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Characterization of Cementitious Materials Through Analysis of Dispersion Curves Using Singular Value Decomposition

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ABSTRACT

Non-destructive characterization of cementitious materials is essential to ensure structural integrity and longevity in civil engineering. This study presents an advanced approach using singular value decomposition (SVD) to analyze ultrasonic dispersion curves, thereby improving the clarity and precision of ultrasonic characterization of cement slabs. We focused on the impact of different water/cement ratios on ultrasonic properties, revealing how these ratios influence the material's microstructure and mechanical properties. Our results show that higher water/cement ratios increase porosity and reduce density, thus slowing down the propagation of ultrasonic waves and potentially reducing mechanical strength. The application of SVD allowed for the isolation of significant signal components, providing a better understanding of the acoustic behavior and mechanical properties of the material. These findings highlight the importance of precise control over the water/cement ratio in the development of durable and robust cementitious structures. The study suggests that SVD can significantly improve the reliability of non-destructive testing methods, making it a valuable tool in materials science and engineering. This research advances current methodologies by offering enhanced accuracy in characterizing the internal structure and mechanical properties of cementitious materials.

1. INTRODUCTION

The rigorous characterization of cementitious materials is of fundamental importance in ensuring structural integrity, durability, and performance in a variety of application areas, including civil engineering, construction, and infrastructure development [1, 2]. Historically, destructive testing methods have been commonly used; however, their drawbacks, such as cost, lack of practicality for in situ assessments, and time-consuming nature, have stimulated the search for alternative approaches [3].

In response to these challenges, ultrasonic techniques have gained popularity as non-destructive tools for characterizing cementitious materials [4-8]. By exploiting the propagation of ultrasonic waves through the material, these techniques provide invaluable information on the internal structure, elastic properties, and durability of the material. Nevertheless, the reliability of these techniques is directly linked to the accuracy with which the acquired signals are interpreted and analyzed, a task often complicated by the presence of echoes, overlapping waveforms, and noise [9]. This has led to the exploration of advanced signal processing methods, such as singular value decomposition (SVD), which recent studies have recognized for their effectiveness in enhancing signal clarity and measurement accuracy [10].

Traditional methods for processing ultrasonic signals are often based on conventional filtering techniques and Fourier transform analysis [11], which may not be sufficient to clearly distinguish the different propagation modes in the presence of significant noise. These methods may also fail to extract relevant information when signals are strongly disturbed by interference or low-amplitude components. In this context, this study proposes to incorporate singular value decomposition (SVD) into ultrasound signal processing [12, 13]. SVD is a mathematical method that decomposes a matrix into its constituent elements, thus revealing the structures and patterns inherent in the data. Applied to ultrasonic data obtained from cement slabs, SVD eliminates unwanted noise, distinguishes between different wave modes, and improves measurement reliability and accuracy. SVD separates the significant components of ultrasonic signals from unwanted noise, improving the clarity of the signals analyzed. By breaking down signals into singular values and vectors, SVD accurately identifies the different modes of propagation of ultrasonic waves, which are essential for characterizing the mechanical properties of materials [14-16].

Thus, by applying the SVD method in our study, we were able to obtain more accurate and reliable results concerning the ultrasonic properties of cementitious materials for different water/cement ratios. This innovative approach has enabled us to improve the characterization of dispersion curves and the determination of longitudinal propagation velocities, thus providing essential information on the internal structure and behavior of cementitious materials.

2. MATERIALS AND METHODS

2.1 Experimental setup and sample preparation

In this study, cement samples are prepared with water/cement ratios of 0.35, 0.40, and 0.50, as shown in Table 1. Drinking water is used for the preparation of all specimens. These samples are cured for 48 hours at a controlled temperature of 25°C to ensure optimal homogeneity.

Table 1. Cement pastes specimens

Specimens	S1	S2	S3
Cement (g)	200	200	200
Water (g)	70	80	100
w/c (%)	0.35	0.4	0.5

The cement used is Portland composite CPJ45 (CEM II), produced in the Souss Massa region of Morocco. Its chemical and physical properties, analyzed in collaboration with the cement plant, are presented in Table 2.

Table 2. Chemical composition and physical properties of Portland cement CPJ 45

Chemical Composition (%)		Physical Properties	
SiO_2	16.7	Blaine (m^2/kg)	409.2
Al_2O_3	5.48	Absolute density (g/cm^3)	3.1593
FeO_3	2.8	Compactness (%)	52.1
CaO	51.8	Porosity (%)	47.9
MgO	3.1	Retention capacity (%)	45
K_2O	0.8	Permeability (ml/s)	11.03825
SO_3	2.7	**Compressive Strength**	
P_2O_5	0.28		
Cl^-	0.02	2 days (MPa)	17.1
Insoluble	11.8	7 days (MPa)	29.9
L.O.I	4.7	28 days (MPa)	46.1

The experimental configuration, as illustrated in Figure 1, includes several precision instruments: two submerged transducers (Panametrics V309), a precision pulse generator (5052 Sofranel PR), a digital oscilloscope (54600B-100 MHz), a computer fitted with a GPIB acquisition card, and a stepper motor.

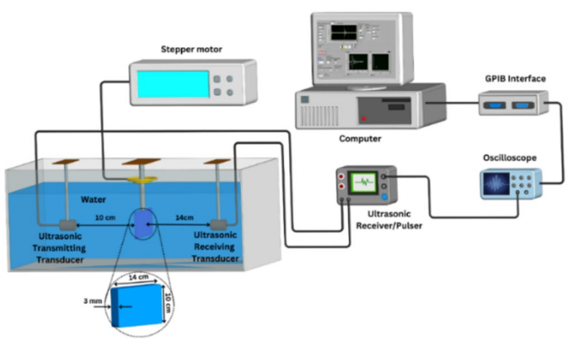


Figure 1. Schema of experimental setup [4, 17]

The pulse generator activates the broadband transducers, which emit ultrasonic waves at a central frequency of 5 MHz using 13 mm diameter piezoelectric emitter pads. These transducers, along with the cement sample, are fully submerged in a water tank to ensure complete immersion and safeguard against parasitic echoes from the tank's edges. Positioned directly opposite each other along the same axis, the transducers apply to both sides of the cement plate, operating within the far field (Fraunhofer) zone, thereby emulating a near-plane wave propagation [17, 18].

This system is designed to allow rotational adjustments around a horizontal axis perpendicular to the ultrasonic beam, enhancing the precision in aligning the beam incidence with the sample surface. The arrangement also supports the capture of signals at varied incidences, facilitated by a stepper motor that rotates the cement plate with an accuracy of 10^{-3} degrees, ensuring precise modification of the incidence angle.

The ultrasonic signals captured are then digitized by the digital oscilloscope (54600B-100 MHz). An average of these signals is computed and relayed to the computer via a National Instruments acquisition card. This setup is managed by LabVIEW software, which also processes signals that are not directly displayable on the oscilloscope.

Additionally, we have developed a custom software component within the LabVIEW environment. This component simplifies the process of signal acquisition and recording, automates the Fast Fourier Transform (FFT) calculations for each signal, and facilitates the subsequent computation of the transmission coefficient.

2.2 Ultrasonic measurements

The use of a system capable of generating and receiving waves is essential for exploring the propagation of wave modes. This study uses an optimized configuration in which two corresponding transducers are positioned opposite each other on either side of the cement slab, aligned along the same axis, rather than side by side. This approach simplifies the traditional configuration while improving the directness and quality of wave generation.

As depicted in Figure 1, the experimental device allows for adjustments in the angle of incidence of the cement plate once the transducers are set. This feature is vital for an accurate experimental evaluation of the dispersion curves in cement. It involves adjusting the incidence angles and systematically recording the transmitted time signals and their spectra at each angle, employing the previously discussed transmission technique.

Indeed, the FFT (Fast Fourier Transform) spectrum, obtained from these time signals using FFT transformation, enables visualization of the resonance characteristics of the cement plate. Consequently, the transmission coefficient is computed by determining the ratio of the Fourier transform of the transmitted signal ($A_{trans}(f)$) to the Fourier transform of the reference signal ($A_{ref}(f)$), which represents the spectrum of the signal transmitted through water [18, 19]. The transmission coefficient is defined by the following equation [4, 20]:

$$T(f) = \frac{A_{trans}(f)}{A_{ref}(f)} \quad (1)$$

Once the transmission coefficients are calculated, as outlined in Eq. (1), for a range of frequencies, water/cement ratios, and angles of incidence, the data are systematically organized. For the water/cement ratio of 0.35, the transmission coefficients from angles between 0° and 70° are compiled into a matrix. This matrix is then transformed into a graphical image that vividly illustrates the dispersion curve for the cement slab at this specific ratio. This analytical method was similarly applied to the other ratios, specifically for w/c = 0.4 and w/c = 0.5. The technique used to obtain the dispersion curves experimentally is illustrated in Figure 2.

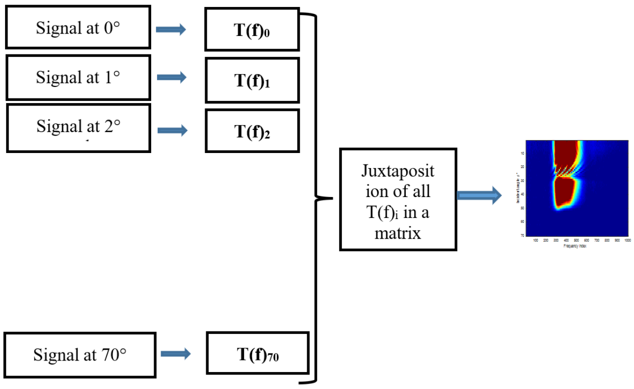


Figure 2. Illustration of the procedure used to create the dispersion curves associated with a given E/C ratio

2.3 Singular value decomposition (SVD) for ultrasonic signal processing

2.3.1 Mathematical principles of SVD

The singular value decomposition (SVD) is a mathematical technique used to decompose a matrix A of size $M \times N$ into three matrices: U, S and V^T. This decomposition is represented as follows [21-23]:

$$A = USV^T \quad (2)$$

where,

U is an orthogonal matrix of dimension M×M.

V is an orthogonal matrix of dimension N×N.

S is a quasi-diagonal matrix of dimension M×N containing singular values σ_i, S is given by:

$$S = \begin{pmatrix} \sigma_1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ \vdots & & \sigma_r & & & \\ & & & 0 & & \vdots \\ & & & & \ddots & \\ 0 & & & & 0 \ \cdots & 0 \end{pmatrix} \quad (3)$$

The singular values σ_i are the square roots of the eigenvalues of the matrices A^TA and AA^T, which are real and non-negative because these matrices are symmetrical and positive semi-definite.

2.3.2 Separation of an energy wave using SVD

The SVD can be used to separate an energetic wave in the presence of lower-energy contributions. This technique is applied to ultrasonic signals recorded through hardened cement plates.

The P contributions received by the ultrasonic transducer include a high-energy wave (index 1), corresponding to the first echo, (P-1) lower-energy waves (indices 2 to P) and noise. The signals received by transducer M can be written in matrix form as follows:

$$X = \underline{a_1}\begin{pmatrix} s_{11} \\ s_{22} \\ \vdots \\ s_{1n} \end{pmatrix} + \sum_{p=2}^{P} \underline{a_p}\begin{pmatrix} s_{p1} \\ s_{p2} \\ \vdots \\ s_{pn} \end{pmatrix} + N \quad (4)$$

We assume that P < (M-1). Let X_D be the matrix representing the contribution of the wave 1 (D for dominant) received by the sensor at each acquisition, and X_{LE} the matrix representing the contributions of waves 2 to P (LE for low energy) received by the sensor [22]. Then:

$$X = X_D + X_{LE} + N \quad (5)$$

Considering that the dominant contribution is perfectly synchronized to the sensor at each acquisition, and if the noise is uncorrelated with the signal, then:

$$X = \sigma_1 \underline{u_1}\ \underline{v_1^T} + \sum_{k=2}^{K} \sigma_k \underline{u_k}\ \underline{v_k^T} + \sum_{k=K}^{L} \sigma_k \underline{u_k}\ \underline{v_k^T} \quad (6)$$

where, K is the rank of the matrix X_{LE} and L = min (number of acquisitions, ΔT). Thus, the matrix X can be decomposed into three subspaces:

- The subspace $\sigma_1 \underline{u_1}\ \underline{v_1^T}$ corresponding to the dominant contribution synchronized.
- The subspace $\sum_{k=2}^{K} \sigma_k \underline{u_k}\ \underline{v_k^T}$ corresponding to the contribution of low energy.
- The subspace $\sum_{k=K}^{L} \sigma_k \underline{u_k}\ \underline{v_k^T}$ corresponding to the noise N.

The contribution corresponding to the highest wave energy can therefore be extracted by retaining only the first SVD section of the received signals.

2.3.2 Application of SVD to ultrasonic signals

In this section, singular value decomposition (SVD) is applied to ultrasonic signals transmitted through cement samples. The steps involved are illustrated using the example of signals transmitted through a cement sample hardened to a w/c ratio of 0.35.

First, the Fast Fourier Transform (FFT) of the original ultrasound signal transmitted through the sample is calculated. The FFT is used instead of the time signal, as the dispersion curves use the transmission coefficient, which is based on the relationship between the FFT of the transmitted signal and the FFT of the reference signal. Figure 3 shows the FFT of the original signal.

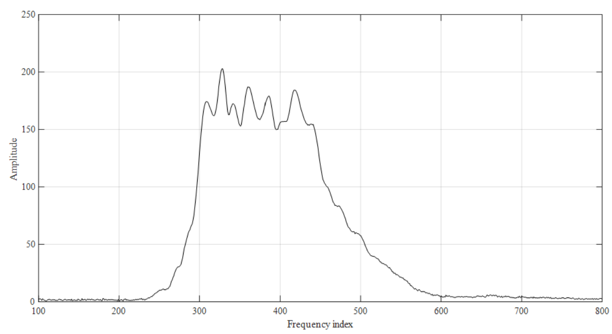


Figure 3. FFT of the original ultrasonic signal of cement with a w/c ratio of 0.35

As can be seen in Figure 4, the gap between the singular values corresponding to the most energetic contribution and the singular values corresponding to low-energy contributions is very large. This makes it easier to separate the contributions.

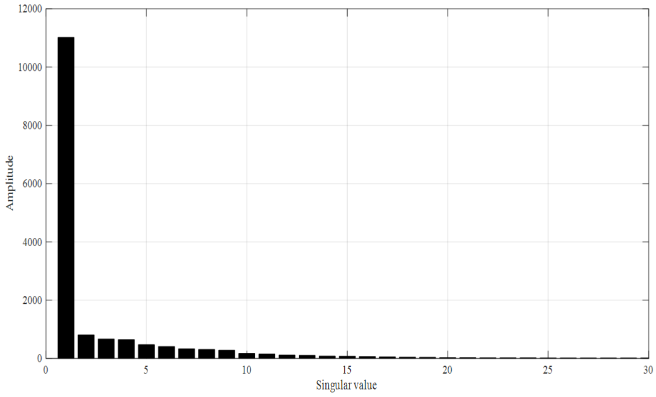


Figure 4. Histogram of singular values

Finally, the dominant energy contributions are extracted by retaining the first singular vector, representing the most energetic wave. Low-energy contributions are also identified and separated from the noise. This step isolates the high-energy wave and filters out the low-energy contributions and noise. Figures 5 and 6 show the FFT of the high-energy and low-energy waves, respectively.

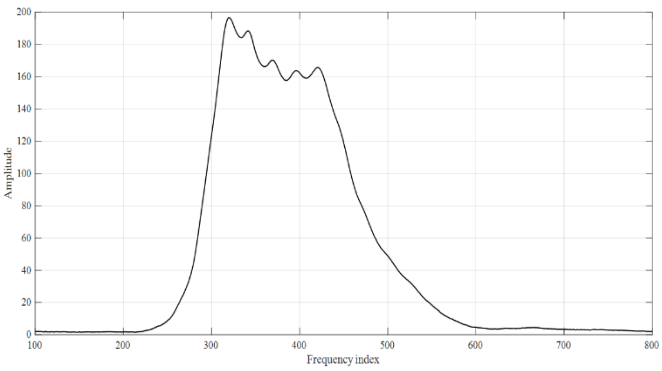


Figure 5. High-energy component after SVD

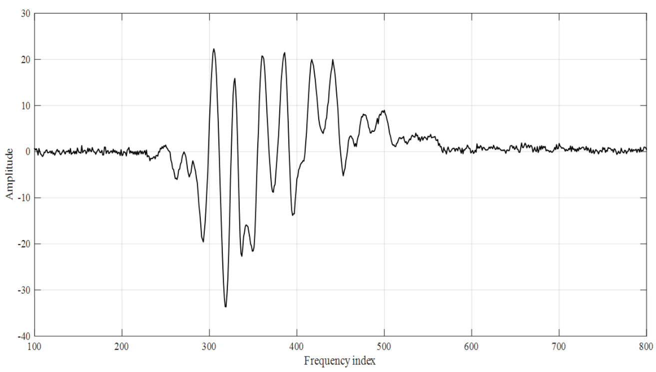


Figure 6. Low-energy component after SVD

This demonstration highlights the undeniable ability of SVD to decompose and differentiate the various energy contributions of a signal. By isolating the dominant wave and filtering out low-energy components and noise, more accurate measurements of the acoustic parameters of cementitious materials should be obtained. This approach is particularly effective in separating high-energy echoes from noise, which should improve the reliability of acoustic property measurements.

3. RESULTS AND DISCUSSION

In this section, we present the results and analyses obtained by applying singular value decomposition (SVD) to experimental dispersion curves of hardened cement plates for w/c ratios of 0.35, 0.4 and 0.5.

The SVD allows us to decompose and differentiate the various energy contributions by isolating the dominant wave and filtering out low-energy components and noise. This method has enabled us to obtain more detailed dispersion curves. The experimental dispersion curves for each w/c ratio, as well as those obtained after applying SVD, are shown in Figures 7, 8 and 9.

Upon close examination of the obtained results, significant variations in the clarity and visibility of dispersion modes can be distinguished based on the water-cement (w/c) ratio. For a w/c ratio of 0.40, it is particularly interesting to note the emergence of additional dispersion modes that are previously masked in the experimentally obtained acoustic signature. These modes, although present, are not distinct and seem to be intertwined with noise in the original image.

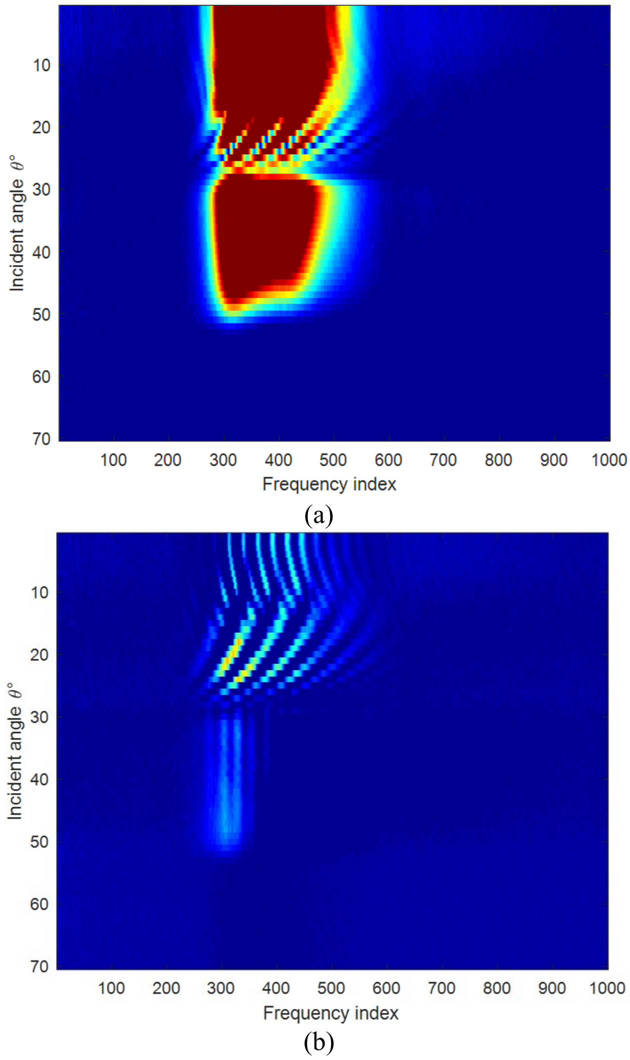
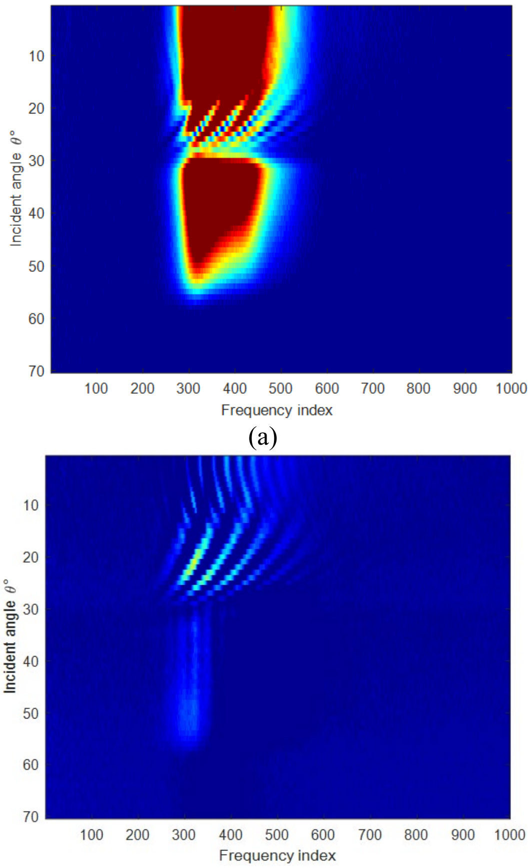


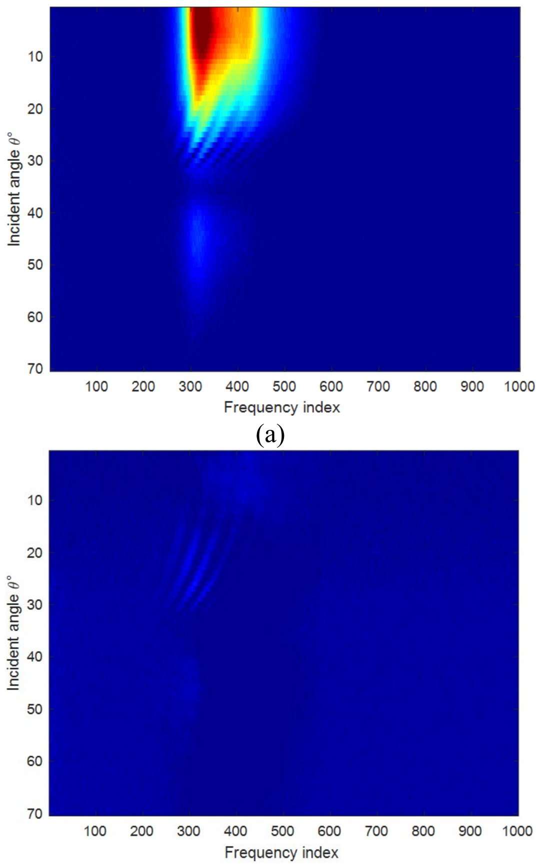
Figure 7. (a) Experimental dispersion curves and (b) curves obtained after applying SVD for w/c=0.35



(a)

(b)

Figure 8. (a) Experimental dispersion curves and (b) curves obtained after applying SVD for w/c=0.40



(a)

(b)

Figure 9. (a) Experimental dispersion curves and (b) curves obtained after applying SVD for w/c=0.50

For a w/c ratio of 0.50, the application of SVD seems to accentuate the sharpness of the modes and increase the spacing between them compared with the original acoustic signature. For a w/c ratio of 0.35, we observe that the transmitted ultrasonic signals are the clearest compared to the other ratios. The modes are distinct, well separated and show minimal interference.

The importance of these observations lies in their ability to improve the characterization of dispersion modes. The modes in the dispersion curves obtained after the application of SVD are clearer and less noisy thanks to the elimination of low energies and noise. This has enabled us to clearly visualize the convergence of modes towards a critical angle, making it easier to determine the longitudinal velocity of cement slabs using the Snell-Descartes laws given by Eq. (7) [17]:

$$V_L = \frac{V_{water}}{\sin(\theta_{cL})} \quad (7)$$

where, V_{water} is the ultrasonic velocity in water and θ_{cL} is the critical angle of the longitudinal wave in the cement slab sample.

With the effectiveness of SVD established for clarifying dispersion modes, we now turn our attention to the impact of the water/cement ratio on the longitudinal ultrasonic velocity. Figure 10 illustrates this crucial relationship for assessing the microstructure and strength of hardened cement slabs.

Figure 10 demonstrates a systematic decrease in longitudinal velocity dependent on the w/c ratio. The ultrasonic velocity decreases from 3083 m/s at a w/c ratio of 0.35 to 2743 m/s at a w/c ratio of 0.50, emphasizing the significant impact of the w/c ratio on the ultrasonic properties of cement and reflecting its microstructure and mechanical strength. An increase in porosity due to a higher w/c ratio results in reduced material density and increases interference in the propagation of ultrasonic waves, thus decreasing their speed. This suggests that higher w/c ratios may compromise the mechanical strength of cement, as lower ultrasonic speeds are typically associated with less compact and less resistant materials. The results align with previous studies that have reported a decrease in ultrasonic wave speed with an increase in the w/c ratio, confirming the adverse effect of excess water on the structural quality of cement.

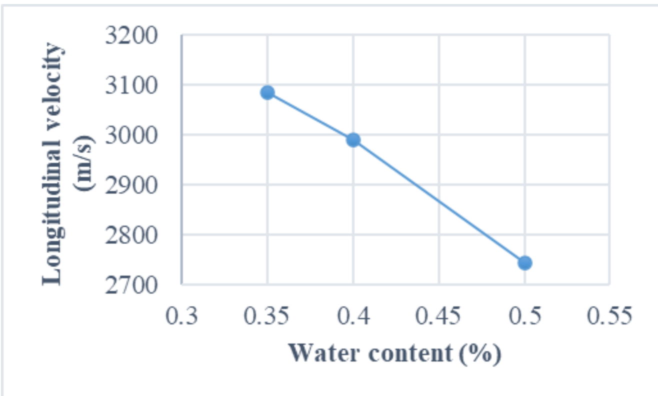


Figure 10. The evolution of ultrasonic longitudinal velocity as a function of water content

The results show that higher water/cement ratios increase porosity, leading to a reduction in density and an increase in ultrasonic wave interference. These microstructural changes directly influence the dispersion curves obtained. The formation of larger and more numerous pores, and the potential presence of microcracks, explain the reduction in the speed of ultrasonic waves and the lower mechanical strength of the material.

These findings underscore the importance of rigorous control over the w/c ratio in preparing cement mixtures, indicating that meticulous adjustments to the ratio can positively affect the durability and strength of cement structures.

4. CONCLUSIONS

This study validates the use of singular value decomposition (SVD) for the analysis of ultrasonic signals in cementitious materials, improving the accuracy and clarity of dispersion curve interpretations. By applying SVD, we obtained more distinct dispersion curves and a more accurate determination of longitudinal ultrasonic velocities, confirming the critical role of water/cement ratios in influencing cement microstructure and mechanical properties. In particular, higher water/cement ratios, which increase porosity and reduce density, lead to more pronounced interference in the propagation of ultrasonic waves, reducing their velocity and potentially compromising material strength.

These advances highlight the practical value of SVD, in particular its application to non-destructive testing and structural condition monitoring. The innovative aspect of this research lies in its ability to clarify dispersion modes and enhance the reliability of acoustic measurements, offering substantial improvements over traditional methods. This makes SVD an invaluable tool for civil engineering, as it provides critical information on material properties that are essential for ensuring the safety and longevity of infrastructures.

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