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Development and Analysis of Embedded Explicit Runge-Kutta Methods for Directly Solving Special Fifth-Order Quasi-Linear Ordinary Differential Equations

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ABSTRACT

This paper introduces two pairs of numerical embedded methods of Runge-Kutta (RK) type, known as ERKMF methods, for directly solving a class of quasi-linear ordinary differential equations (ODEs) of 5th-order with the form $\varphi^{(5)}(\tau) = \Psi(\tau, \varphi(\tau))$. The first pair, ERKMF6(5), is a 5th-order embedded in 6th-order method that satisfies the condition first the same as last (FSAL), and the vector output is represented by the coefficient matrix's final row. The second pair, ERKMF7(6), is also an embedded method. The paper then applies these methods to solve specific 5th-order problems using variable step-size codes and compares the results to those obtained using an existing embedded RK method. The ERKMF methods have shown to be advantageous and effective compared to existing methods, as demonstrated by the numerical findings. In conclusion, these new pairs of ERKMF methods provide a promising approach to directly solve quasi-linear ODEs of 5th-order and could have significant implications for the field of numerical methods for differential equations.

1. INTRODUCTION

Differential equations (DEs) of higher order, whether partial or ordinary, are commonly used in the mathematical modelling of real-world issues in fields such as economics, physics, and engineering [1]. For instance, the fifth-order Korteweg-de Vries (KdV) equation has been applied to various physical phenomena, including capillary-gravity water waves, linked oscillator chains, and magneto acoustic waves in plasma. Another variation of the KdV equation, the Gardner-Kawahara equation, represents weakly nonlinear long internal waves at the interface between two thin layers of different densities [1-3].

While numerical methods have been developed for solving fifth-order ODEs, they often suffer from limitations. Some authors have constructed numerical methods of the Runge-Kutta (RK) type for solving general or special classes of fifth-order ODEs [2]. In contrast, others have developed implicit-block numerical methods for solving general classes of fifth-order ODEs [3]. However, these methods are not always suitable for solving the class of quasi-linear ODEs of the fifth order [4, 5].

To address this problem, we propose embedded explicit ERKMF methods for directly solving quasi-linear ODEs of the fifth order. These methods have several advantages, including their ability to solve higher-order problems directly without reducing them to lower order and their ability to provide an estimate of the local error [6]. Our proposed methods are designed to be effective and efficient in solving fifth-order ODEs, and we compare their performance to existing methods in numerical experiments.

In conclusion, this paper presents a new approach for directly solving a class of quasi-linear ODEs of the fifth order using embedded explicit ERKMF methods. Our proposed methods provide a promising solution to the challenges associated with solving fifth-order ODEs and could have significant implications for numerical methods for differential equations.

2. PRELIMINARY

In this study, we propose a method for directly solving a class of quasi-linear fifth-order ordinary differential equations (ODEs) of the form:

$$\varphi^{(5)}(\tau) = \Psi\big(\tau, \varphi(\tau)\big), a \leq \tau \leq b \tag{1}$$

With initial conditions (ICs)

$$\varphi^{(i)}(\tau) = \zeta i, \tag{2}$$

For $i = 0, 1..., 4$, where, $\Psi: \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$; $\varphi(\tau) = [\varphi_1(\tau), \varphi_2(\tau), \dots, \varphi_N(\tau)]$

For $i = 0, 1..., 4$.

In the past, academics, engineers, and scientists used to solve Eq. (1) by converting a system of first-order ODEs with five additional dimensions into a system of fifth-order ODEs. However, it has been found that using numerical methods for solving the problem quickly and accurately is more efficient

[7-12].

When employing the multi-step approaches mentioned earlier to solve the ODEs in Eq. (1), initial values are necessary [13, 14]. This work aims to directly solve specific fifth-order ODE problems using a one-step technique. The general version of the ERKMF approach with m-stage is used to address the initial value problems (IVPs) in Eq. (1) as follows:

$$\varphi_{n+1} = \varphi_n + \hbar\varphi_n' + \frac{\hbar^2}{2}\varphi_n'' + \frac{\hbar^3}{6}\varphi_n''' + \frac{\hbar^4}{24}\varphi_n'''' + \hbar^5 \sum_{i=1}^{m} \tilde{b}_i\, k_{i,} \tag{3}$$

$$\varphi'_{n+1} = \varphi_n' + \hbar\varphi_n'' + \frac{\hbar^2}{2}\varphi_n''' + \frac{\hbar^3}{6}\varphi_n'''' + \hbar^4 \sum_{i=1}^{m} \tilde{b}_i'\, k_{i,} \tag{4}$$

$$\varphi''_{n+1} = \varphi_n'' + \hbar\varphi_n''' + \frac{\hbar^2}{2}\emptyset_n'''' + \hbar^3 \sum_{i=1}^{m} \tilde{b}_{\mathrm{i}}''\, k_{\mathrm{i},} \tag{5}$$

$$\varphi'''_{n+1} = \varphi_n''' + \hbar\varphi_n'''' + \hbar^2 \sum_{i=1}^{m} \tilde{b}_{\mathrm{i}}'''\, k_{\mathrm{i},} \tag{6}$$

$$\varphi''''_{n+1} = \varphi_n'''' + \hbar \sum_{i=1}^{m} \tilde{b}_i''''\, k_{i,} \tag{7}$$

where,

$$\mathrm{k}_1 = \Psi(\tau_{\mathrm{n}}, \varphi_n) \tag{8}$$

$$k_i = \Psi\left(\tau_{\mathrm{n}} + c_i\hbar, \varphi_n + c_i\hbar\varphi_n' + \frac{\hbar^2}{2}c_i^2\varphi_n'' + \frac{\hbar^3}{6}c_i^3\varphi_n''' + \frac{\hbar^4}{24}c_i^4\varphi_n'''' + \hbar^5 \sum_{j=1}^{m} a_{ij}k_{j,}\right) \tag{9}$$

For i=2, 3…, m [4].

The ERKMF method is an explicit or implicit numerical method that involves a large number of numerical or algebraic computations, most of which were done using Maple software as conducted by Gande and Gruntz [15]. It uses Taylor's series expansion [16-19] to obtain the coefficients of the method specified in Eqs. (2)-(9). We are particularly interested in the derivation of embedded pairs of order $p(q)$ of explicit ERKMF methods, where p is higher than or = q + 1 which offers a low-cost error estimation.

To develop the embedded pairs of explicit ERKMF algorithms, we use a Butcher Tableau that specifies the values of the parameters $(c, A, b, b', b'', b''', b'''')$. These methods provide an approximation of the solution and an estimate of the local error.

In Butcher Tableau, the embedded pair can be stated as below.

$$\begin{array}{c} C \\ A\!: \tilde{b}^{\,T}; \tilde{b}'^{\,T}; \tilde{b}''^{\,T}; \tilde{b}'''^{\,T}; \tilde{b}''''^{\,T} \\ \tilde{b}^{\,T}; \tilde{b}'^{\,T}; \tilde{b}''^{\,T}; \tilde{b}'''^{\,T}; \tilde{b}''''^{\,T} \end{array}$$

Overall, our proposed approach offers a promising solution to the challenges associated with solving quasi-linear fifth-order ODEs and could have significant implications for numerical methods for differential equations. The following part will provide the derivation of the method.

3. CONSTRUCTION OF PROPOSED METHOD

The order conditions (OCs) of ERKMF integrators for solving fifth-order (ODEs) have been derived by Mechee and Kadhim [4], we will present the algebraic (OCs) up to seventh order as follows:

Order conditions for φ:

$$\sum \widetilde{b}_1 = \frac{1}{120}, \sum \widetilde{b}_1 c_1 = \frac{1}{720}, \sum \widetilde{b}_1 c_1^2 = \frac{1}{2520}, \sum \widetilde{b}_1 c_1^3 = \frac{1}{840}. \tag{10}$$

Order conditions for φ':

$$\sum \tilde{b}_i' = \frac{1}{24}, \sum \tilde{b}_i'\, c_i = \frac{1}{120}, \sum \tilde{b}_i' c_i^2 = \frac{1}{360}, \sum \tilde{b}_i' c_i^3 = \frac{1}{840}, \sum \tilde{b}_i' a_{ij} = \frac{1}{5040}, \sum \tilde{b}_i' c_i^4 = \frac{1}{120}. \tag{11}$$

Order conditions for φ'':

$$\sum \tilde{b}_i'' = \frac{1}{6}, \sum \tilde{b}_i''\, c_i = \frac{1}{24}, \sum \tilde{b}_i'' c_i^2 = \frac{1}{60}, \sum \tilde{b}_i'' c_i^3 = \frac{1}{120}, \quad \sum \tilde{b}_i'' a_{ij} = \frac{1}{720}, \sum \tilde{b}_i'' c_i^4 = \frac{1}{210}, \sum \tilde{b}_i'' a_{ij}c_j = \frac{1}{5040}. \tag{12}$$

Order conditions for φ''':

$$\sum \tilde{b}_i''' = \frac{1}{2}, \sum \tilde{b}_i'''\, c_i = \frac{1}{6}, \sum \tilde{b}_i''' c_i^2 = \frac{1}{12}, \sum \tilde{b}_i''' c_i^3 = \frac{1}{20}, \sum \tilde{b}_i''' c_i^4 = \frac{1}{30}, \sum \tilde{b}_i''' a_{ij}c_i = \frac{1}{5040}, \sum \tilde{b}_i''' c_i^5 = \frac{1}{42}. \tag{13}$$

Order conditions for φ'''':

$$\sum \tilde{b}_i'''' = 1, \sum \tilde{b}_i''''\, c_i = \frac{1}{2}, \sum \tilde{b}_i'''' c_i^2 = \frac{1}{3}, \sum \tilde{b}_i''' c_i^3 = \frac{1}{4}, \sum \tilde{b}_i''' c_i^4 = \frac{1}{5}, \sum \tilde{b}_i''' c_i^5 = \frac{1}{6}, \sum \tilde{b}_i''' c_i^5 = \frac{1}{42}, \sum \tilde{b}_1''' a_{ij} = \frac{1}{740}, \sum \tilde{b}_i''' a_{ij}c_i = \frac{1}{5040}, \sum \tilde{b}_i''' c_i a_{ij} = \frac{1}{840}, \sum \tilde{b}_i''' c_i^6 = \frac{1}{7}. \tag{14}$$

In order to create effective pairs, the following techniques are used.

(a) For higher and lower order ERKMF formula, respectively, the quantities of $\Vert \mathcal{O}_g^{(p+1)} \Vert_2$ and $\Vert \mathcal{O}_g^{(p+1)} \Vert_2$ should be as minimal as possible.

$$\left\Vert \mathcal{O}^{(p+1)} \right\Vert_2 = \left(\sum_{i=1}^{n_1} (\mathcal{O}_i^{(p+1)})^2 + \sum_{i=1}^{n_2} (\mathcal{O}_i'^{(p+1)})^2 + \sum_{i=1}^{n_3} (\mathcal{O}_i''^{(p+1)})^2\right)^{\frac{1}{2}}$$

$$\left\Vert \hat{\mathcal{O}}^{(p+1)} \right\Vert_2 = \left(\sum_{i=1}^{n_1} (\hat{\mathcal{O}}_i'^{(p+1)})^2 + \sum_{i=1}^{n_2} (\hat{\mathcal{O}}_i''^{(p+1)})^2 + \sum_{i=1}^{n_3} (\hat{\mathcal{O}}_i'''^{(p+1)})^2\right)^{\frac{1}{2}} \tag{15}$$

where, $\mathcal{O}^{(p+1)}, \mathcal{O}'^{(p+1)}, \mathcal{O}''^{(p+1)}, \mathcal{O}'''^{(p+1)}$ and $\mathcal{O}''''^{(p+1)}$ are called error terms for $\varphi, \varphi', \varphi'', \varphi'''$ and φ'''' respectively.

(b) The following formula yields an estimate of the local error at the position τ_{n+1} : $(LTE) = max\{\Vert \eta_{n+1} \Vert_\infty, \Vert \eta_{n+1}' \Vert_\infty, \Vert \eta_{n+1}'' \Vert_\infty, \Vert \eta_{n+1}''' \Vert_\infty, \Vert \eta_{n+1}'''' \Vert_\infty\}$, where, $\eta_{n+1} = \hat{\mathcal{O}}_{n+1} - \mathcal{O}_{n+1}$, $\eta_{n+1}' = \hat{\mathcal{O}}_{n+1}' - \mathcal{O}_{n+1}'$, $\eta_{n+1}'' = \hat{\mathcal{O}}_{n+1}'' - \mathcal{O}_{n+1}''$, $\eta_{n+1}''' = \hat{\mathcal{O}}_{n+1}''' - \mathcal{O}_{n+1}'''$, $\eta_{n+1}'''' = \hat{\mathcal{O}}_{n+1}'''' - \mathcal{O}_{n+1}''''$, where, $\mathcal{O}_{n+1}$, $\mathcal{O}_{n+1}'$ $\mathcal{O}_{n+1}''$ $\mathcal{O}_{n+1}'''$ and $\hat{\mathcal{O}}_{n+1}$, $\hat{\mathcal{O}}_{n+1}'$, $\hat{\mathcal{O}}_{n+1}''$, $\hat{\mathcal{O}}_{n+1}'''$ and $\hat{\mathcal{O}}_{n+1}''''$ using the higher order and lower order formulas respectively. This local error estimation, LTE can be used to control the step size h by the standard formula as given in the study [10],

$$h_{n+1} = 0.9h_n\left(\frac{Tol}{LTE}\right)^{\frac{1}{q+1}} \tag{16}$$

where, 0.9 is a safety factor indicating the local error estimation at each step, *Tol* is the necessary accuracy, which is the maximum allowed local error, and *Tol* is the necessary accuracy. If *LTE*≤*Tol*, the step is accepted, and we accept the higher order mode (or local extrapolation) technique, which calls for applying the more accurate approximation to advance the integration and updating h using Eq. (16). If *LTE*≤*Tol*, the step is rejected, and the h step size is reduced by half.

3.1 Derivation of proposed pair ERKMF6(5) method

Based on the RKM6 coefficients, which stand for the four-stage sixth-order approach [4]. Then, we'll concentrate on the ERKMF6(5) method's derivation, which has the first identified as the last (FSAL) feature. Then, a singular solution is obtained and is given as follows after solving the algebraic equations of the set of (OCs) up to sixth order with FSAL-conditions, which involved 15 equations and 20 variables:

$$\begin{gathered}
\tilde{b}'_1 = \frac{1}{40} + \frac{\tilde{b}'_3\sqrt{15}}{5},\\
\tilde{b}'_2 = \frac{\tilde{b}'_3\sqrt{15}}{5} + \frac{\tilde{b}'_4\sqrt{15}}{5} + \frac{1}{60} - \tilde{b}'_3 - \tilde{b}'_4,\\
\tilde{b}'_3 = \tilde{b}'_3,\ \tilde{b}'_4 = \tilde{b}'_4,\\
b_1 = \frac{1}{120} - \tilde{b}_2 - \tilde{b}_3 - \tilde{b}_4,\ \tilde{b}_2 = \tilde{b}_2,\\
\tilde{b}_3 = \tilde{b}_3,\ \tilde{b}_4 = \tilde{b}_4,\ \tilde{b}'''_1 = 0,\ \tilde{b}'''_1 = \frac{2}{9},\\
\tilde{b}'''_3 = \frac{-\sqrt{15}}{36} + \frac{5}{36},\ \tilde{b}'''_3 = \frac{5}{36} + \frac{\sqrt{15}}{36},\\
\tilde{b}''_1 = \frac{1}{24} + \frac{\sqrt{15}}{120} + \frac{3\,\tilde{b}'''_4\sqrt{15}}{5} - 3\tilde{b}''_4,\\
\tilde{b}''_2 = \frac{\sqrt{15}}{180} + \frac{2\,\tilde{b}''_4\sqrt{15}}{5} - 2\tilde{b}''_4 + \frac{1}{12},\\
\tilde{b}''_3 = \frac{1}{24} + 4\,\tilde{b}''_4 - \tilde{b}''_4\sqrt{15} - \frac{\sqrt{15}}{72}, \qquad \tilde{b}''_4 = \tilde{b}''_4,\\
\tilde{b}'''_1 = 0,\ \tilde{b}'''_2 = \frac{4}{9},\ \tilde{b}'''_3 = \frac{5}{18},\ \tilde{b}'''_4 = \frac{5}{18}.
\end{gathered} \tag{17}$$

$$\begin{aligned}
\left\|\varphi_g^{(6)}\right\|_2 = \Bigg[&\left(\frac{1}{2}\,\tilde{b}_2 + \tilde{b}_3\left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right) + \tilde{b}_4\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right) - \frac{1}{720}\right)^2 + \left(-\frac{\sqrt{15}}{20}\,\tilde{b}'_4 + \frac{13}{5040} - \frac{1}{4}\,\tilde{b}'_3 - \frac{1}{4}\,\tilde{b}'_4 + \right.\\
&\left.\tilde{b}'_3\left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)^2 + \tilde{b}'_4\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)^2\right)^2 + \left(\frac{\sqrt{15}}{1440} + \frac{\sqrt{15}}{20}\,\tilde{b}''_4 - \frac{1}{4}\,\tilde{b}''_4 + \frac{1}{480} + \left(\frac{1}{24} + 4\,\tilde{b}''_4 - \tilde{b}''_4\sqrt{15} - \right.\right.\\
&\left.\left.\frac{\sqrt{15}}{72}\right)\left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)^3 + \tilde{b}''_4\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)^3\right)^2 + \left(-\frac{7}{360} + \left(-\frac{\sqrt{15}}{36} + \frac{5}{36}\right)\left(\frac{1}{2} + \frac{\sqrt{15}}{10}\right)^4 + \left(\frac{5}{36} + \frac{\sqrt{15}}{36}\right)\left(\frac{1}{2} + \right.\right.\\
&\left.\left.\frac{\sqrt{15}}{10}\right)^4\right)^2 + \left(-\frac{11}{72} + \frac{5\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)^5}{18} + \frac{5\left(\frac{1}{2} - \frac{\sqrt{15}}{10}\right)^5}{18}\right)^2\Bigg]^{\frac{1}{2}}
\end{aligned} \tag{18}$$

Using minimizing commands in the software of Maple, yields the following results:

$\tilde{b}'_3 = 0.135040979817489$, $\tilde{b}'_4 = 1.12226849616564$, $\tilde{b}_2 = 0.281203473883770$, $\tilde{b}_3 = -0.264980891010455$, $\tilde{b}_4 = 0.850956859431792$ and $\tilde{b}''_4 = 0.109346990692354$ and the minimum-value $\left\|\varphi_g^{(6)}\right\|_2$ is 0.01651670981.

Moreover, for the optimized these values, we choose: $\tilde{b}'_3 = \frac{1}{10}$, $\tilde{b}'_4 = \frac{561}{500}$, $-\tilde{b}_2 = \tilde{b}_3 = -0.5\tilde{b}_4 = -\frac{1}{5}$, $\tilde{b}''_2 = \frac{1}{10}$.

The four-stage embedded ERKMF technique coefficients can be stated as follows in Table 1 for all coefficients.

Table 1. Embedded pair ERKMF6(5) method

0	0			
$\frac{1}{2}$	$\frac{1}{2}$	0		
$\frac{1}{2}+\frac{\sqrt{15}}{10}$	0	$\frac{1}{2}$	0	
$\frac{1}{2}-\frac{\sqrt{15}}{10}$	0	$\frac{1}{2}$	$-\frac{350}{200}$	0
	$\frac{1}{180}$	$-\frac{359}{360}$	$\frac{1}{2}$	$\frac{1}{2}$
	$-\frac{67}{4}-\frac{109\sqrt{15}}{360}$	$-\frac{59}{60}-\frac{109\sqrt{450}}{360}$	$\frac{1}{2}$	$\frac{145}{72}+\frac{109\sqrt{15}}{216}$
	0	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{15}}{72}$	$\frac{1}{18}+\frac{\sqrt{15}}{72}$
	0	$\frac{2}{9}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$	$\frac{5}{36}+\frac{\sqrt{15}}{36}$
	0	$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$
	$-\frac{19}{24}$	$\frac{1}{5}$	$-\frac{1}{25}$	$\frac{4}{5}$
	$\frac{1}{40}-\frac{7\sqrt{15}}{50}$	$-\frac{53}{60}+\frac{7\sqrt{15}}{50}$	$\frac{1}{10}$	$\frac{4}{5}$
	$-\frac{31}{120}+\frac{41\sqrt{15}}{600}$	$-\frac{7}{60}+\frac{41\sqrt{15}}{900}$	$\frac{53}{120}-\frac{41\sqrt{15}}{360}$	$\frac{1}{10}$
	0	$\frac{2}{9}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$
	0	$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$

3.2 Derivation of proposed pair ERKMF7(6) method

Based on the RKM7 coefficients, which stand for the five-stage seventh-order approach [4]. Then, we'll concentrate on the ERKMF method's ERKMF7(6) derivation, which has the first identical as the last (FSAL) condition. Then, a singular solution is obtained and is given as follows after solving the equations of order conditions up to order seven with FSAL conditions, which involved 20 equations and 25 variables:

$$\begin{aligned}
\tilde{b}'_1 &= \frac{89}{2520} - \frac{220\,\tilde{b}'_4\sqrt{10434}}{14161} + \frac{220\,\tilde{b}'_5\sqrt{10434}}{14161} + \frac{8373\,\tilde{b}'_4}{14161} + \frac{8373\,\tilde{b}'_5}{14161},\\
\tilde{b}'_2 &= \frac{339\,\tilde{b}'_4\sqrt{10434}}{28322} - \frac{339\,\tilde{b}'_5\sqrt{10434}}{28322} + \frac{29}{2520} - \frac{17812\,\tilde{b}'_4}{14161},\\
\tilde{b}'_3 &= \frac{4722\,\tilde{b}'_4}{14161} - \frac{4722\,\tilde{b}'_5}{14161} + \frac{101\,\tilde{b}'_4\sqrt{10434}}{28322} - \frac{101\,\tilde{b}'_5\sqrt{10434}}{28322} - \frac{13}{2520},\\
\tilde{b}'_4 &= \tilde{b}'_4,\ \tilde{b}'_5, \tilde{b}'_1 = \frac{1}{180} - \frac{\tilde{b}_4\sqrt{10434}}{119} + \frac{\tilde{b}_5\sqrt{10434}}{119} - 2\tilde{b}_3 - \frac{9\tilde{b}_4}{119} - \frac{9\tilde{b}_5}{119},\\
\tilde{b}_2 &= \frac{\tilde{b}_4\sqrt{10434}}{119} - \frac{\tilde{b}_5\sqrt{10434}}{119} + \frac{1}{360} + \tilde{b}_3 - \frac{110\tilde{b}_4}{119} - \frac{110\tilde{b}_5}{119},\\
\tilde{b}_3 &= \tilde{b}_3, \tilde{b}_4 = \tilde{b}_4, \tilde{b}_5 = \tilde{b}_5\,,\\
\tilde{b}'''_1 &= \frac{17}{35}\,, \tilde{b}'''_2 = \frac{307}{1305}, \tilde{b}'''_3 = \frac{3}{1765},\\
\tilde{b}'''_4 &= \frac{18942137\sqrt{10434}}{5607675045} + \frac{4829021}{12898620},\\
\tilde{b}'''_5 &= \frac{18942137\sqrt{10434}}{5607675045} + \frac{4829021}{12898620},\\
\tilde{b}''_1 &= -\frac{13}{420} - \frac{3216510\,\tilde{b}''_5\sqrt{10434}}{11796113} - \frac{339866682\,\tilde{b}''_5}{11796113} - \frac{\sqrt{10434}}{840},\\
\tilde{b}''_2 &= -\frac{1072170\,\tilde{b}''_5\sqrt{10434}}{48869611} - \frac{113288894\,\tilde{b}''_5\sqrt{10434}}{48869611} - \frac{\sqrt{10434}}{10440},
\end{aligned}$$

$$\tilde{b}_3'' = \frac{\sqrt{10434}}{42360} + \frac{339866682\,\tilde{b}_5''}{594861127} + \frac{3216510\,\tilde{b}_5''\sqrt{10434}}{594861127} - \frac{31}{10590},$$
$$\tilde{b}_4'' = \frac{32573\sqrt{10434}}{25797240} + \frac{3569402335481\,\tilde{b}_5''}{120756808781} + \frac{34923793410\,\tilde{b}_5''\sqrt{10434}}{120756808781} - \frac{35739}{286636}$$
$$\boldsymbol{\tilde{b}_5'' = \tilde{b}_5'', \tilde{b}_1'''' = -\frac{17}{35}, \tilde{b}_2'''' = \frac{614}{1305}, \quad \tilde{b}_3'''' = \frac{2}{1765},}$$
$$\boldsymbol{\tilde{b}_4'''' = \frac{1635034}{3224655} + \frac{3468551\sqrt{10434}}{1495380012},}$$
$$\tilde{b}_5'''' = \frac{1635034}{3224655} - \frac{3468551\sqrt{10434}}{1495380012}.$$

Embedded pair ERKMF7(6) method is shown as below:

$$a_{21} = \frac{1}{2}, \ a_{31} = -\frac{1}{2}, \ a_{41} = \frac{1}{2}, \ a_{42} = -\frac{1}{2},$$
$$a_{43} = \frac{1}{2}, \ a_{51} = \frac{1}{2}, \ a_{52} = -\frac{1}{2}, \ a_{53} = \frac{1}{2},$$
$$a_{54} = \frac{7167}{14336}, \ c_2 = \frac{1}{2}, \ c_3 = -\frac{1}{2}, \ c_4 = \frac{55}{119} - \frac{\sqrt{10434}}{238},$$
$$c_5 = \frac{55}{119} + \frac{\sqrt{10434}}{238}, \tilde{b}_1 = \frac{17}{840}, \ \tilde{b}_2 = \frac{307}{250560},$$
$$\tilde{b}_3 = \frac{27}{112960}, \quad \tilde{b}_4 = \frac{2097869}{154783440} + \frac{285688201\sqrt{10434}}{2153347217280},$$
$$\tilde{b}_5 = \frac{2097869}{154783440} - \frac{285688201\sqrt{10434}}{2153347217280},$$
$$\tilde{b}_1' = -\frac{17}{210}, \tilde{b}_2' = \frac{307}{31320}, \tilde{b}_3' = \frac{9}{14120},$$
$$\tilde{b}_4' = \frac{8681773}{154783440} - \frac{4101409\sqrt{10434}}{7476900060},$$
$$\tilde{b}_5' = -\frac{381488}{5055477} - \frac{9931\sqrt{10434}}{13481272},$$
$$\tilde{b}_1''' = -\frac{17}{35}, \tilde{b}_2''' = \frac{307}{1305}, \quad \tilde{b}_3''' = \frac{3}{1765},$$
$$\tilde{b}_4''' = \frac{4829021}{12898620} + \frac{18942137\sqrt{10434}}{5607675045},$$
$$\tilde{b}_5''' = \frac{4829021}{12898620} - \frac{18942137\sqrt{10434}}{5607675045},$$
$$\tilde{b}_1'''' = -\frac{17}{35}, \quad \tilde{b}_2'''' = \frac{614}{1305}, \quad \tilde{b}_3'''' = \frac{2}{1765},$$
$$\tilde{b}_4'''' = \frac{1635034}{3224655} + \frac{3468551\sqrt{10434}}{1495380012},$$
$$\tilde{b}_5'''' = \frac{1635034}{3224655} - \frac{3468551\sqrt{10434}}{1495380012},$$
$$\tilde{b}_1 = \frac{359}{3825} - \frac{103\sqrt{10434}}{11900}, \tilde{b}_2 = -\frac{29309}{30600} + \frac{103\sqrt{10434}}{11900},$$
$$\tilde{b}_3 = \frac{1}{100}, \quad \tilde{b}_4 = \frac{26}{25},$$
$$\tilde{b}_5 = \frac{1}{100}, \tilde{b}_1' = \frac{16423777}{25489800} - \frac{111\sqrt{10434}}{70805},$$
$$\tilde{b}_2' = -\frac{32730113}{25489800} + \frac{34239\sqrt{10434}}{2832200},$$
$$\tilde{b}_3' = \frac{8886083}{25489800} - \frac{34239\sqrt{10434}}{2832200},$$
$$\tilde{b}_4' = \frac{51}{50}, \quad \tilde{b}_5' = \frac{1}{100},$$
$$\tilde{b}_1'' = \frac{528738349}{8847084750} - \frac{10355701\sqrt{10434}}{7077667800},$$
$$\tilde{b}_2'' = \frac{1797030453}{24434805500} - \frac{10355701\sqrt{10434}}{7077667800},$$
$$\tilde{b}_3'' = -\frac{2102196427}{892291690500} + \frac{10355701\sqrt{10434}}{356916676200},$$
$$\tilde{b}_4'' = \frac{18625876710731}{120756808781000} + \frac{337316248673\sqrt{10434}}{217362255805800},$$
$$\tilde{b}_5'' = \frac{1}{1000}, \tilde{b}_1''' = -\frac{17}{35}, \tilde{b}_2''' = \frac{307}{1305}, \tilde{b}_3''' = \frac{3}{1765},$$
$$\tilde{b}_4''' = \frac{4829021}{12898620} + \frac{18942137\sqrt{10434}}{5607675045},$$
$$\tilde{b}_5''' = \frac{4829021}{12898620} - \frac{18942137\sqrt{10434}}{5607675045}, \tilde{b}_1'''' = -\frac{17}{35},$$
$$\tilde{b}_2'''' = \frac{614}{1305}, \tilde{b}_3'''' = \frac{2}{1765}, \tilde{b}_4'''' = \frac{1635034}{3224655} + \frac{3468551\sqrt{10434}}{1495380012},$$
$$\tilde{b}_5'''' = \frac{1635034}{3224655} - \frac{3468551\sqrt{10434}}{1495380012}.$$

4. NUMERICAL EXPERIMENTS

This section presents testing of problems that include the form $\varphi^{(5)}(\tau) = \Psi(\tau, \varphi(\tau))$. The numerical results obtained from solving these problems are compared to the results obtained when the same set of problems is converted into a system of first-order equations and solved using existing Runge-Kutta pairs of the same order. The purpose of the numerical experiments is to compare the performance of different numerical methods for solving ordinary differential equations (ODEs) of the form $\varphi^{(5)}(\tau) = \Psi(\tau, \varphi(\tau))$. The goal is to determine which method is more accurate and efficient for solving different types of ODEs.

- **Tol:** Tolerance.
- **F. N:** Number of the function call.
- **MAXERR:** max $(|\varphi(\tau_n) - \varphi_n|)$ which is the maximum between absolute errors of the exact solutions and the computed solutions.
- **ERKMF6(5):** 6(5) pair of Runge-Kutta type in which derived in this paper.
- **ERKMF7(6):** 7(6) pair of Runge-Kutta type in which derived in this paper.
- **RK6(5):** 6(5) pair of Runge-Kutta type given by Verner [20].
- **RK7(6):** 7(6) pair of Runge-Kutta type derived by Verner [21].
- **RKM6:** Direct Runge-Kutta-Mohammed method of sixth-order derived by Mechee and Kadhim [4].

Problem 1: (Non-Linear Problem)

$$\varphi^{(5)}(\tau) = \cos(\tau), \ t \in [0, 10]$$

(ICs):

$$\varphi'(0) = -\varphi'''(0) = 1, \varphi^{2i}(0) = 0$$

for $i = 0, 1, 2$.

Exact-solution:

$$\varphi(\tau) = sin(\tau).$$

Problem 2: (Linear Problem)

$$\varphi^{(5)}(\tau) = -\varphi(\tau), \ \tau \in [0, 2]$$

(ICs):

$$\varphi(0) = \varphi''(0) = \varphi^{(4)}(0) = 1, \varphi'(0) = \varphi'''(0) = -1.$$

Exact-solution:

$$\varphi(\tau) = e^{-\tau}.$$

Problem 3: (Non-Linear Problem)

$$\varphi^{(5)}(\tau) = -120\,\varphi^{6}(\tau), \ \tau \in [0, 3]$$

(ICs):

$$\varphi(0) = -\varphi'(0) = 1, \ 12\varphi''(0) = -4\,\varphi'''(0) = \varphi^{(4)}(0) = 24.$$

Exact-solution:

$$\varphi(\tau) = \frac{1}{1+\tau}.$$

5. RESULTS AND DISCUSSION

In order to compare the performance of different methods, we conducted calculations to determine the maximum global error and the number of function evaluations required for integration. It is important to note that the methods being compared are of the same order. Specifically, we focused on analyzing the performance of the RK6(5), RK7(6), and RKM6 methods.

The results of our comparison are presented in Tables 2 to 4 and Figures 1 to 3, specifically for the new methods ERKMF6(5) and ERKMF7(6).

Table 2. The numerical results for solving Problem 1

$TOL(h)$	Method	F.C	MAXERR
10^{-2}	ERKFM6(5)	342	$5.82149*10^{-2}$
	RK6(5)	455	$4.68324*10^{-1}$
	ERKFM7(6)	59	$6.34362*10^{-4}$
	RK7(6)	400	$3.89747*10^{-3}$
	RKM6	354	$1.361459*10^{0}$
10^{-4}	ERKFM6(5)	1077	$1.5407*10^{-3}$
	RK6(5)	2835	$9.9535*10^{-2}$
	ERKFM7(6)	100	$1.0298*10^{-4}$
	RK7(6)	700	$1.1602*10^{-4}$
	RKM6	3375	$1.0286*10^{-4}$
10^{-6}	ERKFM6(5)	3381	$5.0539*10^{-5}$
	RK6(5)	28780	$7.1453*10^{-3}$
	ERKFM7(6)	215	$8.7073*10^{-6}$
	RK7(6)	1300	$2.8672*10^{-6}$
	RKM6	33814	$1.0369*10^{-6}$

Table 3. The numerical results for solving Problem 2

$TOL(h)$	Method	F.C	MAXERR
10^{-2}	ERKFM6(5)	70	$4.38138*10^{-2}$
	RK6(5)	175	$7.89853*10^{-3}$
	ERKFM7(6)	20	$5.20234*10^{-3}$
	RK7(6)	150	$1.31482*10^{-3}$
	RKM6	70	$5.50927*10^{-2}$
10^{-4}	ERKFM6(5)	209	$1.2146*10^{-3}$
	RK6(5)	450	$3.9681*10^{-4}$
	ERKFM7(6)	30	$4.9524*10^{-5}$
	RK7(6)	200	$7.6116*10^{-6}$
	RKM6	575	$1.1836*10^{-3}$
10^{-6}	ERKFM6(5)	625	$3.9042*10^{-5}$
	RK6(5)	4075	$3.3854*10^{-5}$
	ERKFM7(6)	55	$8.1794*10^{-6}$
	RK7(6)	300	$2.7631*10^{-7}$
	RKM6	5542	$1.0950*10^{-4}$

Table 4. The numerical results for solving Problem 3

$TOL(h)$	Method	F.C	MAXERR
10^{-2}	ERKFM6(5)	163	$9.89597*10^{-1}$
	RK6(5)	440	$1.33978*10^{-1}$
	ERKFM7(6)	84	$6.61457*10^{-1}$
	RK7(6)	395	$3.42242*10^{-1}$
	RKM6	291	$1.43638*10^{-1}$
10^{-4}	ERKFM6(5)	433	$2.5885*10^{-2}$
	RK6(5)	3885	$9.9982*10^{-3}$
	ERKFM7(6)	84	$4.0104*10^{-3}$
	RK7(6)	645	$4.5198*10^{-4}$
	RKM6	2329	$1.7565*10^{-2}$
10^{-6}	ERKFM6(5)	1317	$8.0144*10^{-4}$
	RK6(5)	32320	$1.0692*10^{-3}$
	ERKFM7(6)	134	$2.4848*10^{-4}$
	RK7(6)	1045	$7.5931*10^{-5}$
	RKM6	22505	$1.775*10^{-3}$

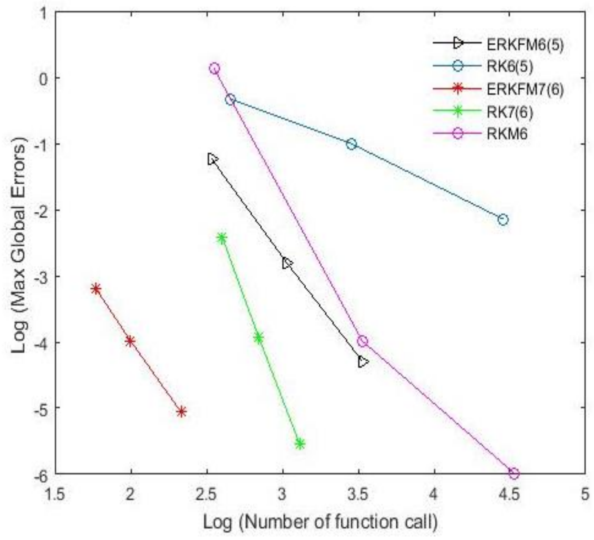

Figure 1. The efficiency curves when solving Problem 1

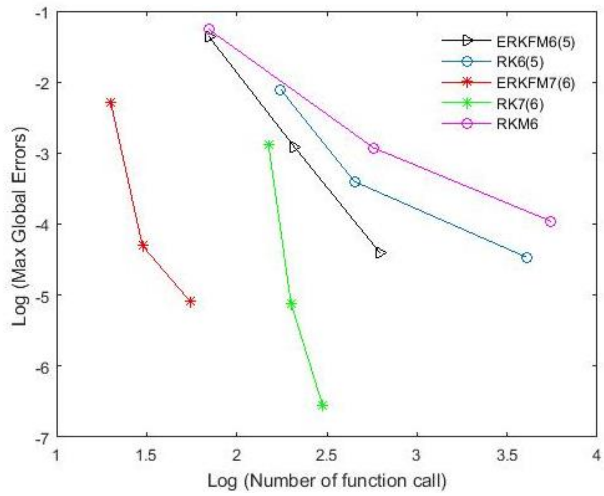

Figure 2. The efficiency curves when solving Problem 2

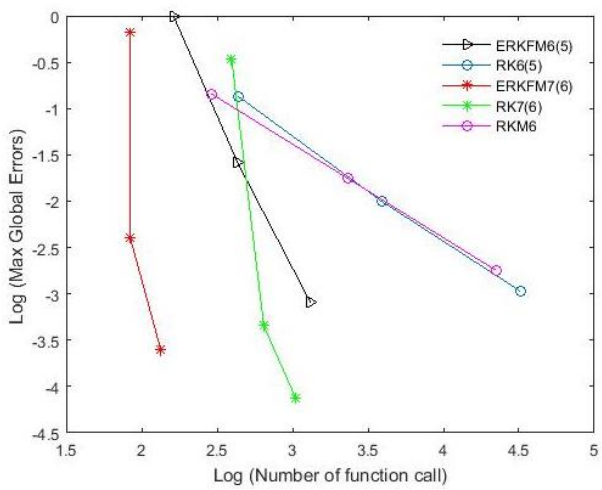

Figure 3. The efficiency curves when solving Problem 3

Table 2 reveals that the newly proposed methods, ERKMF6(5) and ERKMF7(6), demonstrate superior accuracy for various tolerances such as $TOL = 10^{-2}$, $TOL = 10^{-4}$, and $TOL = 10^{-6}$.

Moving on to Table 3, it is observed that ERKMF6(5), ERKMF7(6), RK6(5), RK7(6), and RKM6 exhibit similar maximum error values. However, when it comes to accuracy in terms of function evaluations, the aforementioned methods perform differently. Specifically, ERKMF6(5) and ERKMF7(6) achieve the best accuracy.

Table 4 indicates that all methods produce the same maximum error at $TOL = 10^{-2}$. Nevertheless, the newly proposed methods outperform the others in terms of both maximum error and function evaluations for other tolerances.

6. CONCLUSION

This paper introduces two pairs of embedded ERKMF algorithms, namely ERKMF6(5) and ERKMF7(6), designed to directly solve a specific subclass of quasi-linear fifth-order ordinary differential equations (ODEs) using variable step size codes. To assess their performance, these algorithms are compared to existing Runge-Kutta (RK) methods of the same algebraic order. The comparison criterion used is the maximum error in the solution, which is determined as the maximum absolute difference between the actual and computed solutions.

The numerical results presented in Tables 2 to 4 and Figures 1 to 3 clearly demonstrate that the new ERKMF methods exhibit smaller global errors compared to the existing methods when applied to the direct solution of fifth-order ODEs by transforming them into equivalent first-order systems. Furthermore, the new methods demonstrate lower computational costs and consistently outperform other existing methods in terms of accuracy and efficiency across all considered problems and step sizes for solving fifth-order differential equations.

Based on these findings, it can be concluded that the proposed ERKMF6(5) and ERKMF7(6) algorithms offer significant improvements in solving fifth-order ODEs directly. These methods showcase enhanced accuracy, reduced computational burden, and increased efficiency when compared to alternative techniques.

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REFERENCES

[1] Wazwaz, A.M. (2006). Solitons and periodic solutions for the fifth-order KdV equation. Applied Mathematics Letters, 19(11): 1162-1167. https://doi.org/10.1016/j.aml.2005.07.014

[2] Isaza, P., Linares, F., Ponce, G. (2015). Decay properties for solutions of fifth order nonlinear dispersive equations. Journal of Differential Equations, 258(3): 764-795. https://doi.org/10.1016/j.jde.2014.10.004

[3] Lee, C.T. (2015). Some remarks on the fifth-order KdV equations. Journal of Mathematical Analysis and Applications, 425(1): 281-294. https://doi.org/10.1016/j.jmaa.2014.10.021

[4] Mechee, M.S., Kadhim, M.A. (2016). Explicit direct integrators of RK type for solving special fifth-order ordinary differential equations. American Journal of Applied Sciences, 13(12): 1452-1460. https://doi.org/10.3844/ajassp.2016.1452.1460

[5] Mechee, M.S., Fawzi, F.A. (2021). Generalized Runge-Kutta integrators for solving fifth-order ordinary differential equations. Italian Journal of Pure and Applied Mathematics, 45: 600-610.

[6] Turki, M.Y., Alku, S.Y., Mechee, M.S. (2022). The general implicit-block method with two-points and extra derivatives for solving fifth-order ordinary differential equations. International Journal of Nonlinear Analysis and Applications, 13(1): 1081-1097. https://doi.org/10.22075/IJNAA.2022.5650

[7] Fawzi, F.A., Senu, N., Ismail, F., Majid, Z.A. (2018). A new integrator of Runge-Kutta type for directly solving general third-order odes with application to thin film flow problem. Applied Mathematics & Information Sciences, 12(4): 775-784. https://doi.org/10.18576/amis/120412

[8] Fawzi, F.A., Abdullah, Z.M., Hussein, N.K. (2021). Two embedded pairs for solve directly third-order ordinary differential equation by using Runge-Kutta Type Method (RKTGD). Journal of Physics: Conference Series, 1879(2): 022123. https://doi.org/10.1088/1742-6596/1879/2/022123

[9] Hussain, K., Ismail, F., Senu, N. (2015). Two embedded pairs of Runge-Kutta type methods for direct solution of special fourth-order ordinary differential equations. Mathematical Problems in Engineering, 2015: 893763. https://doi.org/10.1155/2015/893763

[10] Senu, N., Mechee, M., Ismail, F., Siri, Z. (2014). Embedded explicit Runge–Kutta type methods for directly solving special third order differential equations $y''' = f(x, y)$. Applied Mathematics and Computation, 240: 281-293. https://doi.org/10.1016/j.amc.2014.04.094

[11] Awoyemi, D.O., Idowu, O.M. (2005). A class of hybrid collocation methods for third-order ordinary differential equations. International Journal of Computer Mathematics, 82(10): 1287-1293. https://doi.org/10.1080/00207160500112902

[12] Waeleh, N., Majid, Z.A., Ismail, F. (2011). A new algorithm for solving higher order IVPs of ODEs. Applied Mathematical Sciences, 5(56): 2795-2805.

[13] Demba, M.A., Senu, N., Ismail, F. (2016). A four-stage third-order symplectic explicit trigonometrically-fitted Runge–Kutta–Nyström method for the numerical integration of oscillatory initial-value problems. International Journal of Pure and Applied Mathematics, 111(2): 165-178. https://doi.org/10.12732/ijpam.v111i2.3

[14] Fawzi, F.A., Globe, H.M., Ghawadri, N.G. (2024). Efficient embedded diagonal implicit Runge-Kutta method for directly solving third order ODEs. Ibn AL-Haitham Journal for Pure and Applied Sciences, 37(2): 388-408. https://doi.org/10.30526/37.2.3409

[15] Gander, W., Gruntz, D. (1999). Derivation of numerical methods using computer algebra. SIAM Review, 41(3): 577-593. https://doi.org/10.1137/S003614459935093X

[16] Dormand, J.R. (2018). Numerical Methods for Differential Equations: A Computational Approach. CRC Press. https://doi.org/10.1201/9781351075107

[17] Dormand, J.R., Prince, P.J. (1980). A family of embedded Runge-Kutta formulae. Journal of

Computational and Applied Mathematics, 6(1): 19-26. https://doi.org/10.1016/0771-050X(80)90013-3
[18] Prince, P.J., Dormand, J.R. (1981). High order embedded Runge-Kutta formulae. Journal of Computational and Applied Mathematics, 7(1): 67-75. https://doi.org/10.1016/0771-050X(81)90010-3
[19] El-Mikkawy, M., Rahmo, E.D. (2003). A new optimized non-FSAL embedded Runge–Kutta–Nystrom algorithm of orders 6 and 4 in six stages. Applied Mathematics and Computation, 145(1): 33-43. https://doi.org/10.1016/S0096-3003(02)00436-8
[20] Verner, J.H. (2014). Explicit Runge–Kutta pairs with lower stage-order. Numerical Algorithms, 65: 555-577. https://doi.org/10.1007/s11075-013-9783-y
[21] Verner, J.H. (1994). Strategies for deriving new explicit Runge-Kutta pairs. Annals of Numerical Mathematics, 1: 225-244.