IIETA International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 11, No. 8, August, 2024, pp. 2163-2169
Journal homepage: http://iieta.org/journals/mmep

Exploring Chaotic Dynamics in a Fourth-Order Newton Method for Polynomial Root Finding

Wisam K. Ghafil[1], Ghassan A. Al-Juaifri[2*], Anas Al-Haboobi[3]

[1] Department of Mathematics, University of Thi-Qar, Nasiriyah 64001, Iraq
[2] Department of Mathematics, Faculty of Computer Science and Mathematics, University of Kufa, Kufa 54001, Iraq
[3] Department of Postgraduate Studies, University of Kufa, Kufa 54001, Iraq

Corresponding Author Email: ghassana.ali@uokufa.edu.iq

https://doi.org/10.18280/mmep.110818

Received: 23 July 2023
Revised: 10 November 2023
Accepted: 25 November 2023
Available online: 28 August 2024

Keywords:
chaos, nonlinear dynamics, bifurcation, root-finding, quartic polynomials

ABSTRACT

This paper investigates the dynamics of a fourth-order Newtonian iterative method for finding roots of polynomials of degrees three and four. Unlike traditional fourth-order methods requiring third derivatives, this technique avoids them by using the same derivative order in each of its three steps per iteration. When applied to differentiable functions, the method generates chaotic dynamics, as shown for quartic polynomials. Specifically, we apply this root-finding approach to the bifurcation diagram of the logistic map over an interval. Our findings demonstrate the potential for complex behavior even in simple iterative methods, and highlight the usefulness of this approach for exploring polynomial system dynamics. The paper identifies examples of fourth-degree polynomials, explains bifurcation, and chaos.

1. INTRODUCTION

A Cr function $h : \mathbb{R} \to \mathbb{R}$, where $r \geq 1$, can be used to solve the nonlinear equation $h(x) = 0$ using iterative methods. The iterative method starts with an initial guess x_0 and improves it using an iteration of the form $x_{n+1} = \Phi(x_n)$. The analysis of the convergence of the sequence $x_{nn} \geq 0$ to the solution x^* of the equation, as well as the order of convergence, is carried out by imposing certain conditions on both x_0 and h. The most widely used and studied method is Newton's iterative method, which is of the form $x_{n+1} = N_{h(x_n)} = x_n - \frac{h(x_n)}{h'(x_n)}$. The roots of h are fixed points of N_h. The simple roots of the polynomial h are those that are not critical points, and they exhibit quadratic convergence under Newton's method in their vicinity. The polynomial's Newton iteration can be expressed as a rational function on the $\mathbb{R}$. Third order methods in the classical sense are not practical for general use due to their high computational cost. A different approach than the traditional Newton's method for systems is presented by Rodomanov and Nesterov [1]. This refers to a method that can achieve third-order accuracy in finding solutions without using second derivatives. This method involves two iterations of Newton's method with the same derivative and requires only one LU decomposition per iteration, making it more efficient.

The primary aim of this study is to conduct a thorough and exhaustive analysis of the dynamics inherent in a discrete dynamical system. This system is characterized by a unique fourth-order approach that eliminates the reliance on second derivatives. The investigation not only revolves around uncovering the fundamental framework of the system but also delves into the intricate dynamics displayed by nonlinear systems. This encompasses a spectrum of elements, including the computation of Lyapunov exponents, prognostications about future system evolution, and insights into the underlying causal relationships between interactions within the system.

A particularly significant aspect involves delving into the presence of chaos or the susceptibility of the system to initial conditions. This area has attracted notable attention in recent academic investigations [2]. Furthermore, the study aspires to reveal the intricacies associated with fourth-degree polynomial chaos. This pivotal advancement lays the groundwork for further exploration using alternative numerical methodologies, such as the application of techniques like Halley's Method [3].

Recent research efforts [4-6] have intricately explored the importance of chaos and the sensitivity of initial conditions in scholarly discourse. Concurrently, the proposal of Halley's Method as an efficacious numerical strategy for addressing complex polynomial chaos has been put forth [7].

The amalgamation of these diverse aspects within this study significantly enriches the broader comprehension of dynamic systems and their convoluted behaviors [8].

Ángel and José [9] conducted a comprehensive study on the real dynamics of damped Newton's methods when applied to cubic polynomials. Amrein and Wihler [10] engaged in a discussion on the traditional Newton method's application for solving nonlinear operator equations in Banach spaces, contextualized within the continuous Newton method. Hurley and Martin [11] demonstrated that Newton's method for finding roots of a real function f results in chaotic dynamics, including an abundance of periodic points and positive topological entropy, across a wide class of functions. Their work elucidated how the conventional Newton's iterative method can exhibit chaotic behavior when applied to an extensive array of functions [11-13].

The paper is structured into several sections. The second section introduces the fourth-order method and the Scaling Theorem, which provides a useful change of coordinates. The Scaling Theorem states that under certain conditions, the study of the dynamics of iterations of general maps can be simplified to the study of specific families of iterations of simpler maps. This is achieved through a linear scaling transformation on the dependent variable.

The Scaling Theorem was originally proposed by Feigenbaum in 1978 [14] and further developed by Coullet and Tresser in 1978 [15]. It enables the universal quantification and classification of period-doubling cascades and routes to chaos across different dynamical systems. The theorem has been influential in the study of nonlinear and chaotic systems.

In the current paper, the Scaling Theorem is leveraged to transform the complex fourth-order polynomial system into a simplified form. This allows for an in-depth analysis of the fundamental dynamics and nonlinear behaviors exhibited.

The third section explores the dynamics of quartic polynomials, particularly in cases exhibiting chaotic traits like sensitivity to initial conditions. The application of the Scaling Theorem enables a detailed investigation into the transition to chaos and the emergence of chaotic bands and windows in the iterative mappings of the polynomial. The Lyapunov exponents across varying parameters are computed to quantify the chaotic nature.

By utilizing the Scaling Theorem transformation, the research provides novel insights into the intricate nonlinear dynamics and chaos in higher-degree polynomial systems.

2. CHAOTIC PROBLEM

For $\gamma = 1$, a function N_p is standard Newtons method used for solve a quartic Polynomial $P(x) = x^4 + 1$:

$$N_p = x - \frac{P(x)}{P'(x)} = \frac{3x^4 - 1}{4x^3} \tag{1}$$

N_p has fixed points at $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}i}{2}, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}i}{2}$. The relaxed Newtons approach, however, not converge to the roots of provided quartic since $x, \gamma \in R$. We can ask more questions regarding the repetition when considered as a dynamical system, for example: Is repetitions result a chaotic sequence? Is it possible to establish this analytically? The piecewise map $\xi(x) = 4x^3(mod\ 1)$ is shown to be topologically conjugate to N_{p1} (x) as follows:

Definition 2.1 [16] If a homeomorphism φ, which is one-to-one a continuous change of coordinates, connects two maps ν and ξ, then $\varphi \circ \nu \circ \varphi^{-1} = \xi$. This suggests that $\varphi \circ N_{p1} \circ \varphi^{-1} = 4x^3$ in the case of $N_{p1}(\, mod\ 1)$.

$$\varphi(x)=\frac{1}{4}+\frac{1}{\pi}tan^{-1}(\mathrm{x}) \tag{2}$$

$$\begin{tikzcd} R \arrow[r, "Np_1"] \arrow[d, "\xi"'] & R \arrow[d, "\xi"] \\ {[0,1]} \arrow[r, "\varphi"] & {[0,1]} \end{tikzcd}$$

The conjugacy graph that goes with it is as above.

The term "conjugacy" refers to the fact that this diagram commutes. The graphs of N1(x) and$\xi(x)$ may be found in Figure 1. Because $\varphi(x)$ is a homeomorphism: N1(x)→$\xi(x)$, there is a one-to-one connection between the dynamics of Newton's approach and a function of the interval N1(x). To Newton's approach, periodic orbits are used, for example, are translated into equivalent orbits for φ(x). As well see, this relationship is really useful. It enables us to achieve achievements even the best of circumstances occasionally with phase gaps very little effort. We take advantage of the fact that a conjugacy preserves chaos.

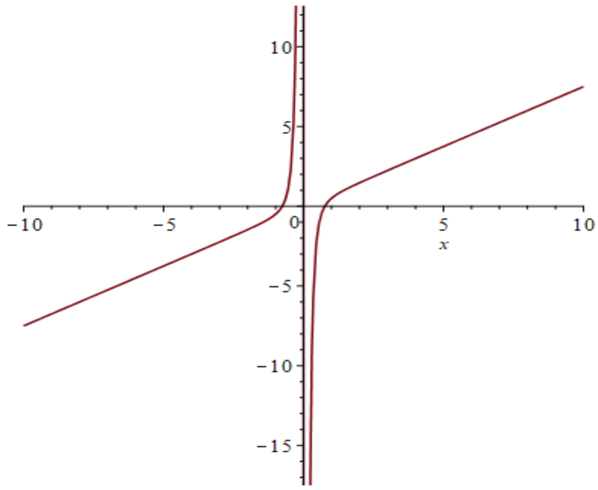

Figure 1. The graph N_1 of $P(x) = x^4 + 1$

Definition 2.2 [16] On a set, a function $p(x)$ is said to be chaotic if

1. $p\ (x)$ is transitive.
2. $p(x)$ has a delicate relationship with the starting conditions.

For example, we prove $\varphi(x) = 4x^3(mod\ 1)$ is chaotic on [0, 1]. solution 1. φ(x) is transitive on [0, 1]. The condition that there is an integer n such that, given any open sets S and D in [0, 1], φ n $(S) \cap D \neq \varphi$ is equal to the definition of transitive. This is due to the fact that ξ is growing. Because each iteration increases the size of each open set by a factor of four, it will finally cover the entire set, intersecting D in the process. The initial conditions have a significant impact on $\xi(x)$. If $\xi(x)$ is positive Lyapunov exponent then its sensitive dependency on initial conditions. The Lyapunov exponent calculates the pace at which orbits in close proximity are moving away from one other at an exponential rate. The derivative calculated along an orbit's natural logarithm is averaged to determine it

$$\Lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln|\xi'(x_i)| = \int_0^1 ln\,|\xi'|d\lambda = ln\ 12.$$

The second assertion is Because the Lebesgue measure is an ergodic, invariant measure for ξ, allowing us to apply the Birkhoff Ergodic theorem, which states that time and the averages of space are equal. Initial conditions definitely influence ln2>0, and thus ξ. Finally, conjugacy preserves the qualities of transitivity and sensitive dependence. The truth is that Lyapunov exponents for topologically conjugate maps are the same. *N1*(*x*) must also be chaotic because 4×3 (mod 1) is chaotic. We produce a family of unit interval functions, ξ beta, by making use of the same conjugacy $\varphi(x)$ to all of the relaxed

Newton's technique functions, N beta. For the families N and H, we'll follow the same steps as in the previous example. We show that the components of these families have chaotic dynamics by demonstrating that they are transitive in their according to phase spaces and that their beginning conditions are heavily contingent on them.

2.1 Devaney chaotic

To show N_γ is transitive using the same conjugacy transformation conjugate to the map $4x^3(mod\ 1)$. The $\xi\gamma$ family of maps is the end result. The picture below depicts the topological conjugacy between $N\gamma$ and $\xi\gamma$. The composite function $\xi\gamma = \varphi \circ N\gamma \circ \varphi^{-1}$ simplifies to:

$$\xi\gamma = \frac{1}{4} + \frac{1}{\pi} tan^{-1} \left(\frac{1 + \gamma + \cos(4x)}{\sin(4x)}\right) \quad (3)$$

As seen in the two cases in, On the interval [0, 1], the $\xi\gamma$ maps are piecewise monotone growth functions. We employ both N_γ and $\xi\gamma$ in the analysis that follows to take advantage of this conjugacy.

Proposition 2.2.1 The function $\xi\gamma$: [0, 1]→[0, 1] is expanding for $0<\gamma<2$.

Proof.

The derivative of ξ_γ is:

$$\xi'_\gamma(x) = \frac{4(1 + (1-\gamma)\, cos(4\pi x))}{1 + (1-\gamma)2 + 2(1-\gamma)cos(4\pi x))} \quad (4)$$

To begin with, notice that the denominator is positive:

$$\begin{gathered} 1 + (1-\gamma)^2 + 2(1-\gamma)\cos(4\pi x)) \\ = (1-\gamma + cos(4\pi x))^2 + (sin(4\pi x))^2 > 0 \end{gathered} \quad (5)$$

Now we'll show that $\xi'_\gamma>1$ for $0<\gamma<2$, a fact that comes from the following chain of inequalities:

$$\begin{gathered} 1 + (1-\gamma)^2 < 4, \\ 1 + (1-\gamma)2 + 2(1-\gamma)cos(4\pi x)) < 4 + 4(1-\gamma)cos(4\pi x), \end{gathered}$$

this leads to,

$$\xi'_\gamma(x) = \frac{4(1+(1-\gamma)\, cos(4\pi x))}{1+(1-\gamma)2+2(1-\gamma)cos(4\pi x))} > 1. \quad (6)$$

As a result, every open set will be strewn in length with each repetition, finally covering the whole set. As a result, there is an integer n such that for any two open sets H and S in [0, 1], then $\xi_{\gamma n}(H) \cap S \neq \varphi$, that is, $\xi_{\gamma n}(x)$ is transitive on [0, 1]. $N\gamma$ is also transitive as a result of the conjugacy.

One way for demonstrating sensitive dependency on initial conditions is to show that adjacent orbits diverge rapidly on a local scale. This is because the map has nonnegative Lyapunov exponent, which is preserved due to conjugacy. The Lyapunov exponent is calculated using the formula:

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} ln|N'_\gamma(x_i)| = \int_{-\infty}^{\infty} \infty\ ln|N'_\gamma| d\rho. \quad (7)$$

The invariant, we can calculate numbers that are sampled and averaged over orbits using an ergodic measure of dynamics. Its existence ensures that the average will be the same for almost all initial conditions. In order to calculate the Lyapunov exponents, first determine this measure.

The Figure 1 represents a diagram that shows the path of the solution in the first root in Newton's method and its convergence to the starting point.

Figure 2 indicates the map of the solution path when $\gamma=\frac{1}{2}$ and the instability of the solution due to the parameter γ, which leads to chaos in the solution path according to Devaney's concept.

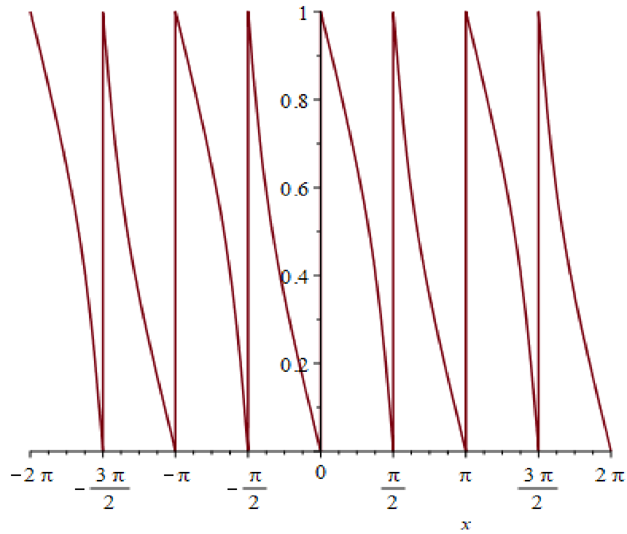

Figure 2. Graphis Bifurcation and chaos of the function $\xi_{\frac{1}{2}}$

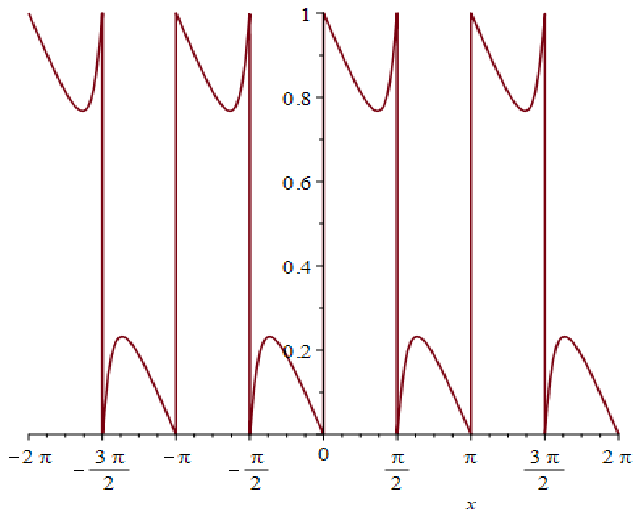

Figure 3. Graphis Bifurcation and chaos of the function $\xi_{\frac{3}{2}}$

Figure 3 indicates the map of the solution path when $\gamma=\frac{3}{2}$ and the instability of the solution due to the parameter γ, which leads to chaos in the solution path according to Devaney's concept.

3. A FOUR-ORDER NEWTON-TYPE ITERATIVE METHOD

We aim to examine the behavior of a root-finding method that is iterative and of fourth order, which is referred to as in previous studies [16-20]:

$$y_n = x_n - \frac{h(x_n)}{h'(x_n)}; x_{n+1} = y_n - \frac{h(x_n)}{h'(x_n)}. \tag{8}$$

The fourth-order iterative root-finding method is defined by the following function:

$$M_h(x) = N_n(x) - \frac{h(N_n(x))}{h'(x_n)} \tag{9}$$

where, N_h (x)=x−uh(x) and $uh(x) = \frac{h(N_n(x))}{h'(x_n)}$. In other meaning, $x_{n+1} = M_h(x_n)$. It is evident that the roots of f correspond to fixed points of M_h. A root α is considered to be simple if $h'(\alpha) \neq 0$. Super-attracting fixed points of M_h are identified as simple roots of h, as $M'h(\alpha) = 0$ and $M''h(\alpha) = 0$. If a solution β of h is not simple, it is classified as an alluring fixed point of M_h, but not super-attracting, with $0 < |M'h(\beta)| < 1$.

It is worth noting that the fixed points of M_h may exceed the roots of h and are known as extraneous fixed points. The primary goal of this paper is to investigate the behavior of the discrete dynamical system defined by M_h and determine if there is any evidence of chaos or sensitivity to initial conditions. It has been demonstrated by Hurley and Martin [11] that Newton's classical iterative method $Nh(x) = x - uh(x)$ The occurrence of chaos in a wide range of functions for the Newton's classical iterative method is attributed to the presence of points x^* where $h'(x^*) = 0$ but $h(x^*) \neq 0$. Likewise, this paper presents evidence that chaos may arise when the iterative method M_h is utilized with a specific one-parameter family of cubic polynomials, leading to the creation of a graph illustrating the bifurcation behavior of the system as a parameter is varied, specifically a period doubling bifurcation diagram. The issue of obtaining a rational map, $M_p(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials, without common factors, is encountered when applying the iterative method M_h to a polynomial. Numerator evaluation is non-zero at those points where the denominator is zero, and these points correspond to the poles of the iterative method. To investigate the dynamics of M_h using graphical analysis, it is treated as a map $\widetilde{M}h$ on [0,1], where $G: R \to [0,1]$ is a homeomorphism from R into [0,1], and $\widetilde{M}h(x) = (G \circ M_h \circ G^{-1})(x)$. We may extend $\widetilde{M}h(x)$ to maps from [0, 1] into itself, with the fixed points at $x = 0$ and $x = 1$ being repelling.

We now possess the subsequent valuable outcome:

Theorem 3.1. [8] Consider an analytic function $h(x)$ and an affine map $T(x) = \alpha x + \beta$, where $\alpha \neq 0$. Let $g(x) = (h \circ T)(x)$. Then, the affine conjugation T relates the iterates of M_h and M_g as $T \circ M_g \circ T^{-1}(x) = M_h(x)$. In other words, M_h and M_g are conjugate by T.

Proof. Let $g(x) = (h \circ T)(x)$. We want to show that $T \circ M_g \circ T^{-1}(x) = M_h(x)$.

First, note that T is an affine map, so T^{-1} is also an affine map. Therefore, $T^{-1}(x) = \frac{1}{\alpha}(x - \beta)$

Next, we have:

$$\begin{gathered} M_g\left(T^{-1}(x)\right) = T^{-1}(x) - \frac{g\left(T^{-1}(x)\right)}{g'\left(T^{-1}(x)\right)} = \frac{1}{\alpha}(x-\beta) \\ - \frac{h(\alpha^{-1}(x-\beta)+\frac{\beta}{\alpha})}{h'(\alpha^{-1}(x-\beta)+\frac{\beta}{\alpha})} = \frac{1}{\alpha}(x-\beta) - \frac{h(\frac{x}{\alpha}-\frac{\beta}{\alpha}+\frac{\beta}{\alpha})}{h'((\frac{x}{\alpha}-\frac{\beta}{\alpha}+\frac{\beta}{\alpha}))} = \frac{1}{\alpha}(x-\beta) - \\ \frac{h(\frac{x}{\alpha})}{h'(\frac{x}{\alpha})} = \alpha\left(\frac{x}{\alpha} - \frac{x}{\beta}\right) - \alpha\frac{h\left(\frac{x}{\alpha}\right)}{h'\left(\frac{x}{\alpha}\right)} + \beta = T^{-1}\left(M_h(x)\right). \end{gathered} \tag{10}$$

Thus, we have shown that $T \circ M_g \circ T^{-1}(x) = M_h(x)$, as required. This means that M_h and M_g are affine conjugated by T.

The Theorem above is also applicable when $g(x) = c(h \circ T)(x)$, where c is a non-zero scalar. Therefore, using a suitable change of coordinates, the study of the dynamics of the iterative methods M_h can be simplified by examining the dynamics of the interval map$\widetilde{M}h_s$. Furthermore, it is possible to reduce any quartic polynomial to one of the most basic cubic polynomials $h_\bullet(x) = x^4$, $h_+(x) = x^4 + x$, $h_-(x) = x^4 - x$, or to a member of the one-parameter family of cubic maps $h\gamma(x) = x^4 + \gamma x + 1$. This can be achieved by putting M_h inside the conjugacy class through an appropriate recalling.

4. QUARTIC POLYNOMIALS

Consider a quartic polynomial $h: R \to R$. By applying an affine transformation $\tau(x) = \alpha x + \beta$, we can convert h into a simpler quartic polynomial, namely $h_\bullet(x) = x^4, h_+(x) = x^4 + x$, $h_-(x) = x^4 - x$, or a one-parameter family of quartic polynomials $h\gamma(x) = x^4 + \gamma x + 1$. Consequently, we can examine the behavior of these less complex quartic polynomials, with the latter family's behavior, the value of γ has an impact on the outcome. Figure 4 shows that the function n is neither chaotic nor bifurcated.

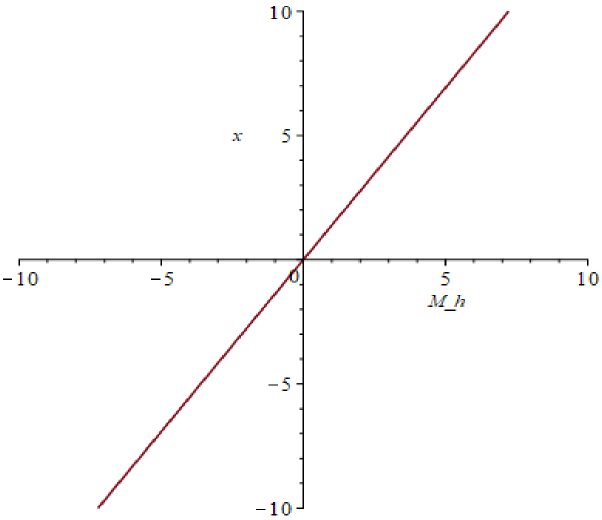

Figure 4. The graph N_1 of $h\bullet(x)$−= x^4

Case $h_\bullet(x) = x^4$

The Quartic polynomial $h_\bullet(x) = x^4$, the associated map $M_{h_\bullet} = \frac{687}{1024}$ is a linear contraction. Thus, the dynamics of this polynomial is straightforward and it is considered to be dynamically trivial. The sole fixed point located at $x = 0$ serves as the only attractor, and while it is a global attractor, it does not act as a super-attractor.

Case $h_\bullet(x)_- = x^4-1$

For $h_\bullet(x)_- = x^4 - 1$, we have that

$$M_{h_\bullet(x)-} = \frac{687x^{16}+404x^{12}-54x^8-12x^4-1}{1024x^{15}} \tag{11}$$

Let us examine the fixed points of $M_{1,h_{\bullet}(x)-}$. The roots of $h_{\bullet}(x)_- = x^4 - 1$ are $x_{1,2} = \pm 1$, which are super-attracting fixed points. Additionally, $h_{\bullet}(x)_-$ has two pairs of complex conjugate roots, namely $x_{3,4} = \pm i$. Therefore, the fixed points of $M_{1,h_{\bullet}(x)-}$ are ± 1 (with super-attracting behavior) and $\pm i$ (with repelling behavior). Since $M_{h_{\bullet}(x)-}$ it lacks real fixed points, the dynamics of the system must be chaotic, as illustrated in Figure 5. Consider the interval $I = [p_1, p_2]$, defined by the repelling fixed points $p_1 = G(x_3)$ and $p_2 = G(x_4)$ of $M\,h-$. The visualization in Figure 5 depicts the constrained behavior of $M\,h-$within the interval I.

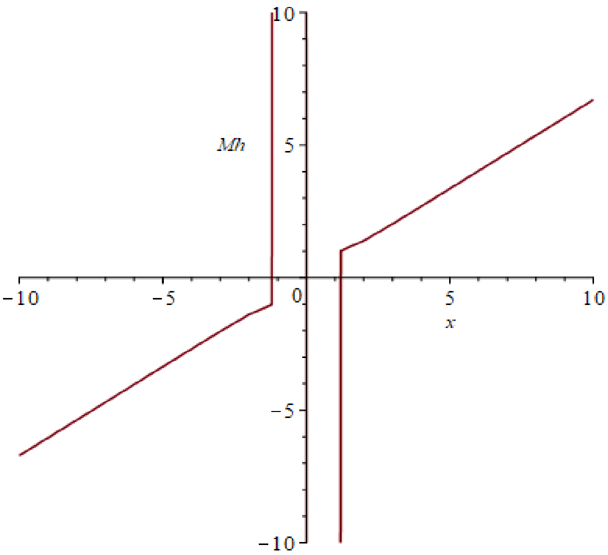

Figure 5. The graph N_1 of $h_{\bullet}(x)_- = x^4 - 1$

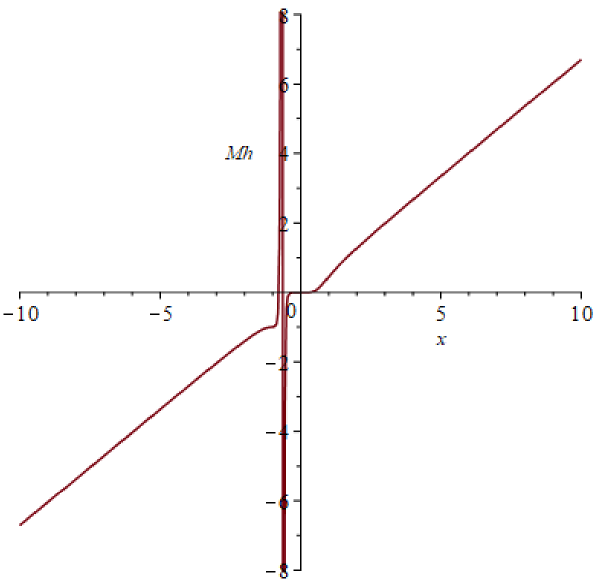

Figure 6. The graph N_1 of $h\bullet(x)_- = x^4+1$

Case $h_{\bullet}(x)_+ = x^4 + 1$

Since the polynomial $h_{\bullet}(x)_+ = x^4 + 1$ has no real roots, as a consequence, $M_{h\bullet(x)+}$ does not have any real fixed points, and its dynamics are chaotic, as evidenced in Figure 5, it becomes apparent that $M_{h\bullet(x)+}$ follows a chaotic trajectory, attributed to the fact that its solid points diverge from real values, while the initial point remains real. This combination ultimately gives rise to the observed chaotic path.

Case $h_{\bullet}(x)_+ = x^4 + x$

In this particular case, h_+ has only two real roots located at $x = 0$ and $x = -1$.

Furthermore, we have:

$$M_{h_+(x)} = \frac{3x^7(229\,x^9 + 192\,x^6 + 48\,x^3 + 4)}{(4\,x^3 + 1)^5} \qquad (12)$$

Because $h'_+(x) = 4\,x^3 + 1 > 0$ for all $x > 0$, we can conclude that h_+ has no critical points. Therefore, $M_{h_+}(x)$ is a global homeomorphism from R to itself and there are no additional or extraneous fixed points present. Therefore, the dynamics of $M_{h_+}(x)$ are straightforward: there exists a unique fixed point at $x = 0$, which acts as a global super-attracting fixed point. In Figure 6, a clearly chaotic path is shown.

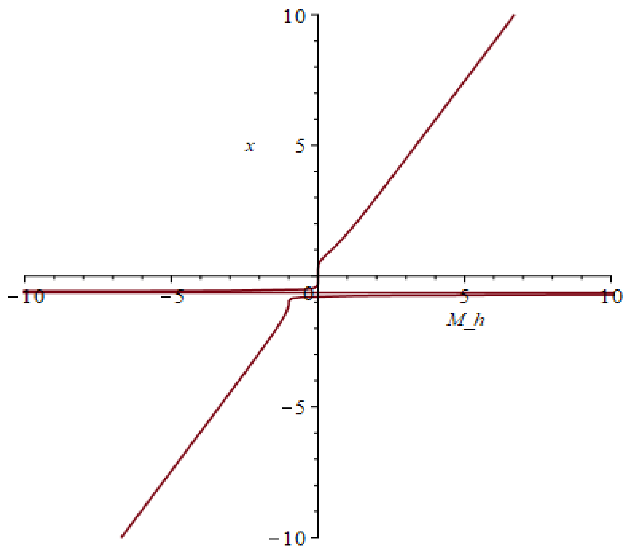

Figure 7. The graph N_1 of $h\bullet(x)_- = x^4 + x$

Case $h_{\bullet}(x)_- = x^4 - x$

The status M_{f-} defined as follows:

$$M_{h_-(x)} = \frac{3\,x^7(229\,x^9 - 192\,x^6 + 48\,x^3 - 4)}{(4\,x^3 - 1)^5} \qquad (13)$$

The roots of h, namely $x_0 = 0$ and $x_1 = 1$, are super-attracting fixed points of $M'_{h-}(x)$. In addition, there are extraneous fixed points that can be found by evaluating at the points x_j for $j = 2, \ldots, 8$. It can be observed that $|M'_{h-}(x_j)| > 1$, indicating that these fixed points are repelling.

Furthermore, $M_{h_-(x)}$ has asymptotes at $a_1 = -\frac{(\sqrt[3]{4})^2}{4}$ and $a_2 = \frac{(\sqrt[3]{4})^2}{4}$. As approaches a_1, M_{h_-} tends towards positive infinity, while as approaches a_2, M_{h_-} tends towards negative infinity. The dynamics of M_{h_-} in this scenario are straightforward as evidenced in Figure 3, it becomes apparent that $M_{h_-(x)}$ follows a chaotic trajectory, attributed to the fact that its solid points diverge from real values, while the initial point remains real. This combination ultimately gives rise to the observed chaotic path.

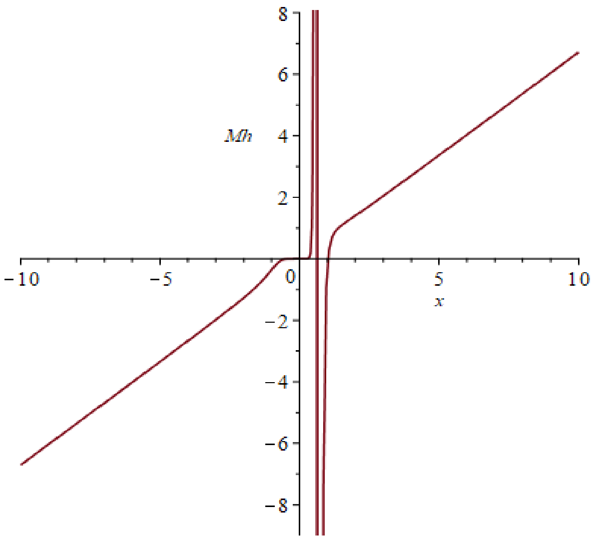

Figure 8. The graph N_1 of h•x+=$x^4 - 1$

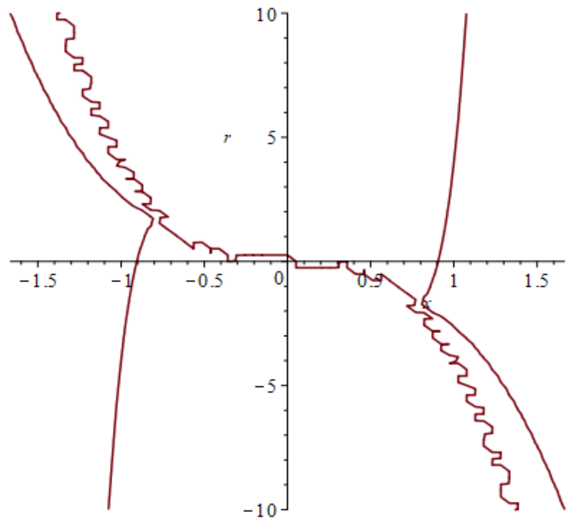

Figure 9. Graphis bifurcation and chaos of the function $M_{\gamma sn}$

Case M_h

To simplify notation, we use M_γ to refer to $M_{h\gamma}$. The analysis of the behavior of the iterative process used to find roots of a function M_γ is possible by considering its dependence on the parameter γ. The function $M_\gamma(x)$ is given by:

$$M_\gamma(x) = \frac{687x^{16} - 576\gamma x^{13} + 144\gamma^2 x^{10} - 404x^{12} + 12\gamma^3 x^7 - 448\gamma x^9}{(3x^4+\gamma)^5} + \frac{-144\gamma^2 x^6 - 54x^8 - 20\gamma^3 x^3 - \gamma^4 + 12x^4 - 1}{(3x^4+\gamma)^5} \quad (14)$$

The first derivative of $M'_\gamma(x)$ is:

$$M'_\gamma(x) = \frac{-1131x^{16} - 1152\gamma x^{13} - 288\gamma^2 x^{10} + 228x^{12} - 24\gamma^3 x^7 + 640\gamma x^9}{(3x^4+\gamma)^6} + \frac{192\gamma^2 x^6 + 27054x^8 + 24\gamma^3 x^3 + \gamma^4 - 60x^4 + 5}{(3x^4+\gamma)^6} \quad (15)$$

And the second derivative is:

$$M''_\gamma(x) = \frac{2178x^{16} + 3456\gamma x^{13} + 864\gamma^2 x^{10} + 1192x^{12} + 72\gamma^3 x^7 - 1664\gamma x^9}{(3x^4+\gamma)^7} + \frac{-480\gamma^2 x^6 - 1620x^8 - 56\gamma^3 x^3 - 2\gamma^4 + 360x^4 - 30}{(3x^4+\gamma)^7} \quad (16)$$

For $\gamma \neq 0$, let $m_\gamma = M_\gamma(0) = \frac{-\gamma^4-1}{\gamma^5}$. The critical point $x = 0$ for $\gamma > 0$, M_γ is the corresponding critical value of $M\gamma$, which acts as a local maximum because $M''_\gamma(0) = \frac{-2\gamma^4-30}{\gamma^6}$. As γ approaches zero, the local maximum m_γ becomes negative and increases. When γ exceeds a value of approximately 0.9, denoted as γ_{sn}, M_γ possesses a single fixed point $x_{sa,\gamma}$ that corresponds to the only real solution of h_γ. Furthermore, all points in R converge to the unique fixed point $x_{sa,\gamma}$ under iteration of M_γ. Thus, the dynamics of M_γ is straightforward in this instance. It is worth noting that the function $\gamma \rightarrow x_{sa,\gamma}$ is monotonically lessening as γ approaches zero.

For all $\gamma > \gamma_{sn}$, M_γ has another fixed point, indicated by $x_{sa,\gamma}$, which is a saddle-node fixed point. This means that $M''_\gamma(x_{sn}) = 1$. As γ exceeds a certain threshold γ_{sn}, the rational iterative root-finding method M_γ exhibits two fixed points: a super-attracting fixed point $x_{sa,\gamma}$ that corresponds to the sole root of h_γ, and an additional saddle-node fixed point $x_{sa,\gamma}$, which is extraneous. A visual representation of this can be seen in Figure 7. In this scenario, all iterates of each point in R under M_γ will converge to $x_{sa,\gamma}$. It is worth noting that as γ approaches zero, the function $\gamma \rightarrow x_{sa,\gamma}$ decreases. The dynamics illustrated in Figure 8 manifest distinctly as the gamma value is altered. The depiction in this Figure 9 highlights that as the gamma value approaches 0.9, the emergence of a chaotic trajectory becomes evident.

5. CONCLUSIONS

Our study delved into the behavior and evolution of a fourth-order iterative method used for estimating roots in nonlinear equations, known as their dynamics. We investigated whether chaos and bifurcations arise when applying this method to a family of fourth-degree polynomials, akin to observations in Newton's method. Our findings indicate that under specific conditions, bifurcations and chaos indeed emerge, highlighting the intricate behavior even in seemingly simple iterative approaches.

Calculating convergence regions, areas where the method aligns with nonlinear equation solutions, poses numerical challenges. Nonetheless, our study provides crucial insights into the method's behavior, including the potential for chaos and bifurcations in specific scenarios. This insight guides future numerical analysis research, aiding in optimizing starting point choices and computing stable solution convergence areas.

Given Newton's method's significance for rapid root convergence, our findings offer a fresh perspective on enhancing its efficiency through improved starting point selection. Overall, our research advances the understanding of complex behaviors in numerical methods and facilitates their effective application in practical problem-solving.

REFERENCES

[1] Rodomanov, A., Nesterov, Y. (2022). Rates of superlinear convergence for classical quasi-Newton methods. Mathematical Programming, 194: 159-190. https://doi.org/10.1007/s10107-021-01622-5

[2] Cushing, J.M. (2003). Chaos in Ecology: Experimental Nonlinear Dynamics. Elsevier, Vol. 1.

[3] Proinov, P.D., Ivanov, S.I. (2015). On the convergence of Halley's method for multiple polynomial zeros. Mediterranean Journal of Mathematics, 12: 555-572. https://doi.org/10.1007/s00009-014-0400-7

[4] Kolen, J., Pollack, J. (1990). Back propagation is sensitive to initial conditions. Advances in Neural Information Processing Systems, 3.

[5] Glasner, E., Weiss, B. (1993). Sensitive dependence on initial conditions. Nonlinearity, 6(6): 1067. https://doi.org/10.1088/0951-7715/6/6/014

[6] Banks, J., Brooks, J., Cairns, G., Davis, G., Stacey, P. (1992). On Devaney's definition of chaos. The American Mathematical Monthly, 99(4): 332-334. https://doi.org/10.1080/00029890.1992.11995856

[7] Pickover, C.A. (1988). A note on chaos and Halley's method. Communications of the ACM, 31(11): 1326-1329. https://doi.org/10.1145/50087.50093

[8] Guckenheimer, J., Holmes, P. (2013). Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer Science & Business Media, vol. 42.

[9] Ángel, A.M., José, M.G. (2015). Real dynamics for damped Newton's method applied to cubic polynomials. Journal of Computational and Applied Mathematics, (275): 527-538. https://doi.org/10.1016/j.cam.2013.11.019

[10] Amrein, M., Wihler, T.P. (2014). An adaptive Newton-method based on a dynamical systems approach. Communications in Nonlinear Science and Numerical Simulation, 19(9): 2958-2973. https://doi.org/10.1016/j.cnsns.2014.02.010

[11] Hurley, M., Martin, C. (1984). Newton's algorithm and chaotic dynamical systems. SIAM Journal on Mathematical Analysis, 15(2): 238-252. https://doi.org/10.1137/0515020

[12] Saari, D.G., Urenko, J.B. (1984). Newton's method, circle maps, and chaotic motion. The American Mathematical Monthly, 91(1): 3-17. https://doi.org/10.1080/00029890.1984.11971328

[13] Hurley, M. (1986). Attracting orbits in newtons method. Transactions of the American Mathematical Society, 297(1): 143-158. https://doi.org/10.1090/S0002-9947-1986-0849472-6

[14] Feigenbaum, M.J. (1978). Quantitative universality for a class of nonlinear transformations. Journal of Statistical Physics, 19(1): 25-52. https://doi.org/10.1007/BF01020332

[15] Coullet, P., Tresser, C. (1978). Iterations d'endomorphismes et groupe de renormalisation. Le Journal de Physique Colloques, 39(C5): C5-25. https://doi.org/10.1051/jphyscol:1978513

[16] Wang, Z., Ahmadi, A., Tian, H.G., Jafari, S., Chen, G.R. (2023). Lower-dimensional simple chaotic systems with spectacular features. Chaos, Solitons & Fractals, 169: 113299. https://doi.org/10.1016/j.chaos.2023.113299

[17] Kurka, P. (2003). Topological and symbolic dynamics. Société mathématique de France.

[18] Motsa, S.S., Dlamini, P., Khumalo, M. (2013). A new multistage spectral relaxation method for solving chaotic initial value systems. Nonlinear Dynamics, 72: 265-283. https://doi.org/10.1007/s11071-012-0712-8

[19] Amat, S., Busquier, S. (2007). Third-order iterative methods under Kantorovich conditions. Journal of Mathematical Analysis and Applications, 336(1): 243-261. https://doi.org/10.1016/j.jmaa.2007.02.052

[20] Hasan, M.S., Mondal, R.N., Lorenzini, G. (2019). Centrifugal instability with convective heat transfer through a tightly coiled square duct. Mathematical Modelling of Engineering Problems, 6(3): 397-408. https://doi.org/10.18280/mmep.060311