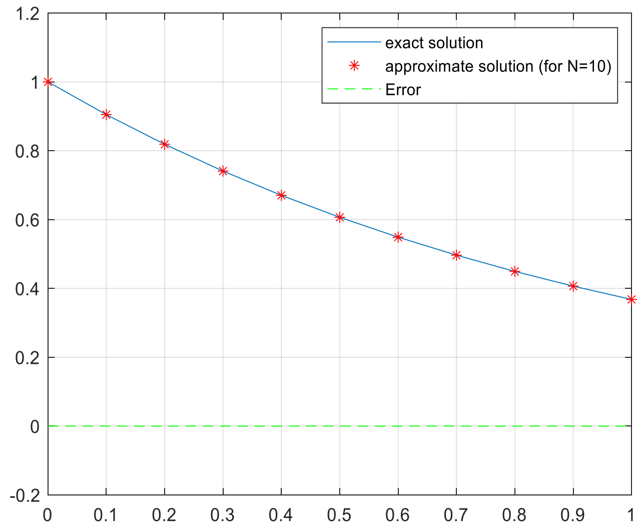

Figure 6. Solutions and absolute errors corresponding to Example 5

Table 6. Solutions of Example 5

t	Ex.S	App.S (N=6)	App.S (N=8)	App.S (N=10)
0	1.000000000000	1.000000000000	0.999999999999	1.000000000035
0.1	0.904837418035	0.904836651302	0.904837416431	0.904837418064
0.2	0.818730753077	0.818729435860	0.818730751057	0.818730753098
0.3	0.740818220681	0.740816851984	0.740818218442	0.740818220691
0.4	0.670320046035	0.670318599040	0.670320043229	0.670320046033
0.5	0.606530659712	0.606528829431	0.606530656568	0.606530659698
0.6	0.548811636094	0.548809400569	0.548811632614	0.548811636069
0.7	0.496585303791	0.496583126859	0.496585299618	0.496585303757
0.8	0.449328964117	0.449327031668	0.449328960136	0.449328964077
0.9	0.406569659740	0.406565599311	0.406569654869	0.406569659699
1	0.367879441171	0.367864027021	0.367879404664	0.367879441080

Table 7. Absolute errors of current method for Example 5

t	Ab.E (N=6)	Ab.E (N=8)	Ab.E (N=10)
0	$3.33e-17$	$4.97e-14$	$3.52e-11$
0.1	$7.76e-07$	$1.60e-09$	$2.84e-11$
0.2	$1.32e-06$	$2.02e-09$	$2.03e-11$
0.3	$1.37e-06$	$2.23e-09$	$9.71e-12$
0.4	$1.45e-06$	$2.80e-09$	$2.03e-12$
0.5	$1.83e-06$	$3.14e-09$	$1.39e-11$
0.6	$2.24e-06$	$3.47e-09$	$2.48e-11$
0.7	$2.18e-06$	$4.17e-09$	$3.36e-11$
0.8	$1.93e-06$	$3.98e-09$	$3.98e-11$
0.9	$4.06e-06$	$4.87e-09$	$4.13e-11$
1	$1.54e-05$	$3.65e-08$	$9.05e-11$

Example 5.
Consider the following LFIDE:

$$\begin{cases} x'(t)+x(t)-\int_0^1 x(z)dz = e^{-t}-1, \quad x(0)=1 \\ x(t)=e^{-t} \end{cases}$$

Table 6 and Figure 6 show the Exact Solution and the numerical results obtained by current method for Example 5. The results related to the Absolute Errors are listed in Table 7, these results show that errors decrease when N increases.

Example 6.
Consider the following example:

$$\begin{cases} x^{(4)}(t)-\int_0^1 (t-z)x(z)dz = \frac{1}{4}+\left(1-2\ln(2)\right)t-\frac{6}{(1+t)^4} \\ x(t)=\ln(t+1) \end{cases}$$

Under the conditions:

$$x(0)=0, x'(0)=1, x''(0)=-1, x^{(3)}(0)=2$$

Table 8. Results for Example 6

t	Ex.S	App.S (N=10)	Ab.E (N=10)
0.1	0.09531017980	0.09531017401	5.79600758e-09
0.2	0.18232155679	0.18232146464	9.21490075e-08
0.3	0.26236426447	0.26236388224	3.82226043e-07
0.4	0.33647223662	0.33647124574	9.90881794e-07
0.5	0.40546510810	0.40546307014	2.03797158e-06
0.6	0.47000362925	0.46999998874	3.64050549e-06
0.7	0.53062825106	0.53062238591	5.86515076e-06
0.8	0.58778666490	0.58777984812	6.81678700e-06
0.9	0.64185388617	0.64186210865	8.22247401e-06

Table 9. Results for Example 6

t	Power.S (N=10)	Chebychev.S (N=10)	Current.M (N=8)
0.1	2.1000e-07	3.8052e-04	5.9395e-08
0.2	1.3650e-06	3.6884e-04	1.1358e-06
0.3	2.2653e-05	3.4487e-04	5.1853e-06
0.4	2.3341e-05	3.4228e-04	1.3760e-05
0.5	2.7115e-05	4.2488e-03	2.8711e-05
0.6	2.8371e-05	2.4947e-03	5.1232e-05
0.7	3.2837e-05	2.1345e-03	6.8496e-05
0.8	1.7805e-04	1.5534e-03	1.3236e-05
0.9	1.6954e-04	1.4867e-03	3.2054e-04

Table 8 and Figure 7 display the Exact Solution (Ex.S), the Approximate Solution (App.S) and Absolute Errors (Ab.E) obtained by the current method for Example 6, with N=10.

Table 9 represents a comparison of the Absolute Errors (Ab.E) obtained by the current method (for N=8) with those from the Power and Chebychev Series [30] corresponding to Example 6. It is observed that the results obtained using Genocchi polynomials are more accurate than the study conducted by Gegele et al. [30].

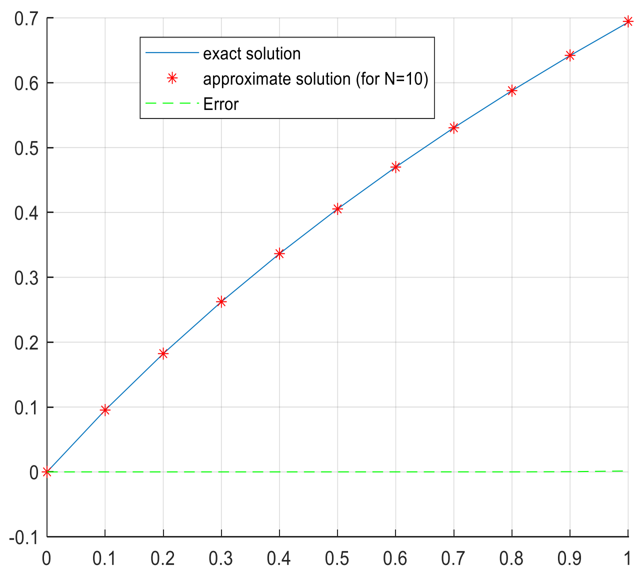

Figure 7. Solutions and absolute errors corresponding to Example 6

6. RESULTS AND DISCUSSION

Obtained results in Section 5 demonstrate the accuracy, efficiency and speed of the current method.

In the first example, due to the advantages of the Genocchi polynomials, such as smaller coefficients and lesser terms, we achieved an Approximate Solution for N=2. In contrast, Boole's polynomials [29] provided the same result but for N=3.

The results obtained in Examples 2, 3, 4, 5 and 6, presented in the tables and figures above, show the efficiency of current method, with Absolute Error decreasing as N increases. Moreover, comparing the numerical results obtained by Genocchi polynomials with some existing methods demonstrate that the proposed method achieves better accuracy.

7. CONCLUSIONS

In this study, we have used a new collocation method based on Genocchi truncated series to solve high-order LFIDEs. The examples and results presented in section 5 have proved the efficiency and effectiveness of the current method, showcasing its superiority over existing methods [9, 11-17, 29, 30]

Moreover, the present method is easy to use and quick to apply using MATLAB tool. This is due to the advantages of Genocchi polynomials, which include fewer terms and smaller coefficients compared to other polynomials. In future research, we aim to expand the application of our method to other types of IDEs.

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