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Using Hermite Polynomials to Solve Volterra-Fredholm Integral Equation of the Second Kind

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ABSTRACT

The objective of this study is to solve Linear Volterra-Fredholm Integral Equations of the second kind numerically using Hermite polynomials. We will present an approximate solution as a series that converges towards the exact solution. Several examples are provided to illustrate the numerical results, specifically comparing the exact and numerical solutions. These comparisons are shown in tables, demonstrating that the error between the exact and numerical solutions is negligible. Additionally, diagrams highlight how closely the numerical solution matches the exact solution, underscoring the accuracy of the grouping method used to solve the Volterra-Fredholm Integral Equation with the MATLAB program. This method is noted for its simplicity, speed, and high accuracy in obtaining numerical results.

1. INTRODUCTION

The integral equations form the basis of several mathematical models in many scientific domains, such as engineering, mathematics, and chemistry. Numerous areas in engineering and applied mathematics are connected to integral equations as defined in the study conducted by Wazwaz [1] and Semenova et al. [2].

The Volterra-Fredholm Integral Equations have played a major role in developing many crucial in applied mathematics, engineering, and physics [3], and have found extensive application in various scientific fields. The Volterra and Fredholm integral equations are combined to create these integral equations.

We define the Linear Volterra-Fredholm Integral Equations of the second kind (LVFIES for short) as follows [3-5]:

$$\lambda(\tau)-\int_{-1}^{\tau} h_1(\tau,\nu)\lambda(\nu)\,d\nu-\int_{-1}^{1} h_2(\tau,\nu)\lambda(\nu)\,d\nu=f(\tau) \qquad -1\le\tau,\nu\le 1 \tag{1}$$

In the case where $f(\tau)$, $h_1(\tau, \nu)$ and $h_2(\tau, \nu)$ are functions. The function $\lambda(\tau)$ is the unknown function that must be determined. There are many techniques for properly effectively resolving such problems [6, 7], such as Adomian's method [8, 9], Chebyshev polynomials [10-12], They serviced the Volterra Friedolm integral equation using a Chebychev polynomial with summation points. Then they proposed a new summation method to solve the Volterra Fredholm equation, Euler series to solve the Fredholm integral equation [13], Laplace transform method [14, 15].

In this context, Salman Mustfaf [16] presented an innovative method for solving linear fractional Volterra-Fredholm Integro-Differential Equations (LFVFIDEs) with fractional derivatives in the Caputo sense. To validate the effectiveness of this new approach, it is compared with existing techniques, demonstrating its superior performance in addressing these complex problems. Varol et al. [17] analyzed the application of Laguerre polynomials in constructing a numerical approximation method for solving fractional linear integral-differential equations (IDEs) of the Fredholm-Volterra type.

Negarchi and Nouri [18], and Nemati [19] discussed solutions to the integral equation using the Legendre polynomial.

In our work, we use Hermite polynomials to solve the Volterra-Fredholm equation integral. In the context of numerical methods, convergence analysis is crucial to ensure that the method reliably approximates the exact solution as the number of terms increases.

The paper's structure is: introductory section contains an introduction to the LVFIES, in the second one, we demonstrate the existence and the uniqueness of the solution of the LVFIES, and then we introduce definitions about the Hermite polynomials, which is the basis of our work in the third section. In the fourth one, we put the description of the collocation Hermite method to solve LVFIES we search for an approximate solution in the form of series. In the fifth section, numerical examples are presented to demonstrate the accuracy and efficiency of the current method. Finally, the conclusions of this work are summarized.

2. EXISTENCE AND UNIQUENESS OF LVFIES SOLUTIONS

There are multiple theorems that demonstrate the existence and unicity of the LVFIES solutions, we present one of them Theorem suppose the Eq. (1), such that:

$$f \in C[-1,1]$$
$$h_1(\tau,\nu) \in C(K),\ K = \left\{(\tau,\nu) \in R^2, -1 \le \tau \le \nu \le 1\right\}$$
$$\lambda \in C([-1,1],\ h_2(\tau,\nu) \in C[-1,1]\times[-1,1])$$
$$N_1 = \max h_1(\tau,\ \nu)\ \ [-1,1]\times[-1,1]$$
$$N_2 = \max h_2(\tau,\ \nu)\ \ [-1,1]\times[-1,1]$$

Under the continuity conditions above suppose that there is a constant C>0 such that:

$$\frac{1}{C}\left(N_1 + N_2 e^{2C}\right)$$

Then the Eq. (1) has a unique solution $\lambda \in C(-1, 1]$ and this solution can be obtained by the successive approximation method.

Proof

Let the integral operator ζ: $C([-1, 1] \rightarrow C([-1, 1])$ defined by:

$$\zeta\lambda(\tau) = \int_{-1}^{\tau} h_1(\tau,\nu)\lambda(\nu)d\nu - \int h_2(\tau,\nu)\lambda(\nu)d\nu = f(\tau)$$

We have:

$$\left|\zeta\lambda(\tau) - \zeta\psi(\tau)\right| =$$
$$\left|\int_{-1}^{\tau} h_1(\tau,\nu)(\lambda(\nu)-\psi(\nu))d\nu - \int_{-1}^{1} h_2(\tau,\nu)(\lambda(\nu)-\psi(\nu))d\nu\right|$$
$$\le \int_{-1}^{\tau}\left|h_1(\tau,\nu)\right|\left|\lambda(\nu)-\psi(\nu)\right|d\nu + \int_{-1}^{1}\left|h_2(\tau,\nu)\right|\left|\lambda(\nu)-\psi(\nu)\right|d\nu$$
$$\le N_1\int_{-1}^{\tau}\left|\lambda(\nu)-\psi(\nu)\right|e^{(-C(\nu+1))}e^{(C(\nu+1))}d\nu$$
$$+ N_2\int_{-1}^{1}\left|\lambda(\nu)-\psi(\nu)\right|e^{(-C(\nu+1))}e^{(C(\nu+1))}d\nu$$
$$\le \frac{N_1}{C}\left(e^{(C(\tau+1)}-1\right) + \frac{N_2}{C}(e^{2C}-1)\left\|\lambda-\psi\right\|$$
$$\le \frac{N_1}{C}e^{(C(\tau+1)} + \frac{N_2}{C}(e^{(C(\tau-\tau+1+1)}-1)\left\|\lambda-\psi\right\|$$
$$\le \frac{1}{C}e^{C(\tau+1)}(N_1 + N_2(e^{C(1-\tau)}))\left\|\lambda-\psi\right\|$$
$$\le \frac{1}{C}e^{C(\tau+1)}(N_1 + N_2(e^{2C}))\left\|\lambda-\psi\right\|$$

Consequently:

$$\left|\zeta\lambda(\tau) - \zeta\psi(\tau)\right|e^{(C(\tau+1))} \le \frac{1}{C}(N_1 + N_2(e^{2C})\left\|\lambda-\psi\right\|$$

for every $t \in [-1, 1]$:

So:

$$\left\|\zeta\lambda - A\psi\right\| \le \frac{1}{C}(N_1 + N_2(e^{2C})\left\|\zeta-\psi\right\|$$

We conclude that the operator ζ is Lipschitzian with constant:

$$k = \frac{1}{C}(N_1 + N_2 e^{2C})$$

The assumed condition guarantees that ζ is a contraction. So, the operator ζ has a unique solution which is the solution of Eq. (1).

3. HERMITE POLYNOMIALS

Hermite polynomials are a series of orthogonal polynomials that appear in probability theory, physics, and numerical analysis.

Hermite polynomials are defined by its general form [20, 21]:

$$Ht_n(\tau) = (-1)^n e^{\frac{\tau^2}{2}}\frac{d^n}{d\tau^n}e^{-\frac{\tau^2}{2}},$$
$$n = 0,1,2.....,-\infty < \tau < +\infty$$

The first seven Hermite polynomials are:

$$Ht_0(\tau) = 1,$$
$$Ht_1(\tau) = 2\tau,$$
$$Ht_2(\tau) = 4\tau^2 - 2,$$
$$Ht_3(\tau) = 8\tau^3 - 12\tau,$$
$$Ht_4(\tau) = 16\tau^4 - 48\tau^2 + 12,$$
$$Ht_5(\tau) = 32\tau^5 - 160\tau^3 + 120\tau,$$
$$Ht_6(\tau) = 64\tau^6 - 480\tau^4 + 720\tau^2 - 120,$$

There is orthogonality in the polynomials $Ht_n(\tau)$ with respect to the weight function $S(\tau) = e^{-\frac{\tau^2}{2}}$ with the following condition:

$$\int_{-\infty}^{+\infty} Ht_n(\tau)Ht_m(\tau)S(\tau)d\tau = n!\sqrt{2\pi}\,\delta_{n,m} \qquad (2)$$

$\delta_{n,m}$ denoted the Kronecker delta.

Let:

$$\omega_n(\tau) = \frac{1}{\sqrt{n!\sqrt{2\pi}}} \times Ht_n \qquad (3)$$

So:

$$\left\langle \omega_n, \omega_m \right\rangle =$$
$$\frac{1}{n!\sqrt{2\pi}}\int_{-\infty}^{+\infty} Ht_n(\tau)Ht_m(\tau)S(\tau)d\tau = \begin{cases}1\ si\ n = m\\ 0\ si\ n \ne m\end{cases} \qquad (4)$$

4. METHOD OF APPROXIMATE SOLUTION OF LVFIES BY HERMITE COLLOCATION METHOD

The Hermite collocation method is a numerical technique that approximates solutions to Linear Volterra-Fredholm Integral Equations of the second kind (LVFIEs) using Hermite polynomials. Consider the LVFIES:

$$\lambda(\tau) - \int_{-1}^{\tau} h_1(\tau,\nu)\lambda(\nu)d\nu - \int_{-1}^{1} h_2(\tau,\nu)\lambda(\nu)d\nu = f(\tau)$$

Now, we use the collocation method [22-24] to the Eq. (1), we approximate the unknown function $\lambda(\tau)$, in the form of finite sum as follows:

$$\lambda(\tau) = \sum_{k=0}^{N} a_k \,\omega_k \tag{5}$$

where, ω_k is Hermite polynomials of degree k, a_k are unknown parameters, after substituting expression (5) in Eq. (1), it results:

$$\sum_{k=0}^{\infty} a_k \,\omega_k - \int_{-1}^{\tau} h_1(\tau,\nu) \sum_{k=0}^{N} a_k \,\omega_k(\nu) - \int_{-1}^{1} h_2(\tau,\nu) \sum_{k=0}^{N} a_k \,\omega_k(\tau) = f(\tau) \tag{6}$$

The Eq. (6) is transformed into a system of linear equations.

$$EA=F$$

where, the conjugate matrix of (3) is written as follows:

$$\begin{bmatrix} E_0(\tau_0) & E_0(\tau_0) & \dots & E_n(\tau_0) \\ E_0(\tau_1) & E_0(\tau_1) & \dots & E_n(\tau_1) \\ .. & .. & \dots & \dots \\ .. & .. & \dots & \dots \\ .. & .. & \dots & \dots \\ E_0(\tau_n) & E_0(\tau_n) & \dots & E_n(\tau_n) \end{bmatrix} \tag{7}$$

Such that:

$$E_j(\tau) = \omega_j(\tau_i) - \int_{-1}^{\tau} h_1(\tau_i,\nu)\omega_j(\tau)d\nu - \int_{-1}^{1} h_2(\tau_i,\nu)\omega_j(r)d\nu \tag{8}$$

$$E_0(\tau_0) = \omega_0(\tau_0) - \int_{-1}^{\tau} h_1(\tau_0,\nu)\omega_0(\nu_0)d\nu - \int_{-1}^{1} h_2(\tau_0,\nu)\omega_0(\nu)d\nu$$

$$E_n(\tau_0) = \omega_n(\tau_0) - \int_{-1}^{\tau} k_1(\tau_0,\nu)\,\omega_n(\nu)d\nu - \int_{-1}^{1} k_2(\tau_0,\nu)\omega_n(\nu)d\nu$$

$$E_0(\tau_1) = \omega_0(\tau_1) - \int_{-1}^{\tau} h_1(\tau_1,\nu)\omega_0(\nu)d\nu - \int_{-1}^{1} h_2(\tau_1,\nu)\omega_0(\nu)d\nu$$

$$E_n(\tau_1) = \omega_n(\tau_1) - \int_{-1}^{\tau} h_1(\tau_1,\nu)\omega_n(\nu)d\nu - \int_{-1}^{1} h_2(\tau_1,\nu)\omega_n(\nu)d\nu$$

$$E_0(\tau_n) = \omega_0(\tau_n) - \int_{-1}^{\tau} h_1(\tau_n,\nu)\omega_0(\nu)d\nu - \int_{-1}^{1} h_2(\tau_n,\nu)\omega_0(\nu)d\nu$$

$$E_n(\tau_n) = \omega_n(\tau_n) - \int_{-1}^{\tau} h_1(\tau_n,\nu)\omega_n(\nu)d\nu - \int_{-1}^{1} h_2(\tau_n,\nu)\omega_n(\nu)d\nu$$

For all $i=0, 1, \dots, N$, $j=0, 1, \dots, N$. The vectors A and F are given by:

$$A = \begin{bmatrix} a_0 \\ a_1 \\ .. \\ .. \\ .. \\ a_n \end{bmatrix} \qquad \mathrm{F} = \begin{bmatrix} f(\tau_0) \\ f(\tau_1) \\ . \\ . \\ . \\ f(\tau_n) \end{bmatrix} \tag{9}$$

computing the coefficients a_k so that the interval [-1, 1] satisfies Eq. (6). The equidistant collocation points are taken as follows for this technique:

$$S_j = -1 + \frac{2j}{N} \qquad j = 0,\dots\dots\dots N \tag{10}$$

The residual is defined as following:

$$Z_N(\tau) = \sum_{k=0}^{\infty} a_k \,\omega_k - \int_{-1}^{\tau} h_1(\tau,\nu) \sum_{k=0}^{N} a_k \,\omega_k - \int_{-1}^{1} h_2(\tau,\nu) \sum_{k=0}^{N} a_k \,\omega_k(\tau) - f(\tau) \tag{11}$$

and applying conditions at collocation points, you get:

$$Z_N(\tau_j) = 0 \qquad j = 0,1,\dots\dots\dots N \tag{12}$$

5. NUMERICAL EXAMPLES

In this part, we give some illustrative examples to clarify our work

Example 1

Consider the LVFIES

$$\lambda(\tau) - \int_{0}^{\tau} (2\tau^2\nu + 1)\lambda(\nu)d\nu - \int_{0}^{1} \tau(\nu+1)\lambda(\nu)d\nu = f(\tau)$$

If we take:

$$f(\tau) = -\frac{2}{5}\tau^7 - \frac{5}{4}\tau^4 + \tau^3 - \frac{59}{20}\tau + 1$$

We get:

$$\lambda(\tau) = \tau^3 + 1$$

Applying the Hermite collocation method to approximate the solution $\lambda(\tau)$.

Table 1. Results for Example 1

Val of τ	Ex. Sol	Error (N=3)	Error [25] (N=3)
0.0625	1.0010e+00	1.3323e-15	2.3315e-14
0.01875	1.0080e+00	4.4409e-16	2.3537e-14
0.3125	1.0270e+00	4.4409e-16	2.4425e-14
0.4375	1.0640e+00	8.8818e-16	2.5313e-14
0.5625	1.1250e+00	1.3327e-15	2.6867e-14
0.6875	1.2160e+00	1.1102e-15	2.8644e-14
0.8125	1.3430e+00	1.9984e-15	3.0864e-14
0.9375	1.5120e+00	2.2204e-15	3.3307e-14

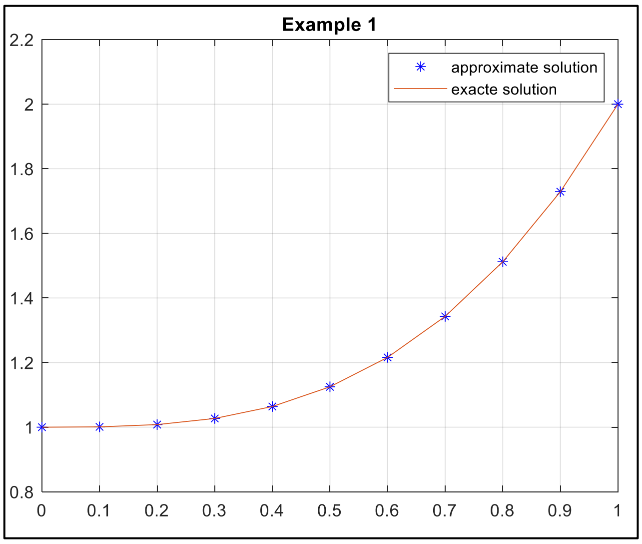

Figure 1. The exact and the approximate solutions for Example 1 with n=3

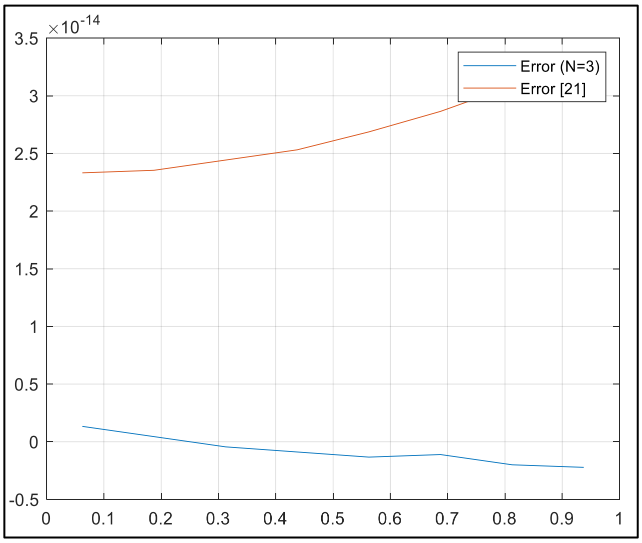

Figure 2. The absolute errors for Example 1

In Table 1, we present the error for Example 1 for N=3 then comparing the results with one of Pull Lucas method [25].

Figures 1 and 2 present respectively comparisons of the exact and approximate solution for n=3, and the absolute errors for Example 1.

Example 2

Consider the LVFIES

$$\lambda(\tau)-\int_{-1}^{\tau} e^{-(\tau+\nu)}\lambda(\nu)d\nu-\int_{0}^{1}\lambda(\nu)d\nu=f(\tau)$$

If we take:

$$f(\tau)=e^{-\tau}+e^{-\frac{3\tau}{2}}+e^{-1}-e^{1}-e^{\frac{2-\tau}{2}}$$

We get:

$$\lambda(\tau)=e^{-\tau}$$

Table 2. Comparison of the errors for Example 1

Val. of τ	Ex. Sol	Error (N=8)	Error (N=10)
-1.00e+00	2.7183e+00	3.4542e-08	0.0909e-09
-0.80e+00	2.2255e+00	2.5994e-08	2.6259e-09
-0.60e+00	1.8221e+00	5.3301e-09	3.2360e-09
-0.40e+00	1.4981e+00	2.0799e-09	7.8511e-09
-0.02e+00	1.2214e+00	1.4442e-08	1.0946e-08
0.00e+00	1.000e+00	2.1173e-08	1.2330e-08
0.20e+00	8.1873e-01	2.5802e-08	1.1896e-08
0.40e+00	6.7032e-01	3.4290e-08	9.5981e-09
0.60e+00	5.4881e-01	2.7952e-08	5.4841e-09
0.80e+00	4.4933e-01	4.7688e-08	2.5731e-10
1.00e+00	3.6788e-01	3.7477e-08	7.2157e-09

In Table 2, a comparison of the exact and the approximate solution for N=8, N=10 is done.

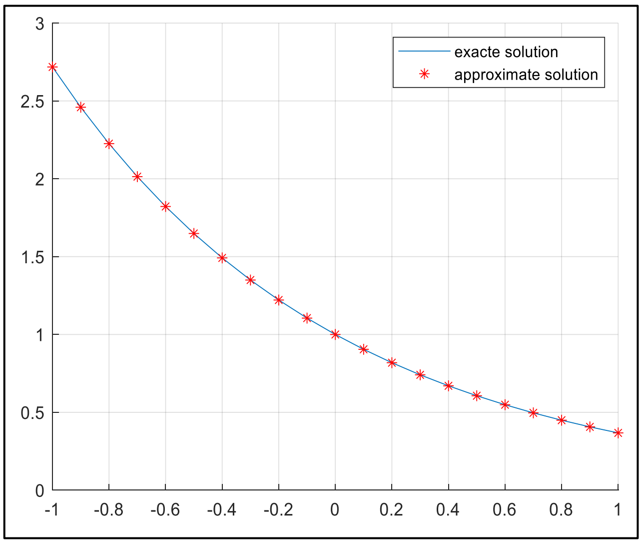

Figure 3. Comparison of the exact and approximate solutions for Example 2

In Figure 3, we depicted the exact solution and the approximate solution for n=8 of the problem in Example 2 obtained by Hermit collocation method.

Example 3

Consider the LVFIES

$$\lambda(\tau)-\int_{-1}^{\tau} e^{-(\tau+\nu)}\lambda(\nu)d\nu-\int_{0}^{1}2\cosh\lambda(\nu)d\nu=f(\tau)$$

If we take:

$$f(\tau)=\frac{70368744177664}{325586356356385549}\cosh(\tau)$$
$$-\frac{70368744177664}{325586356356385549}(\cosh(\tau)(\sin(2)+2))$$
$$+\frac{70368744177664}{325586356356385549}\tau *$$
$$(e^{(-\tau)/2}-e^{-1}+e^{\tau/2}+r(e^{-\tau/2}-e^{\tau/2}))$$

We get:

$$\lambda(\tau)=\frac{\cosh\tau}{\sinh 2\tau+1}$$

Table 3. Comparison of errors for Example 3

Val. of τ	Error (N=6)	Error (N=8)	Error (N=10)
-1.00e+00	9.4467e-08	3.7348e-10	1.0230e-09
-0.80e+00	6.3854e-08	6.4632e-12	5.4004e-10
-0.60e+00	8.5473e-08	3.3289e-10	3.5960e-12
-0.40e+00	6.5937e-08	1.9525e-10	4.9380e-10
-0.02e+00	4.5408e-08	2.0440e-10	8.3413e-10
0.00e+00	4.6028e-08	1.7791e-10	9.6040e-10
0.20e+00	3.6391e-08	1.6898e-10	8.4951e-10
0.40e+00	5.3720e-08	1.2672e-10	5.2245e-10
0.60e+00	7.5637e-08	2.5954e-10	4.1285e-11
0.80e+00	7.8065e-08	4.7925e-11	4.9974e-11
1.00e+00	3.8624e-08	1.1737e-10	9.8771e-10

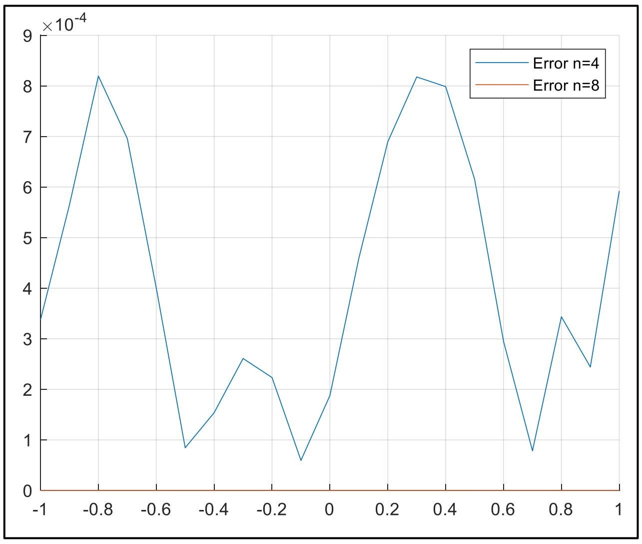

Figure 4. The absolute errors for Example 3

The comparison of the error of Example 3 is presented in Table 3, according to different values of N such that 6, 8 and 10. It can be noted that the absolute error decreasing as the value of the integer N increasing.

Results presented in the previous table are shown in the Figure 4, we remark that error decreases when N increases.

Example 4

Consider the LVFIES:

$$\lambda(\tau)-\int_{-1}^{\tau}\cos(\tau-\nu)\lambda(\nu)d\nu-\int_{-1}^{1}\sin(\tau-\nu)\lambda(\nu)d\nu=f(\tau)$$

If we take:

$$\begin{aligned}f(\tau)=e^{\tau/2}&+e^{-1}(\cos(\tau+1)-\sin(\tau+1))\\&+e^{-1}(\cos(\tau+1)+\sin(\tau+1))\\&-e^{1}(\cos(\tau-1)+\sin(\tau-1))/2\end{aligned}$$

We get:

$$\lambda(\tau)=e^{\tau}$$

Table 4 presents the comparison of the error of Example 4 for N=8 and N=10.

In Figure 5, we depicted the exact solution and the approximate solution for n=6 of the problem in Example 4 obtained by Hermit collocation method.

Table 4. Comparison of the errors for Example 4

Val. of t	Ex. Sol	Error (N=8)	Error N=10
-1.00e+00	3.6788e-01	7.4912e-09	2.2612e-09
-0.80e+00	4.4933e-01	7.6597e-09	1.0637e-09
-0.60e+00	5.4881e-01	9.7107e-09	2.0521e-10
-0.40e+00	6.7032e-01	2.4644e-09	1.3040e-09
-0.02e+00	8.1873e-01	1.4784e-10	2.0384e-09
0.00e+00	1.0000e+00	3.7148e-09	2.2841e-09
0.20e+00	1.2214e+00	7.4832e-09	2.0025e-09
0.40e+00	1.4918e+00	5.2958e-09	1.2455e-09
0.60e+00	1.8221e+00	1.9362e-08	1.4861e-10
0.80e+00	2.2255e+00	1.0192e-11	1.0859e-09
1.00e+00	2.1783e+00	1.8840e-08	2.2178e-09

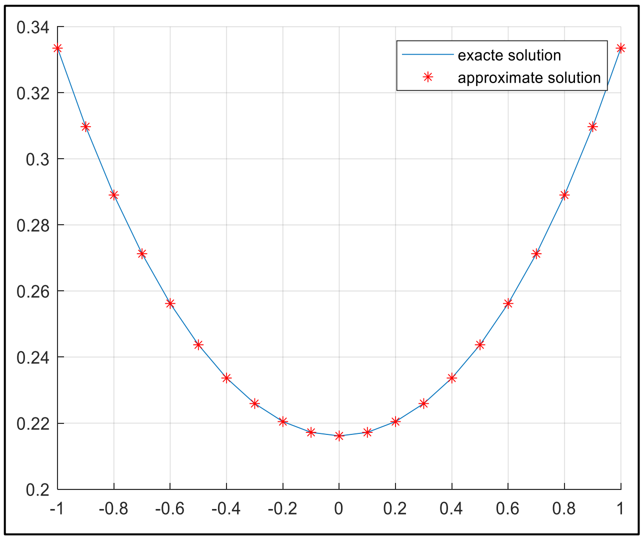

Figure 5. The absolute errors for Example 4

Example 5

Consider the LVFIES

$$\lambda(\tau)-\int_{-1}^{\tau}(\tau-\nu)\lambda(\nu)d\nu-\int_{-1}^{1}\tau\lambda(\nu)d\nu=f(\tau)$$

If we take:

$$f(\tau)=2e^{\tau}-5e^{-1}+e^{-1}-4\tau e^{-1}(e^{2}-1)$$

We get:

$$\lambda(\tau)=\tau e^{\tau}$$

Table 5. Results for Example 5

Val. of τ	Ex. Sol	Sol Appro. N=10	Error N=10
-1.00e+00	-3.6788e-01	-3.6788e-01	1.1805 -09
-0.80e+00	-3.5946e-01	-3.5946e-01	5.6785e-10
-0.60e+00	-3.9229e-01	-3.9229e-01	6.7250e-11
-0.40e+00	-2.6813e-01	-2.6813e-01	6.2277e-10
-0.02e+00	-1.6375e-01	-1.6375e-01	1.0300e-09
0.00e+00	0.0000e+00	-1.2419e-09	1.2419e-09
0.20e+00	2.4428e-01	2.4428e-01	1.2438e-09
0.40e+00	5.9673-01	5.9673e-01	1.0556e-09
0.60e+00	1.0933e+00	1.0933e+00	7.2827e-10
0.80e+00	1.7804e+00	1.7804e+00	3.4034e-10
1.00e+00	2.1783e+00	2.7183e+00	3.9580e-11

The comparison of the exact and the approximate solutions for N=10. The results for Example 5 are presented in Table 5.

Example 6
Consider the LVFIES

$$\lambda(\tau)-\int_{0}^{\tau}\nu\lambda(\nu)d\nu-\int_{-1}^{1}\nu\lambda(\nu)d\nu=f(\tau)$$

If we take:

$$f(\tau)=3\tau+4\tau^{2}-\tau^{3}+\tau^{4}-2$$

We get:

$$f(\tau)=3\tau+4\tau^{2}$$

The comparison of the exact and approximate solutions for N=3 for Example 5 is presented in Figure 6.

Figures 7 and 8 present respectively the absolute errors for Example 6 and the absolute errors for Example 7. The results for Example 5 are presented in Table 6.

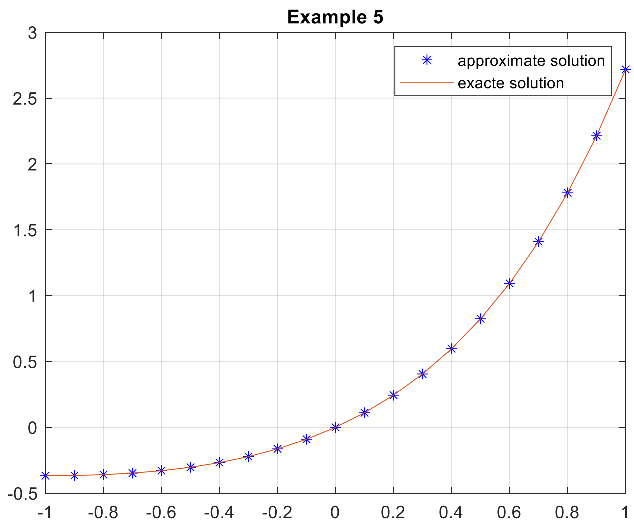

Figure 6. The exact and approximate solutions for Example 5

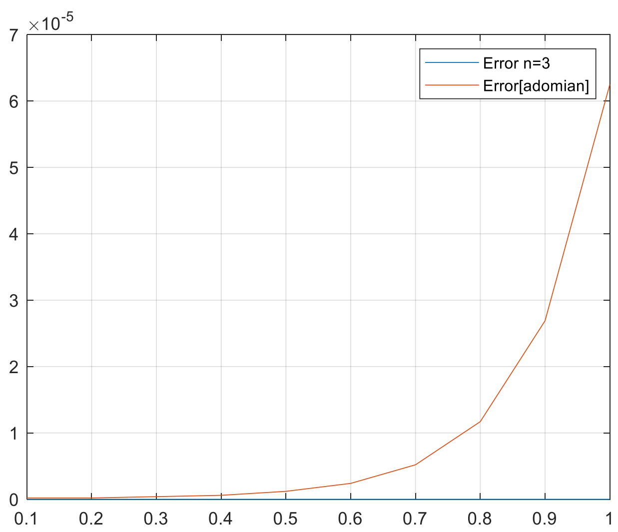

Figure 7. The absolute errors for Example 6

Table 6. Results for Example 6

Val. of τ	Ex. Sol	Error N=3	Error [8] (Adomian)
0.10e-01	3.4000e-01	7.3275e-15	2.0000e-07
0.20e-01	7.6000e-01	7.6605e-15	2.0000e-07
0.30e-01	1.2600e-01	7.9936e-15	4.0000e-07
0.40e-01	1.8400e-01	7.99360e-15	6.0000e-07
0.50e-01	2.5000e+00	8.8818e-15	1.2000e-06
0.60e-01	3.2400e-01	1.0658e-14	2.4000e-06
0.70e-01	4.0600-01	9.7700e-15	5.2000e-06
0.80e-01	4.9600e+00	1.1546e-14	1.1700e-05
0.90e-01	5.9400e+00	1.2434e-14	2.6900e-05
1.00e+00	7.0000e+00	1.4211e-14	6.2500e-05

Example 7
Consider the LVFIES

$$\lambda(\tau)-\int_{0}^{\tau}(\tau-\nu)\lambda(\nu)d\nu-\int_{0}^{1}\tau\lambda(\nu)d\nu=f(\tau)$$

If we take:

$$f(\tau)=-2-2\tau+e^{\tau}$$

We get:

$$\lambda(\tau)=\tau e^{\tau}$$

The comparison of the exact and the approximate solutions for N=4 and N=8. Tables 7 and 8 present respectively the results for Example 7 and a comparison of errors for Example 7 with Adomian. Figure 8 presents the absolute errors for all errors of the same example.

Table 7. Results for Example 7

Val. of τ	Ex. Sol	App. Sol N=4	App. Sol N=8
00e+00	0	5.7732e-15	-6.3288e-09
0.10e-01	1.1052e-01	1.1027e-01	1.1052e-01
0.20e-01	2.4428e-01	2.4418e-01	2.4428e-01
0.30e-01	4.0496e-01	4.0501e-01	4.0496e-01
0.40e-01	5.9673e-01	5.9681e-01	5.9673e-01
0.50e-01	8.2436e-01	8.2434e-01	8.2436e-01
0.60e-01	1.0933e+00	1.0931e+00	1.0933e+00
0.70e-01	1.4096e+00	1.4095e+00	1.4096e+00
0.80e-01	1.7804e+00	1.7805e+00	1.7804e+00
0.90e-01	2.2136e+00	2.2139e+00	2.2136e+00
1.00e+00	2.7183e+00	2.7182e+00	2.7183e+00

Table 8. Comparison of errors for Example 7 with Adomian

Val. of τ	Error N=4	Error N=8	Error [8] [Adomian]
0.10e-01	2.5191e-04	5.1900e-09	1.1984e-03
0.20e-01	1.0304e-04	2.8525e-09	2.4176e-03
0.30e-01	5.6762e-05	1.3126e-09	3.6788e-03
0.40e-01	7.6581e-05	7.4250e-10	5.0039e-03
0.50e-01	2.3198e-05	2.6313e-09	6.4158e-03
0.60e-01	1.2701e-04	4.4717e-09	7.9393e-03
0.70e-01	1.0983e-04	6.3900e-09	9.6006e-03
0.80e-01	6.6356e-05	7.6748e-09	1.1429e-02
0.90e-01	2.4402e-04	9.7343e-09	1.3455e-02
1.00e+00	5.1972e-05	1.0413e-08	1.5716e-02

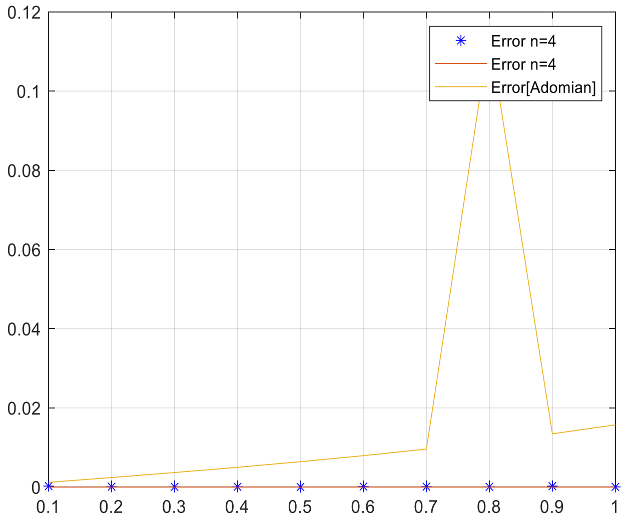

Figure 8. The absolute errors for Example 7

Convergence analysis

Hermite polynomials, owing to their orthogonality and recurrence relations, exhibit strong convergence properties when applied to functions that can be well-represented.

Error bounds

Error bounds provide a quantitative measure of how close the numerical solution is to the exact solution. For Hermite polynomials, the error bound is influenced by the smoothness of the target function and its behavior at the boundaries. This error bound illustrates that as N increases, the error decreases factorially, indicating high accuracy for sufficiently large N.

Stability

Stability is a critical aspect in numerical methods to ensure that errors do not amplify through computations. Hermite polynomials benefit from their orthogonality, which inherently contributes to numerical stability. This property ensures that the polynomial terms do not interact destructively, maintaining the integrity of the approximation over a wide range of values.

Moreover, the stability of Hermite polynomial-based methods is reinforced by their recurrence relations, which allow for the efficient and stable computation of higher-order terms. These relations mitigate the risk of numerical instability that might arise from direct computation methods, making Hermite polynomials a robust choice for high-precision applications.

6. CONCLUSIONS

In this study, we approximate the solution of linear Fredholm-Volterra integral equations using the Hermite collocation method. Different examples are mentioned in order to show the efficiency of the Hermite collocation method.

Obtained results are presented through tables and figures that illustrate both the exact and numerical solutions. By comparing these solutions, we demonstrate that the numerical solution closely matches the exact solution, confirming the efficiency of the presented method.

Future research could focus on enhancing the accuracy and computational efficiency of the Hermite combination method by incorporating adaptive algorithms or higher-order interpolation techniques.

A comparative study with other numerical methods for solving Fredholm-Volterra integral equations, such as Galerkin methods, or spline-based approaches, could provide deeper insights into the relative strengths and weaknesses of the Hermite combination method.

Exploring the applicability of the Hermite combination method to other types of integral equations, such as non-linear Fredholm or Volterra integral equations, may reveal new potential uses and limitations.

Incorporating parallel computing techniques into the Hermite combination method could significantly reduce computation time, making it more feasible for large-scale problems.

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