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Galerkin Solutions for the Saint-Venant Torsion of Prismatic Bars with Rectangular Cross-sections

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ABSTRACT

In this work, the Saint-Venant torsion problem of prismatic bars with rectangular cross-sections was presented as a boundary value problem (BVP) of the theory of elasticity. The governing partial differential equation was formulated and shown to be a Poisson equation in terms of the Prandtl stress functions. The Poisson equation governing the Saint-Venant torsion problem was expressed in variational form using Galerkin variational method. The trial function that apriori satisfies the boundary conditions was chosen as a trigonometric cosine series of infinite terms; and in terms of unknown undetermined coefficients or parameters. The unknown parameters were determined by solving the Galerkin variational integral; thus fully determining the Prandtl stress function. The shear stresses were then determined. The maximum shear stress was also obtained. The moment of the cross-section was determined and found to depend on non – dimensional torsional parameters F1(a/b). The maximum shear stress was also found to depend on dimensionless torsional parameters F2(a/b) which were determined and tabulated. It was found that the solutions obtained using the Galerkin method were mathematically closed form solutions because the exact shape functions were used to approximate the trial solution. Expressions obtained for the Prandtl stress function, shear stresses and moment of cross-section were exact and agreed with solutions in the technical literature.

1. INTRODUCTION

When a torque is applied to a beam with non-circular cross-section, the cross-section rotates about the longitudinal axis of the beam and simultaneously undergoes a significant distortion. The cross-section thus undergoes both twisting and warping deformations [1-8]. Such problems are formulated using theory of elasticity principles. The foundational assumptions of the formulation are the strain-displacement (kinematic) relations of infinitesimal/small deformation assumptions, the stress-strain laws, the differential equations of equilibrium and compatibility requirements. Saint-Venant formulated the problem using theory of elasticity and Prandtl solved the problem in terms of Prandtl's stress functions.

Prandtl's formulation of the Saint-Venant torsion problem leads to a Poisson type partial differential equation (PDE); which can be solved using analytical or numerical methods. Available analytical techniques include the method of separation of variables, eigenfunction expansion methods, integral transform methods and Green's function methods. The numerical methods that can be used to solve the torsion problem are the numerical methods available for solving the boundary value problems (BVP) in engineering. Some of the numerical techniques for solving BVP which are applicable to the Poisson type PDE are Galerkin's variational method, Extended Galerkin's variational method, Ritz's method, Finite Element method (FEM), Finite difference method (FDM); and Boundary element methods (BEM) [1–13].

In this work, the Saint-Venant problem of torsion of prismatic bars with rectangular cross-section will be formulated in variational form, and solved using the Galerkin variational method.

Research aim and objectives

The research aim is to use the Galerkin variational method to solve the Saint-Venant torsion problem for prismatic bars with rectangular cross-sections. The specific objectives are:

(1) to formulate the Saint-Venant torsional problem for prismatic bars with rectangular cross-sections using theory of elasticity principles and techniques

(2) to show that the formulated Saint-Venant torsional problem is governed by a partial differential equation called the Poisson equation when expressed in terms of the Airy's and Prandtl's stress functions of elasticity

(3) to express the boundary value problem (BVP) in variational form using the Galerkin variational method

(4) to solve the Galerkin variational equation for the problem and thus obtain solutions for the unknown parameters in the assumed (trial) solutions for the Airy's or Prandtl's stress function

(5) to obtain analytical expressions for the torque, shear stresses and torsional parameters

(6) to show that Galerkin's variational method can be used to obtain closed form mathematical solutions to the BVP of Saint-Venant torsion for prismatic bars with rectangular cross-sections.

2. THEORETICAL FRAMEWORK

The study considered an isotropic, homogeneous long bar with prismatic cross-section denoted by R^2 on the yz coordinate plane. The longitudinal axis is coincident with the x – Cartesian coordinate axis. The bar is fixed at $x = 0$. The end at $x = l$ is subject to a torsional moment which twists it by an angle $l\theta'$ where θ' is the twist rate and l is the length of the bar [14–21]. Other relevant literature can be found in Roohi et al. [22] and Heydari et al. [23].

The assumptions of the formulation are as follows:

(1) the cross-sections in the yz coordinate plane undergo rotation as a rigid body. For non-circular cross-sections, the cross-section will experience twisting. It is deflected in the x – coordinate direction

(2) the deflection and twist rate are constant along the longitudinal axis of the bar. This renders the problem a two-dimensional (2D) problem in the theory of elasticity

(3) the material of the bar is isotropic and homogeneous.

2.1 Displacement field

The three dimensional (3D) Cartesian components of the displacement field, following Saint-Venant hypothesis [14–21] are given by:

$$u(x, y, z) = \theta'\varphi(y, z) = \beta(y, z) \tag{1}$$

$$v(x, y, z) = -\theta' xz = -\beta xz \tag{2}$$

$$w(x, y, z) = -\theta' xy = \beta xy \tag{3}$$

where $\varphi(y, z)$ is an unknown function which is used to define the deflection and is a function of the y and z Cartesian coordinate variables. u, v, and w are the components of displacement in the x, y and z Cartesian coordinate directions, respectively. $\beta = \theta'$ is the twist rate.

2.2 Strain field

Using the strain-displacement equations for infinitesimally small deformation, the strain fields are obtained as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \tag{4}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{5}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \tag{6}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \beta\left(\frac{\partial \varphi}{\partial y} - z\right) \tag{7}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \beta\left(\frac{\partial \varphi}{\partial z} + y\right) \tag{8}$$

$$\gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\beta x + \beta x = 0 \tag{9}$$

ε_{xx}, ε_{yy}, ε_{zz} are the normal strains while γ_{xy}, γ_{yz} and γ_{xz} are the shear strains. Thus, the strain – compatibility equation becomes:

$$\frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial}{\partial y}\beta\left(\frac{\partial \varphi}{\partial z} + y\right) - \frac{\partial}{\partial z}\beta\left(\frac{\partial \varphi}{\partial y} - z\right) = \beta\left(\frac{\partial^2 \varphi}{\partial y \partial z} + 1 - \frac{\partial^2 \varphi}{\partial z \partial y} - 1\right) = 2\beta \tag{10}$$

$$\text{Provided } \frac{\partial^2 \varphi}{\partial y \partial z} = \frac{\partial^2 \varphi}{\partial z \partial y} \tag{11}$$

Eq. (11) implies that $\varphi(y, z)$ is required to be a continuous function of y and z.

2.3 Stress fields

The generalized Hooke's law of linear isotropic elasticity is given generally by

$$\tau_{ij} = \lambda \partial_{ij}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2G\varepsilon_{ij} = \lambda \partial_{ij}\varepsilon_v + 2G\varepsilon_{ij} \tag{12}$$

where $\partial_{ij} = 1$ for $i = j$; $\partial_{ij} = 0$ for $i \neq j$

λ and G are the Lamé's constants G is the shear modulus. ε_v is the volumetric strain.

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0 \tag{13}$$

The stress fields are given by

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \tag{14}$$

$$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy} = \beta G\left(\frac{\partial \varphi}{\partial y} - z\right) \tag{15}$$

$$\tau_{xz} = G\gamma_{xz} = 2G\varepsilon_{xz} = \beta G\left(\frac{\partial \varphi}{\partial z} + y\right) \tag{16}$$

$$\tau_{yz} = G\gamma_{yz} = 0 \tag{17}$$

where σ_{xx}, σ_{yy}, σ_{zz} are normal stresses τ_{xy}, τ_{yz}, τ_{xz} are shear stresses.

2.4 Differential equations of equilibrium

The differential equations of equilibrium in the absence of body forces f_i given in general by Eq. (18).

$$\sum_j \partial_j \tau_{ij} = f_i = 0 \tag{18}$$

Simplify to become Equations (19)-(21).

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{19}$$

$$\frac{\partial \tau_{xy}}{\partial x} = 0 \tag{20}$$

$$\frac{\partial \tau_{xz}}{\partial x} = 0 \tag{21}$$

2.5 Prandtl's stress function $\phi(y, z)$

Prandtl defined stress functions $\phi(y, z)$ which are functions of the y and z coordinate variables of the cross-section, and independent of x such that the differential equations of equilibrium are satisfied by the non-vanishing stress components τ_{xy} and τ_{xz} as follows:

$$\tau_{xy} = G\beta \frac{\partial \phi}{\partial z}(y, z) \tag{22}$$

$$\tau_{xz} = -G\beta \frac{\partial \phi}{\partial y}(y, z) \tag{23}$$

It is observed that for Prandtl's stress functions Eqns. (22) and (23), Eq. (19) becomes:

$$\frac{\partial}{\partial y}\left(G\beta \frac{\partial \phi}{\partial z}\right) + \frac{\partial}{\partial z}\left(-G\beta \frac{\partial \phi}{\partial y}\right) = G\beta \frac{\partial^2 \phi}{\partial y \partial z} - G\beta \frac{\partial^2 \phi}{\partial z \partial y} = 0 \tag{24}$$

if

$$\frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y} \tag{25}$$

Prandtl's stress functions are solutions of the differential equation of equilibrium if the functions are continuous. The strain components are given in terms of the Prandtl stress function $\phi(x, z)$ as:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \beta \frac{\partial \phi}{\partial z} \tag{26}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = -\beta \frac{\partial \phi}{\partial y} \tag{27}$$

Then the strain compatibility equation is

$$\frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} = -\beta \frac{\partial^2 \phi}{\partial y^2} - \beta \frac{\partial^2 \phi}{\partial z^2} = -\beta\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) \tag{28}$$

From Eq. (10), Eq. (25) can be expressed as:

$$-\beta\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) = 2\beta \tag{29}$$

Simplifying,

$$\Delta\phi = \frac{\partial^2 \phi(y, z)}{\partial y^2} + \frac{\partial^2 \phi(y, z)}{\partial z^2} = \nabla^2 \phi(y, z) = -2 \tag{30}$$

where

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{31}$$

Δ or ∇^2 is the Laplace differential operator.

2.6 Boundary condition

The boundary condition for the Prandtl stress function for a cross-sectional profile with no holes is

$$\phi(y, z) = 0 \tag{32}$$

on the boundary Γ.

2.7 Torque, section moment and shear stresses

The torque or torsional moment, M is computed as the double integral over the cross-section

$$M = \iint_{R^2} (-\tau_{xy} z + \tau_{xz} y) dydz \tag{33}$$

where R^2 is the cross-section of the bar.

$$M = \iint_{R^2} \left(-G\beta \frac{\partial \phi}{\partial z} z - G\beta \frac{\partial \phi}{\partial y} y\right) dydz \tag{34}$$

$$M = -G\beta \iint_{R^2} \left(\frac{\partial \phi}{\partial z} z + \frac{\partial \phi}{\partial y} y\right) dydz \tag{35}$$

Using the method of integration by parts,

$$\iint_{R^2} \frac{\partial \phi}{\partial z} z \, dydz = \int_{\Gamma} \phi z \, n_z \, ds - \iint_{R^2} \phi(y, z) \, dydz = -\iint_{R^2} \phi(y, z) \, dydz \tag{36}$$

$$\iint_{R^2} \frac{\partial \phi}{\partial y} y \, dydz = -\iint_{R^2} \phi(y, z) \, dydz \tag{37}$$

$$M = 2G\beta \iint_{R^2} \phi(y, z) \, dydz = G\beta J \tag{38}$$

where

$$J = 2\iint_{R^2} \phi(y, z) dydz \tag{39}$$

J is the moment of the cross-section, or torsional constant. The modulus of the shear stress is

$$|T| = \left(\tau_{xy}^2 + \tau_{xz}^2\right)^{1/2} \tag{40}$$

3. METHODOLOGY

For a rectangular cross-section on the yz Cartesian coordinate plane defined by

$$-\frac{a}{2} \leq y \leq \frac{a}{2}; -\frac{b}{2} \leq z \leq \frac{b}{2}$$

where $a \geq b > 0$ the Prandtl stress function that satisfies the boundary condition Eq. (32) is assumed in terms of the unknown parameters C_{mn} as the infinite series:

$$\phi(y,z) = \sum_{m}^{\infty}\sum_{n}^{\infty} C_{mn}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b} \quad (41)$$

$m = 1, 3, 5, 7, 9, \ldots$; $n = 1, 3, 5, 7, 9, \ldots$

Since

$$\phi\left(y = \pm\frac{a}{2}, z\right) = \phi\left(y, z = \pm\frac{b}{2}\right) = 0 \quad (42)$$

The Galerkin variational integral becomes:

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (\nabla^2\phi + 2)\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz = 0 \quad (43)$$

Expanding,

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left\{\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\sum_{m}^{\infty}\sum_{n}^{\infty} C_{mn}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\right)+2\right\}\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz = 0 \quad (44)$$

4. RESULTS

The Galerkin variational integral is

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\sum_{m}^{\infty}\sum_{n}^{\infty}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(c_{mn}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\right)\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} -2\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz \quad (45)$$

$$\sum_{m}^{\infty}\sum_{n}^{\infty} -C_{mn}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} -2\cos\frac{n'\pi z}{b}\cos\frac{m'\pi y}{a}\,dydz \quad (46)$$

-2 is expanded in Fourier cosine series as:

$$-2 = \sum_{m}^{\infty}\sum_{n}^{\infty} a_{mn}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b} \quad (47)$$

where

$$a_{mn} = \frac{4}{ab}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(-2)\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\,dydz \quad (48)$$

$$a_{mn} = \frac{4}{ab}\cdot -2\cdot\frac{2a}{m\pi}(-1)^{\frac{m-1}{2}}\frac{2b}{n\pi}(-1)^{\frac{n-1}{2}} \quad (49)$$

$$a_{mn} = \frac{(-2)4^2}{mn\pi^2}(-1)^{\frac{m+n-2}{2}} = \frac{(-2)4^2(-1)^{\frac{m+n}{2}-1}}{mn\pi^2} \quad (50)$$

So,

$$-2 = \sum_{m}^{\infty}\sum_{n}^{\infty}(-1)^{\frac{m+n}{2}-1}\frac{(-2)4^2}{mn\pi^2}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b} \quad (51)$$

So,

$$\sum_{m}^{\infty}\sum_{n}^{\infty} -C_{mn}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz = \sum_{m}^{\infty}\sum_{n}^{\infty}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\frac{(-1)^{\frac{m+n}{2}-1}(-2)(4)^2}{mn\pi^2}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\cos\frac{m'\pi y}{a}\cos\frac{n'\pi z}{b}\,dydz \quad (52)$$

$$C_{mn} = \frac{(2)4^2(-1)^{\frac{m+n-2}{2}}}{mn\pi^2}\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^{-1} = \frac{2^5(-1)^{\frac{m+n}{2}-1}}{mn\pi^2}\left(\frac{1}{\pi^2\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)}\right) = \frac{2^5(-1)^{\frac{m+n}{2}-1}}{mn\pi^4}\left(\frac{1}{\left(\frac{b^2m^2+n^2a^2}{a^2b^2}\right)}\right) \quad (53)$$

$$C_{mn} = \frac{2^5(-1)^{\frac{m+n}{2}-1}a^2b^2}{\pi^4}\frac{1}{mn(m^2b^2+n^2a^2)} \quad (54)$$

The Prandtl stress function is then:

$$\phi(y,z) = \frac{2^5a^2b^2}{\pi^4}\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{(-1)^{\frac{m+n}{2}-1}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}}{mn(m^2b^2+n^2a^2)} \quad (55)$$

$\phi(y, z)$ is obtained as a trigonometric cosine series of infinite terms. The series is a convergent series since

$|C_{mn}\cos\frac{m\pi y}{a}\cos\frac{m\pi z}{b}| \leq |C_{mn}| \leq \text{constant}\left(\frac{1}{m^2n^2}\right)$ and, $\sum_m^\infty\sum_n^\infty\frac{1}{m^2n^2} = \left(\sum_m^\infty\frac{1}{m^2}\right)\left(\sum_n^\infty\frac{1}{n^2}\right)$

4.1 Moment of the cross-section (torsional constant, *j*)

From Eq. (39),

$$J = 2\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\frac{2^5a^2b^2}{\pi^4}\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{(-1)^{\frac{m+n}{2}-1}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}}{mn(m^2b^2+n^2a^2)}\,dydz \quad (56)$$

$$J = \sum_{m}^{\infty}\sum_{n}^{\infty}\frac{2^6a^2b^2}{\pi^4}\frac{(-1)^{\frac{m+n}{2}-1}}{mn(m^2b^2+n^2a^2)}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\,dydz \quad (57)$$

$$J = \frac{2^8a^3b^3}{\pi^6}\sum_{m}^{\infty}\sum_{n}^{\infty}\frac{1}{m^2n^2(m^2b^2+n^2a^2)} \quad (58)$$

Let

$$\frac{a}{b} = r \tag{59}$$

or

$$a = br \tag{60}$$

then,

$$J = \frac{2^8 b^6 r^3}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 (m^2 b^2 + n^2 b^2 r^2)} \tag{61}$$

$$J = \frac{2^8 b^6 r^3}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 b^2 (m^2 + n^2 r^2)} \tag{62}$$

$$J = \frac{2^8 b^4 r^3}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 (m^2 + n^2 r^2)} \tag{63}$$

Alternatively,

$$J = \frac{2^8 a^6}{r^3 \pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 \left(\frac{m^2 a^2}{r^2} + n^2 a^2 \right)} \tag{64}$$

$$J = \frac{2^8 a^6}{r^3 \pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 \left(\frac{m^2 a^2 + n^2 a^2 r^2}{r^2} \right)} \tag{65}$$

$$J = \frac{2^8 a^6}{r^3 \pi^6} \sum_m^\infty \sum_n^\infty \frac{r^2}{m^2 n^2 a^2 (m^2 + n^2 r^2)} \tag{66}$$

$$J = \frac{2^8 a^6}{r \pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 a^2 (m^2 + n^2 r^2)} \tag{67}$$

$$J = \frac{2^8 a^4}{r \pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 (m^2 + n^2 r^2)} \tag{68}$$

or

$$J = \frac{2^8 a^3 b^3}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{b^2 m^2 n^2 \left(\frac{m^2 b^2 + n^2 a^2}{b^2} \right)} \tag{69}$$

$$J = \frac{2^8 a^3 b}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{1}{m^2 n^2 \left(m^2 + \frac{n^2 a^2}{b^2} \right)} \tag{70}$$

$$J = \frac{2^8}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{(a/b)^2}{m^2 n^2 \left(m^2 + \frac{n^2 a^2}{b^2} \right)} \cdot ab^3 \tag{71}$$

$$J = F_1\left(\frac{a}{b}\right) ab^3 \tag{72}$$

where

$$F_1\left(\frac{a}{b}\right) = \frac{2^8}{\pi^6} \sum_m^\infty \sum_n^\infty \frac{(a/b)^2}{m^2 n^2 \left(m^2 + \frac{n^2 a^2}{b^2} \right)} \tag{73}$$

4.2 Shear stress tensors

The shear stresses are found from the Prandtl stress function as:

$$\tau_{xy}(y,z) = G\beta \frac{\partial \phi}{\partial z}(y,z) = G\beta \frac{2^5 a^2 b^2}{\pi^4} \sum_m^\infty \sum_n^\infty \frac{(-1)^{\frac{m+n}{2}-1}}{mn(m^2 b^2 + n^2 a^2)} \cos\frac{m\pi y}{a} \frac{\partial}{\partial z} \cos\frac{n\pi z}{b} \tag{74}$$

$$\tau_{xz}(y,z) = -G\beta \frac{\partial \phi}{\partial y}(y,z) = -G\beta \frac{2^5 a^2 b^2}{\pi^4} \sum_m^\infty \sum_n^\infty \frac{(-1)^{\frac{m+n}{2}-1}}{mn(m^2 b^2 + n^2 a^2)} \cos\frac{n\pi z}{b} \frac{\partial}{\partial y} \left(\cos\frac{m\pi y}{a} \right) \tag{75}$$

$$\tau_{xy}(y,z) = \frac{\beta G 2^5 a^2 b}{\pi^3} \sum_m^\infty \sum_n^\infty \frac{(-1)^{\frac{m+n}{2}}}{m(m^2 b^2 + n^2 a^2)} \cos\frac{m\pi y}{a} \sin\frac{n\pi z}{b} \tag{76}$$

$m = 1, 3, 5, 7, 9, \ldots$; n = 1, 3, 5, 7, 9, …

$$\tau_{xz}(y,z) = -G\beta \frac{2^5 ab^2}{\pi^3} \sum_m^\infty \sum_n^\infty \frac{(-1)^{\frac{m+n}{2}}}{n(m^2 b^2 + n^2 a^2)} \sin\frac{m\pi y}{a} \cos\frac{n\pi z}{b} \tag{77}$$

$m = 1, 3, 5, 7, 9, \ldots$; n = 1, 3, 5, 7, 9, …

The maximum shear stress τ_{max} is found as:

$$\tau_{max} = \frac{G\beta 2^5 a^2}{\pi^3 b} \sum_m^\infty \sum_n^\infty \left\{ \frac{(-1)^{\frac{m-1}{2}}}{m\left(m^2 + n^2 \frac{a^2}{b^2} \right)} \right\} \tag{78}$$

$$\tau_{max} = G\beta ab^2 F_2\left(\frac{a}{b}\right) \tag{79}$$

where

$$F_2\left(\frac{a}{b}\right) = F_1\left(\frac{a}{b}\right) \frac{\pi^3}{2^5 \left(\frac{a}{b}\right)^2} \left\{ \sum_m^\infty \sum_n^\infty \frac{(-1)^{\frac{m-1}{2}}}{m\left(m^2 + \frac{n^2 a^2}{b^2} \right)} \right\} \tag{80}$$

The non-dimensional torsional parameters $F_1(\frac{a}{b})$ and $F_2(\frac{a}{b})$ for the Saint Venant torsion of prismatic bars with rectangular cross-sections are tabulated as Tables 1 and 2 for various values of the ratio a/b for the present study and $\bar{F}_1, \bar{F}_2$ are for results from Jan Francu et al. [3].

Table 1. Variation of torsional parameter F_1 with a/b

$r = a/b$	$F_1(a/b)$	$\overline{F_1}(a/b)$ [3]
1	0.1406	0.141
1.2	0.1661	
1.5	0.1958	0.196
2	0.2287	0.229
2.5	0.2494	
3	0.2633	0.263
4	0.2808	0.281
5	0.2913	0.291
6	0.298	0.298
8	0.307	0.307
10	0.3123	0.312
∞	1/3	1/3

Table 2. Variation of torsional parameter F_2 for with a/b for Saint Venant torsion of bar with rectangular section

$r = a/b$	$F_2(a/b)$	$\overline{F_2}(a/b)$ [3]
1	0.208	0.208
1.5	0.231	0.231
2	0.246	0.246
3	0.267	0.267
4	0.282	0.282
5	0.292	0.292
6	0.299	0.299
8	0.307	0.307
10	0.313	0.313
∞	1/3	1/3

The deflection function φ(y, z) is obtained from solving the following equations obtained from Eqns. (7), (8), (26) and (27):

$$\frac{\partial \varphi}{\partial y} - z = \frac{\partial \phi}{\partial z} \tag{81}$$

and

$$\frac{\partial \varphi}{\partial z} + y = -\frac{\partial \phi}{\partial y} \tag{82}$$

Thus,

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \phi}{\partial z} + z \tag{83}$$

or,

$$\frac{\partial \varphi}{\partial z} = -\frac{\partial \phi}{\partial y} - y \tag{84}$$

By integration of Eq. (83) we obtain Eq. (85) as φ(y, z)

$$\varphi(y,z) = \frac{2^5 a^3 b}{\pi^4} \sum_m^{\infty} \sum_n^{\infty} \frac{(-1)^{\frac{m+n}{2}} \sin\frac{m\pi y}{a} \sin\frac{n\pi y}{b}}{m^2(m^2b^2 + n^2a^2)} + yz \tag{85}$$

4.3 Numerical problem

A numerical problem to illustrate the validity of the results obtained in this study considers the calculation of torsional constant J, given by the expressions in Eqns. (72) and (73) where Eq. (73) is presented as Table 1 in terms of a/b. The torsional contant is important in torsion problems since it determines the torsional rigidity D_t as follows:

$$D_t = GJ \tag{86}$$

We compare our results with results from Jan Francu et al. [3] who presented a Navier series solution of the torsion problem leading to results that are identical with the results from the present study which employed the Galerkin method.

Numerical solutions are presented for various values of a and b as follows and compared with results from Jan Francu et al. [3].

Table 3. Results for torsional stiffness for various cross-sections, and comparison with results from Jan Francu et al. [3]

Cross-section		a/b	F_1	$J=ab^3F_1$ Present study	F_1	$J = ab^3\overline{F_1}$ Jan Francu et al [3]
a (cm)	b (cm)			(cm^4)		(cm^4)
2	2	1	0.1406	2.2496	0.141	2.256
4	2	2	0.2287	7.3184	0.229	7.328
6	2	3	0.2633	12.6384	0.263	12.624
8	2	4	0.2808	17.9712	0.281	17.984
10	2	5	0.2913	23.304	0.292	23.28
12	2	6	0.298	28.604	0.298	28.604
16	2	8	0.307	39.296	0.307	39.296
20	2	10	0.3123	49.968	0.312	49.92
∞	2	∞	1/3	∞	1/3	∞

5. DISCUSSION

This work has successfully presented the Saint-Venant torsion problem of prismatic bars with rectangular cross-section as a boundary value problem (BVP) of the theory of elasticity using Prandtl's stress function ϕ(y, z). The resulting BVP was observed to be a Poisson type partial differential equation in terms of the Prandtl's stress functions. The basis (shape) functions that satisfies the boundary conditions given in terms of trigonometric (cosine) functions; and the assumed (trial) Prandtl stress function used was given as Eq. (45) – a double trigonometric cosine series of infinite terms.

The Galerkin variational formulation of the Saint-Venant torsion equation was obtained as Eq. (45). The unknown parameters of the Galerkin formulation was obtained by solving the Galerkin variational statement of the Poisson equation as Eq. (54). The Prandtl stress function was thus completely determined as Eq. (55), which was found to be a rapidly convergent double trigonometric cosine series with infinite terms.

The moment of the cross-section was obtained in terms of the ratio of the cross-sectional dimensions (a/b) as Eqns. (71), and (72) where Eq. (72) is expressed in terms of the non-dimensional torsion parameter, $F_1(a/b)$ $F_1(a/b)$ is found to depend on the ratio (a/b) as Eq. (73). Values of $F_1(a/b)$ for various values of a/b were calculated and shown in Table 1.

The non-vanishing shear stress fields were found as Eqns. (76) and (77). The maximum shear stress was obtained as Eq.

(78) and presented in terms of the dimensionless torsion parameter $F_2(a/b)$ as Eq. (79). The dimensionless torsion parameter $F_2(a/b)$ was calculated for various values of (a/b) and presented as Table 2.

The numerical results obtained for the torsional constant J for various values of the cross-sectional dimensions were identical with the results obtained by Jan Francu et al. [3] who used Navier series method.

6. CONCLUSIONS

The conclusions of this study are as follows:

(1) the Galerkin variational method has been successfully used to present the Poisson equation for the Saint-Venant torsion of prismatic bars with rectangular cross-section in variational form.

(2) the Galerkin solutions yielded mathematically closed form and exact solutions to the Saint-Venant torsion problem of prismatic bars with rectangular cross-section.

(3) Exact solutions were obtained because the shape functions used were exact shape functions which apriori satisfied all the boundary conditions.

(4) the exact solutions obtained for the Prandtl stress function, the shear stresses and moment of the cross-section were convergent series with infinite terms.

(5) the solutions obtained were closely similar to the solutions obtained by Jan Francu et al. [3] who used Navier series method.

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NOMENCLATURE

x, y, z	Cartesian coordinates in three dimensions
$\left.\begin{array}{l} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{array}\right\}$	Displacement field components in the x,
y, and z	Cartesian coordinate directions
$\theta' = \beta$	twist rate
$\varphi(y, z)$	unknown function related to deflection and used to define the deflection
ε_{xx}, ε_{yy}, ε_{zz}	normal strains
γ_{xy}, γ_{xz}, γ_{yz}	shear strains
λ	Lamé's content
G	shear modulus or modulus of rigidity
D_t	torsional rigidity
ε_v	volumetric strain
∂_{ij}	Kronecker's delta
τ_{ij}	stress using indicial notation
ε_{ij}	strain using indicial notation
σ_{xx}, σ_{yy}, σ_{zz}	normal stresses
τ_{xy}, τ_{xz}, τ_{yz}	shear stresses
$\phi(y, z)$	Prandtl's stress function
R^2	cross-section of the bar
M	torque, torsional moment
Γ	boundary of the cross-section
J	moment of the cross-section, St Venant torsional constant
m, n, m', n'	integers
a,b	in-plane dimensions (length and width)
C_{mn}	unknown parameter of the Prandtl's stress function
a_{mn}	cosine series parameter
r	aspect ratio
$\left.\begin{array}{l} F_1(a/b) \\ F_2(a/b) \end{array}\right\}$	dimensionless torsion parameters
$\left.\begin{array}{l} \bar{F}_1(a/b) \\ \bar{F}_2(a/b) \end{array}\right\}$	dimensionless torsion parameters obtained by Jan Francu et al. [3] using Navier series method
2D	two dimensional
3D	three dimensional
BEM	boundary element method
BVP	boundary value problem
PDE	partial differential equation
FEM	finite element method
FDM	finite difference method

MATHEMATICAL SYMBOLS

$\sum$	summation
$\sum\sum$	double summation
$\int$	integration (integral)
$\iint$	double integration (double integral)
$\frac{\partial}{\partial x}$	partial derivative with respect to x
$\frac{\partial}{\partial y}$	partial derivative with respect to y
$\frac{\partial^2}{\partial x \partial y}$	mixed partial derivative
$\Delta = \nabla^2$	Laplacian