Mathematical Modelling of Engineering Problems
Vol. 11, No. 6, June, 2024, pp. 1672-1678
Journal homepage: http://iieta.org/journals/mmep

Roughness in $S_{|U|}$-Submodules

Sangeetha Sivakumar, Shakeela Sathish*

Department of Mathematics, SRM Institute of Science and Technology, Ramapuram 600089, India

Corresponding Author Email: shakeels@srmist.edu.in

https://doi.org/10.18280/mmep.110630

Received: 30 November 2023
Revised: 22 February 2024
Accepted: 15 March 2024
Available online: 22 June 2024

Keywords:
rough action, rough $S_{|U|}$-submodule, rough σ-stabilize

ABSTRACT

A rough set theory (RST) was developed by Zdzislaw Pawlak to handle vagueness and uncertainty in data analysis. An approximation of a vague concept consists of two precise concepts a lower and an upper approximation. These approximations are two basic operations in rough set theory. An upper approximation contains all objects that may possibly belong to a concept, and a lower approximation contains all objects that certainly belong. The boundary region is the difference between the upper and lower approximations. Thus, rough set theory expresses vagueness by using a boundary region of a set rather than by using membership. By using the pair of sets, rough set theory extends traditional set theory by defining a subset of a universe. The properties of any set can be clearly understood if an algebraic structure is developed. This paper considers an approximation space with a finite universe and introduces a rough action by a symmetric group $S_{|U|}$ acting on all rough sets in this space. Also, we proved that the number of orbits of the symmetric group $S_{|U|}$ in rough sets is one. We then introduced the $S_{|U|}$-submodule and proved that the kernel of rough homomorphism is a rough $S_{|U|}$ submodule. An example of how rough action can be used to find missing values in sample cancer data has also been provided.

1. INTRODUCTION

Numerous mathematical concepts are delivered exclusively using set theory as a key method of presenting all of mathematics. Rough sets were introduced by Pawlak [1]. With the help of lower and upper approximations, this theory can approximate a subset of a universe. The two main approaches for developing rough set theory are constructive and axiomatic. Using constructive methods, primitive notions, such as binary relations on universes, partitions of universes and neighborhood systems, are used to construct lower and upper approximation operators. In contrast, the axiomatic approach focuses on the primitive notions of upper and lower approximation operators, which are appropriate for examining rough set algebras.

An algebraic structure provides a rigorous framework for analyzing and manipulating rough sets by formalizing operations and relationships. The concept of an approximation space is a fundamental algebraic structure in rough set theory, which consists of a universe of discourse, attributes, and binary relations that define the indiscernibility of objects. Furthermore, algebraic structures, including semigroups, monoids, and groups, have been employed in the study of rough sets, particularly within the context of algebraic approaches. By means of these algebraic systems, it becomes possible to study the properties and relationships of rough sets and their connections with other mathematical ideas. Thus, the algebraic structures of rough set theory provide a useful mathematical framework for studying the properties, relationships, and operations within rough sets, offering insights into the nature of uncertainty and approximation in data analysis and knowledge discovery.

The algebraic aspects of rough set theory are proposed by Bonikowski [2]. The structures of the lower and upper approximations based on arbitrary binary relations was given by Liu and Zhu [3]. Rough sets and their properties were applied to rings, modules, semi-groups, groups, ideals, and graphs [4-8]

According to Pomykala and Pomykala [9], rough sets form Stone algebra. Comer [10] discussed several algebras related to algebraic logic, including stone algebras and relation algebras. By considering the upper approximation, Biswas and Nanda [11] gave the concept rough subgroups. A rough ideal into a semigroup was introduced by Kuroki [12] in 1997, containing rough left and right ideals with appropriate examples. With respect to normal subgroups Kuroki and Wang [13] explored lower and upper approximations. The roughness of gamma subsemigroups and ideals in gamma-semigroups were discussed by Jun [14]. In addition, rough ideals are discussed as a generalization of ideals in BCK-algebras by Jun [15].

A topological approach was given by Al-Shami [16] to generate new rough set models. Through ideals, Guler et al. [17] provided rough approximations based on different topologies. The concept of generalized rough approximation spaces based on maximal neighborhoods and ideals was discussed by Hosny et al. [18]. A power set approximation operator for a given set was defined by Mordeson in 2001 [19] using covers of the universal set. The roughness based on fuzzy ideals given by Davvaz [20]. An introduction to rough prime ideals and rough fuzzy prime ideals in a semigroup was made by Xiao and Zhang in 2006 [21]. Sangeetha and Sathish [22] defined rough groups using upper and lower approximations to rough sets within a finite universe. Bağırmaz et al. [23] introduced the notion of topological rough groups, and Altassan et al. [24] introduced rough action on topological rough groups.

In this paper, we present a rough action by a symmetric group $S_{|U|}$ acting on all rough sets in an approximation space with a finite universe. Moreover, we prove that number of $S_{|U|}$ orbits in rough sets is 1. This led us to introduce a rough $S_{|U|}$ submodules and prove results related to some homomorphisms. We have also provided an example of using rough action to find missing values in cancer sample data.

A brief review of rough set theory and rough groups is provided in Section 2 of this paper. In Section 3, we presented rough actions and their properties along with suitable examples. Our discussion of rough $S_{|U|}$ submodules cover Section 4. An example of rough action is provided in Section 5. Conclusion explains the relevance of this work.

2. BASICS OF ROUGH SET THEORY

Approximation space, lower and upper approximations of a given set and results relating to approximations and rough groups are discussed in this section.

Definition 2.1 [1]

Approximation space is composed of a finite set "Λ" ($\neq \phi$) called universe set along with "ζ"an equivalence relation on "Λ" and is represented by $K = ($"Λ", "ζ"$)$.

Definition 2.2 [1]

A family of subsets $E = \{E_1, E_2, \mathrm{E} \dots \dots E_n\}$ of "Λ" are said to be a classification of "Λ" if

- $E \cup E_2 \cup \dots . \cup E_n =$ "Λ"
- $E_i \cap E_j = \phi$, for $i \neq j$

Definition 2.3 [1]

Consider an approximation space $K = ($"Λ", "ζ"$)$. And A is any subset of Λ, then

- "Λ"$^A = \{a_i | [a_i]_\zeta \cap A \neq \phi\}$
- "Λ"$_A = \{a_i | [a_i]_\zeta \subseteq A\}$
- $BN_A =$ "Λ"$^A -$ "Λ"$_A$

are called approximations of upper, lower & boundary region of A with respect to "ζ", respectively, and A is said to be rough if BN_A is non empty otherwise it is crisp.

Definition 2.4 [1]

If $A, B \subseteq$ "Λ", then the following results are due to [1]:

- "Λ"$_A \subseteq A \subseteq$ "Λ"^{A}
- "Λ"$_{A \cap B} =$ "Λ"$_A \cap$ "Λ"$_B$
- "Λ"$_{A \cup B} \supseteq$ "Λ"$_A \cup$ "Λ"$_B$
- "Λ"$^{A \cup B} =$ "Λ"$^A \cup$ "Λ"^{B}
- "Λ"$^{A \cap B} \subseteq$ "Λ"$^A \cap$ "Λ"^{B}
- $A \subseteq B \Longrightarrow$ "Λ"$_A \subseteq$ "Λ"$_B$&"Λ"$^A \subseteq$ "Λ"^{B}

Definition 2.5 [25] **Group**

Groups are non-empty sets with binary operation $*$ that satisfy closure, associativity, identity, and inverse properties under $*$.

Definition 2.6 [25] **Power Set**

Collection of all possible subsets of G forms a Power set represented by 2^G which forms an abelian group along with operation $\triangle$.

Definition 2.7 [22] **Rough Group**

(U, R) be an approximation space, where U has n elements ($n \in N$). $\left(2^{|U|}, \triangle\right)$ is an abelian group, and $R(U)$, a collection rough sets in U with respect to R. $R(U)$ is a rough group if $\overline{R(U)} \cup \underline{R(U)}$ with respect to $\triangle$ forms a subgroup of $(2^U, \triangle)$ and represented by ro_g.

Theorem 2.1 [22]

If ro_{g_1} and ro_{g_2} are two rough groups then $ro_{g_1} \cap ro_{g_2}$ is also rough group.

Theorem 2.2 [22]

If ro_{g_1} and ro_{g_2} are two rough groups then $\overline{ro_{g_1} \cap ro_{g_2}} \subseteq ro_{g_1} \cap ro_{g_2}$.

Theorem 2.3 [22]

If ro_{g_1} and ro_{g_2} are two rough groups then $\overline{ro_{g_1} \cup ro_{g_2}} \subseteq ro_{g_1} \cup ro_{g_2}$.

Theorem 2.4 [22]

If ro_{g_1} and ro_{g_2} are two rough groups then $\underline{ro_{g_1} \cap ro_{g_2}} = ro_{g_1} \cap ro_{g_2}$.

3. ROUGH ACTION

This section introduces rough action, rough stabilizer, rough orbit, and rough homomorphism.

Definition 3.1 [25] ***G*-Set**

Let G be a group and J be any set. G acts on J if : $G \times J \to J$ given by $(g, j) \to g \cdot j \in J$, also,

$$e \cdot j = j, \forall j \in J;\ g \cdot (h \cdot j) = (g \cdot h) \cdot j$$

Then J is said to be G-Set.

Definition 3.2

Let (U, R) be an approximation space. $S_{|U|}$ be symmetric group on $|U|$ elements. $Ro(U)$ be the set of all rough sets. We say $S_{|U|}$ acts on $Ro(U)$ if : $S_{|U|} \times Ro(U) \to Ro(U)$ where $(\sigma, W) \to \sigma\{W\} = \{\sigma(w_n)\} \in Ro(U)$, such that:

$$e \cdot A_i = A_i, \forall A_i \in Ro(U)$$
$$\sigma_1 \cdot (\sigma_2 \cdot A_i) = (\sigma_1 \sigma_2) \cdot A_i$$

Definition 3.3 Rough *σ*-Stabilizer

$Ro(U_1)$, set of all rough sets in $U_1$with respect to R_1 and $S_{|U|}$ acts on $Ro(U_1)$. Let $X_n \in Ro(U_1)$ then $X^\sigma = \{X_n \in Ro(U_1) | \sigma(x_n) = x_n, \forall x_n \in X_n\}$.

Definition 3.4

If $W \in Ro(U_1)$ then, $\sum_{g\in S_{|U_1|}} |W^g| = |S_{|U_1|}|$. As demonstrated in the following example, this result is true.

Example 3.1

Given $U_1 = \{1,2,3\}$ and R be any equivalence relation on U_1.

Equivalence class of U_1 with respect to R is given by:

$$U_1/R = \{\{1,2\},\{3\}\}$$
$$2^{U_1} = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\phi\}$$

represented as X_i, where $i = 1,2,3,\dots 8$.

The rough sets are given by $Ro(U) = \{X_1, X_2, X_5, X_6\}$, where,

$$X_1 = \{1\}, X_2 = \{2\}, X_5 = \{1,3\}, X_6 = \{2,3\}$$

- $R^{X_1} = \{1,2\} \& R_{X_1} = \phi$
- $R^{X_2} = \{1,2\} \& R_{X_2} = \phi$
- $R^{X_5} = \{1,2,3\} \& R_{X_5} = \{3\}$
- $R^{X_6} = \{1,2,3\} \& R_{X_6} = \{3\}$

$$\overline{R(U)} \cup \underline{R(U)} = \{\phi,\{3\},\{1,2\},\{1,2,3\}\}$$

where,

$$R(U) = \{\{1\},\{2\},\{1,3\},\{2,3\}\},$$
$$\overline{R(U)} = \{\{1,2\},\{1,2,3\}\}$$
$$\underline{R(U)} = \{\phi,\{3\}\}$$

$\left(\overline{R(U)} \cup \underline{R(U)}, \triangle\right)$ forms subgroup of $\left(2^{|U|}, \triangle\right)$.

Table 1. Cayley table of $R(U_1)$

	$\boldsymbol{\phi}$	**{3}**	**{1,2}**	**{1,2,3}**
ϕ	ϕ	{3}	{1,2}	{1,2,3}
{3}	{3}	ϕ	{1,2,3}	{1,2}
{1,2}	{1,2}	{1,2,3}	ϕ	{3}
{1,2,3}	{1,2,3}	{1,2}	{3}	ϕ

Hence $R(U)$ is a rough group (Table 1).

Define $\sigma.X_n = 1$, if $\sigma.x = x, \forall x \in X_n$ otherwise 0 (Table 2).

Table 2. Rough action on $R(U_1)$

	$\boldsymbol{X_1}$	$\boldsymbol{X_2}$	$\boldsymbol{X_6}$	$\boldsymbol{X_7}$
(e)	1	1	1	1
(12)	0	0	0	0
(13)	0	1	0	0
(23)	1	0	0	0
(123)	0	0	0	0
(132)	0	0	0	0

- $X^e = \{X_1, X_2, X_6, X_7\}$
- $X^{\sigma_1} = \phi$
- $X^{\sigma_2} = \{X_2\}$
- $X^{\sigma_3} = \{X_1\}$
- $X^{\sigma_4} = \phi, X^{\sigma_5} = \phi$
- $\sum_{g\in S_{|U|}} |X^g| = 6 = S_{|U|}$

Then we have the following result: The number of $S_{|U|}$ orbits in $R(U)$ is 1 since:

$$\frac{\sum_{g\in S_{|U|}} |X^g|}{S_{|U|}} = \frac{6}{6} = 1$$

Example 3.2

Let $U = \{1,2,3,4\}, R$, an equivalence on U.

$$U/R = \{\{1,2\},\{3\},\{4\}\}$$

The possible subsets of U are {1}, {2}, {3}, {4}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}, {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}, {1,2,3,4}, {} represented as W_i, where i=1,2,…16.

The upper and lower approximation of W_i are given by:

$$R^{W_1} = \{1,2\} \;\&\; R_{W_1} = \{\}$$
$$R^{W_2} = \{1,2\} \;\&\; R_{W_2} = \{\}$$
$$R^{W_3} = \{3\} \;\&\; R_{W_3} = \{3\}$$
$$R^{W_4} = \{4\} \;\&\; R_{W_4} = \{4\}$$
$$R^{W_5} = \{1.2\} \;\&\; R_{W_5} = \{1,2\}$$
$$R^{W_6} = \{1,2,3\} \;\&\; R_{W_6} = \{3\}$$
$$R^{W_7} = \{1,2,4\} \;\&\; R_{W_7} = \{4\}$$
$$R^{W_8} = \{1,2,3\} \;\&\; R_{W_8} = \{3\}$$
$$R^{W_9} = \{1,2,4\} \;\&\; R_{W_1} = \{4\}$$
$$R^{W_{10}} = \{3,4\} \;\&\; R_{W_{10}} = \{3,4\}$$
$$R^{W_{11}} = \{1,2,3\} \;\&\; R_{W_1} = \{1,2,3\}$$
$$R^{W_{12}} = \{1,2,4\} \;\&\; R_{W_{12}} = \{1,2,4\}$$
$$R^{W_{13}} = \{1,2,3,4\} \;\&\; R_{W_{13}} = \{3,4\}$$
$$R^{W_{14}} = \{1,2,3,4\} \;\&\; R_{W_{14}} = \{3,4\}$$
$$R^{W_{15}} = \{1,2,3,4\} \;\&\; R_{W_{15}} = \{1,2,3,4\}$$
$$R^{W_{16}} = \{\} \;\&\; R_{W_{13}} = \{\}$$

The Rough sets are given by:

$$Ro(U) = \{W_1, W_2, W_6, W_7, W_8, W_9, W_{13}, W_{14}\}$$
$$W_1 = \{1\}, W_2 = \{2\}, W_6 = \{1,3\}, W_7 = \{1,4\}$$
$$W_8 = \{1,2,3\}, W_9 = \{2,4\}, W_{13} = \{1,3,4\}, W_{14} = \{2,3,4\}$$
$$\overline{R(U)} = \{\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}$$
$$\underline{R(U)} = \{\phi,\{3\},\{4\},\{3,4\}\}$$
$$\overline{R(U)} \cup \underline{R(U)} = \{\phi,\{3\},\{4\},\{3,4\},\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}$$

$\left(\overline{R(U)} \cup \underline{R(U)}, \triangle\right)$ forms subgroup of $\left(2^{|U|}, \triangle\right)$ (Table 3).

Table 3. Cayley table of rough group $R(U_2)$

$\triangle$	$\boldsymbol{\phi}$	**{3}**	**{4}**	**{3,4}**	**{1,2}**	**{1,2,3}**	**{1,2,4}**	**{1,2,3,4}**
ϕ	ϕ	{3}	{4}	{3,4}	{1,2}	{1,2,3}	{1,2,4}	{1,2,3,4}
{3}	{3}	ϕ	{3,4}	{4}	{1,2,3}	{1,2}	{1,2,3,4}	{1,2,4}
{4}	{4}	{3,4,}	ϕ	{3}	{1,2,4}	{1,2,3,4}	{1,2}	{1,2,3}
{3,4}	{3,4}	{4}	{3}	ϕ	{1,2,3,4}	{1,2,4}	{1,2,3}	{1,2}
{1,2}	{1,2}	{1,2,3}	{1,2,4}	{1,2,3,4}	ϕ	{3}	{4}	{3,4}
{1,2,3}	{1,2,3}	{1,2}	{1,2,4}	{1,2}	{3}	ϕ	{3,4}	{4}
{1,2,4}	{1,2,4}	{1,2,3,4}	{1,2,}	{1,2,3}	{4}	{3,4}	$\boldsymbol{\phi}$	{3}
{1,2,3,4}	{1,2,3,4}	{1,2,4}	{1,2,3}	{1,2}	{3,4}	{4}	{3}	ϕ

Define $\sigma.W_n = s$, if $\sigma.x = x, \forall x \in W_n$, otherwise (Table 4).

Table 4. Rough action on $R(U_2)$

	W_1	W_2	W_6	W_7	W_8	W_9	W_{13}	W_{14}
Identity	s	S	s	S	s	s	s	s
permu1	-	-	-	-	-	-	-	-
permu2	-	S	-	-	-	-	-	-
permu3	-	-	-	-	-	-	-	-
permu4	-	-	-	-	-	-	-	-
permu5	-	-	-	-	-	-	-	-
permu6	-	-	-	-	-	-	-	-
permu7	-	-	-	-	-	-	-	-
permu8	-	-	-	-	-	-	-	-
permu9	-	s	-	-	-	-	-	-
permu10	-	-	-	-	-	-	-	-
permu11	s	-	-	-	-	-	-	-
permu12	-	-	-	-	-	-	-	-
permu13	-	s	-	-	s	-	-	-
permu14	-	-	-	-	-	-	-	-
permu15	s	-	-	-	-	-	-	-
permu16	-	-	-	-	-	-	-	-
permu17	s	-	-	-	-	-	-	-
permu18	s	s	-	s	-	-	-	-
permu19	s	-	s	-	-	-	-	-
permu20	-	s	-	-	s	-	-	-
permu21	-	s	-	-	-	s	-	-
permu22	-	-	-	-	-	-	-	-
permu23	-	-	-	-	-	-	-	-

$$W^e = \{W_1, W_2, W_6, W_7, W_8, W_9, W_{13}, W_{14}\}$$
$$W^{per_1}, W^{per_3}, \quad W^{per_4},\ W^{per_5}, W^{per_6}, W^{per_7},$$
$$W^{per_8},\ W^{per_{10}}, W^{per_{12}},$$
$$, W^{per_{14}}\ W^{per_{16}}, W^{per_{22}}\ \&\ W^{per_{23}} = \phi$$
$$W^{per_2} = \{W_2\}, W^{per_9} = \{W_2\}, W^{per_{11}} = \{W_1\},$$
$$W^{per_{13}} = \{W_2, W_8\},\ W^{per_{15}} = \{W_1\},\ W^{per_{17}} = \{W_1\},$$
$$W^{per_{18}} = \{W_1, W_2, W_7\},\ W^{per_{19}} = \{W_1, W_6\},$$
$$W^{per_{20}} = \{W_2, W_8\},\ W^{per_{21}} = \{W_2, W_9\},$$
$$\sum_{g \in S_{|U|}} |X^g| = 24 = S_{|U|}$$

Theorem 3.1

Every action of $S_{|U|}$ on $Ro(U)$ induces a homomorphism from $S_{|U|} \to Sym\big(Ro(U)\big)$

Proof:

Let $S_{|U|}$ acts on $Ro(U)$, $.: S_{|U|} \times Ro(U) \to Ro(U)$, where $(\sigma, W) \to \sigma\{W\} \in Ro(U)$ such that: $e.W = W, \forall W \in Ro(U)$; $\sigma_1.(\sigma_2.W) = (\sigma_1\sigma_2).W$.

Consider the map $\phi: S_{|U|} \to Sym(Ro(U))$, where, $Sym\big(Ro(U)\big): Ro(U) \to Ro(U)$ is defined as:

$$\Sigma_\sigma(W) = \sigma(W)$$
$$\text{Let } \Sigma_\sigma(X) = \Sigma_\sigma(Y) \Rightarrow \sigma\{X\} = \sigma\{Y\}$$
$$\{\sigma(x_n)\} = \{\sigma(y_n)\}, \forall x_n \in X \& \forall y_n \in Y$$
$$\sigma^{-1}\{\sigma(x_n)\} = \sigma^{-1}\{\sigma(y_n)\}, \{\sigma^{-1}\sigma(x_n)\} = \{\sigma^{-1}\sigma(y_n)\}$$
$$X = Y \Rightarrow .\text{ is } 1-1.$$
$$\forall Y \in Ro(U), \exists \sigma^{-1}(\{Y\}) \in Ro(U) \text{ such that}$$
$$\Sigma_\sigma \sigma^{-1}(\{Y\}) = \sigma\sigma^{-1}(\{Y\}) = Y, \Rightarrow .\text{ is onto}$$

Hence $\Sigma_\sigma \in Sym\ (Ro(U))$.

Define $\phi: S_{|U|} \to Sym\ \big(Ro(U)\big)$ as $\phi(\sigma_1\sigma_2) = \Sigma_{\sigma_1\sigma_2}$, $\Sigma_{\sigma_1\sigma_2}(X) = \{\sigma_1\sigma_2(X)\} = \sigma_1\{\sigma_2(X)\} = \Sigma_{\sigma_1}\Sigma_{\sigma_2}$.

Conversely,

Let $\phi : S_{|U|} \to Sym\ \big(Ro(U)\big)$ be a homomorphism where $\phi(\sigma) = \Sigma_\sigma$.

Define $.: S_{|U|} \times Ro(U) \to Ro(U)$ as: $(\sigma, W) = \Sigma_\sigma(W)$

It defines a rough action since:

$$(e, W) = \Sigma_e(W) = e(W) = W$$
$$5\sigma_1\big(\sigma_2(W)\big) = \sigma_1(\Sigma_{\sigma_2}(W))$$
$$\Sigma_{\sigma_1}.\Sigma_{\sigma_2}(W) = \Sigma_{\sigma_1\sigma_2} = \sigma_1\sigma_2(W)$$

So, . defines the R Action.

Definition 3.5 Stabilizer of R Action

Approximation space is composed of a finite set $U \neq \phi$, has n elements. Let $S_{|U|}$ be symmetric group on n elements and $Ro(U)$ collection of all rough sets. If $S_{|U|}$ acts on $Ro(U)$ then

$$\boldsymbol{stab}(.) = \{\boldsymbol{\sigma} \in \boldsymbol{S}_{|U|} | \boldsymbol{\sigma}(\boldsymbol{x_n}) = \boldsymbol{x_n}, \forall \boldsymbol{x_n} \in \boldsymbol{X}, \forall \boldsymbol{X} \in \boldsymbol{Ro}(\boldsymbol{U})\}$$

from above examples we have $stab(.) = \{e\}$.

So, in an R Action, $stab(.)$ is a trivial subgroup of $S_{|U|}$.

Result 3.1

If $S_{|U|}$ acts on $Ro(U)$ then $\sigma.\underline{X} \subseteq \underline{\sigma.X}$ and $\overline{\sigma.X} \subseteq \sigma.\overline{X}$

Example 3.3

Given $U_1 = \{1,2,3\}$ and R be any equivalence relation on U_1.

Equivalence class of U_1 with respect to R is given by:

$$U_1/R = \{\{1,2\}, \{3\}\}$$
$$2^{U_1} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \phi\}$$

represented as X_i, where $i = 1,2,3, \dots 8$. The rough sets are given by $Ro(U) = \{X_1, X_2, X_5, X_6\}$,

$$X_1 = \{1\}, X_2 = \{2\}, X_5 = \{1,3\}, X_6 = \{2,3\}$$

- $R^{X_1} = \{1,2\} \& R_{X_1} = \phi$
- $R^{X_2} = \{1,2\} \& R_{X_2} = \phi$
- $R^{X_5} = \{1,2,3\} \& R_{X_5} = \{3\}$
- $R^{X_6} = \{1,2,3\} \& R_{X_6} = \{3\}$

$$S_3 = \{e, (12), (13), (23), (123), (132)\}$$

Represented as $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$. Consider $X_6 = \{2,3\}$ where $\underline{X_6} = \{3\} \& \overline{X_6} = \{1,2,3\}$.

$$\sigma_0.\underline{X_6} = \{3\},\ \sigma_0.\overline{X_6} = \{1,2,3\}$$
$$\sigma_1.\underline{X_6} = \{3\},\ \sigma_1.\overline{X_6} = \{1,2,3\}$$
$$\sigma_2.\underline{X_6} = \{1\},\ \sigma_2.\overline{X_6} = \{1,2,3\}$$
$$\sigma_3.\underline{X_6} = \{2\},\ \sigma_3.\overline{X_6} = \{1,2,3\}$$
$$\sigma_4.\underline{X_6} = \{1\},\ \sigma_4.\overline{X_6} = \{1,2,3\}$$
$$\sigma_5.\underline{X_6} = \{2\},\ \sigma_5.\overline{X_6} = \{1,2,3\}$$
$$\underline{\sigma_0.X_6} = \{2,3\},\ \overline{\sigma_0.X_6} = \{1,2,3\}$$
$$\underline{\sigma_1.X_6} = \{1,3\},\ \overline{\sigma_1.X_6} = \{1,2,3\}$$
$$\underline{\sigma_2.X_6} = \{2,1\},\ \overline{\sigma_2.X_6} = \{1,2,3\}$$
$$\underline{\sigma_3.X_6} = \{3,2\},\ \overline{\sigma_3.X_6} = \{1,2\}$$
$$\underline{\sigma_4.X_6} = \{3,1\},\ \overline{\sigma_4.X_6} = \{1,2,3\}$$
$$\underline{\sigma_5.X_6} = \{1,2\},\ \overline{\sigma_5.X_6} = \{1,2\}$$

From above, $\sigma_i.\underline{X_6} \subseteq \underline{\sigma_i.X_6}$, and $\overline{\sigma_i.X_6} \subseteq \sigma_i.\overline{X_6}$ for all $i = 0,1,2,3,4,5$.

4. ROUGH $S_{|U|}$ SUBMODULES

The rough $S_{|U|}$ submodule, rough quotient $S_{|U|}$ submodules and rough $S_{|U|}$homomorphism have been introduced in this section.

Definition 4.1 [7]

Let G be any group and M be any abelian group. We say M is G-Module if G acts linearly on M. Define: $G \times M \to M$ such that

- $e \cdot x = x, \forall x \in X$
- $g \cdot (h \cdot m) = (g.h) \cdot m$
- $g \cdot (m_1 + m_2) = g \cdot m_1 + g \cdot m_2$

Definition 4.2

$S_U = \{f : U \to U, \text{bijective functions}\}$ and it forms a permuation group. $(2^U, \triangle)$ forms an abelian group. Define **.**: $S_U \times 2^U \to 2^U$ as $\sigma.X_n = \{\sigma(x_n)\}, \forall x_n \in X_n$ such that:

- $\boldsymbol{e.X = X}, \forall \boldsymbol{X \in 2^U}$
- $\boldsymbol{\sigma_1.(\sigma_2.X_n) = (\sigma_1\sigma_2).X_n}$
- $\boldsymbol{\sigma.(X \triangle Y) = \sigma.X \triangle \sigma.Y}$

Then 2^U is $S_{|U|}$ submodule.

Definition 4.3

Let $R(U)$ be a rough group since ro_g is a subgroup of 2^U. $R(U)$ is said to be a rough $S_{|U|}$-submodule if we define:

$.: S_U \times ro_g \to ro_g$ such that:

- $e.X_n = X_n, \forall X_n \in ro_g$
- $\sigma_1.(\sigma_2.X_n) = (\sigma_1\sigma_2).X_n$
- $\sigma(X_n \triangle X_m) = \sigma.X_n \triangle \sigma.X_m$

Then $R(U)$ is a rough $S_{|U|}$-submodule.

Theorem 4.1

Let (U, R_1) and (U, R_2) are approximation spaces where U is a finite universe with respect to equivalence relations R_1 and R_2. $R_1(U) \cap R_2(U)$ is a rough $S_{|U|}$-submodule if $\overline{R_1(U)} \cap \overline{R_2(U)} = \overline{R_1(U) \cap R_2(U)}$

Proof:

Let $X, Y \in R_1(U) \cap R_2(U)$ $X, Y \in R_1(U)$ and $R_2(U) \Rightarrow \overline{X}, \overline{Y} \in \overline{R_1(U)}$ and $\overline{R_2(U)}$. Also $\underline{X}$, $\underline{Y} \in \underline{R_1(U)}$ and $\underline{R_2(U)}$. Since, $\overline{R_1(U)} \cap \overline{R_2(U)} = \overline{R_1(U) \cap R_2(U)}$.

$$\overline{X}, \overline{Y} \in \overline{R_1(U) \cap R_2(U)} \;\&\; \underline{X}, \underline{Y} \in \underline{R_1(U) \cap R_2(U)}$$
$$\overline{R_1(U) \cap R_2(U)} \cup \underline{R_1(U) \cap R_2(U)} \leq (2^U, \Delta)$$

Hence $R_1(U) \cap R_2(U)$ is a rough S_U-submodule.

Definition 4.4 Rough Quotient $S_{|U|}$ −submodule

Let $R(U)$ be a rough $S_{|U|}$ submodule, ro_g is abelian subgroup of $(2^U, \Delta)$ and hence $R(U)$ is normal in 2^U. Then $\left(\frac{2^U}{R(U)}, \Delta\right) = \{R(U)\Delta Y \mid Y \in 2^U\}$ forms rough quotient group and hence it is rough quotient $S_{|U|}$ module.

Definition 4.5 Rough Homomorphism

$R_1(U)$, $R_2(U)$ are rough $S_{|U|}$ submodules.

Define $\phi: R_1(U) \to R_2(U)$ by $\phi(X) = Y$, if $|X| = |Y|$. Then ϕ defines a homomorphism since:

$$\phi(\{\}) = \{\} \;\&$$
$$\phi(X \triangle Y) = \phi(X) \triangle \phi(Y)$$

This homomorphism is said to be rough $S_{|U|}$ submodule homomorphism.

Example 4.2

Let $\boldsymbol{U = [1, 2, 3\}}$ and $\boldsymbol{\eta_1}$& $\boldsymbol{\eta_2}$ be two equivalence relations on $\boldsymbol{U}$.

$$\boldsymbol{\frac{U}{\eta_1} = \{\{1, 2\}, \{3\}\}, U/\eta_2 = \{\{2, 3\}, \{1\}\}}$$
$$\boldsymbol{2^U = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{\}\}}$$

With respect to $\boldsymbol{\eta_1}$, the rough sets are $\boldsymbol{X_1 = \{1\}, X_2 = \{2\}, X_5 = \{1, 3\}, X_6 = \{2, 3\}}$, since,

- $\boldsymbol{\eta_1^{X_1} = \{1, 2\}}$, $\quad \boldsymbol{\eta_{1_{X_1}} = \{\}}$
- $\boldsymbol{\eta_1^{X_2} = \{1, 2\}}$,, $\quad \boldsymbol{\eta_{X_2} = \{\}}$
- $\boldsymbol{\eta_1^{X_5} = \{1, 2, 3\}}$, $\quad \boldsymbol{\eta_{1_{X_5}} = \{3\}}$
- $\boldsymbol{\eta_1^{X_6} = \{1, 2, 3\}}$, $\quad \boldsymbol{\eta_{X_6} = \{3\}}$

$$\boldsymbol{\overline{\eta_1(U)} \cup \underline{\eta_1(U)} = \{\{1, 2\}, \{3\}, \{1, 2, 3\}, \{\}\}}$$

With respect to $\boldsymbol{\eta_2}$, the rough sets are $\boldsymbol{X_2 = \{2\}, X_3 = \{3\}, X_4 = \{1, 2\}}$ $\boldsymbol{and}$ $\boldsymbol{X_5 = \{1, 3\}}$.

- $\boldsymbol{\eta_2^{X_2} = \{2.3\}}$, $\quad \boldsymbol{\eta_{2_{X_2}} = \{\}}$
- $\boldsymbol{\eta_2^{X_3} = \{2, 3\}}$, $\quad \boldsymbol{\eta_{2_{X_3}} = \{\}}$
- $\boldsymbol{\eta_2^{X_4} = \{1, 2, 3\}}$, $\quad \boldsymbol{\eta_{2_{X_4}} = \{1\}}$
- $\boldsymbol{\eta_2^{X_5} = \{1, 2, 3\}}$, $\quad \boldsymbol{\eta_{2_{X_5}} = \{1\}}$

$$\boldsymbol{\overline{\eta_1(U)} \cup \underline{\eta_1(U)} = \{\{2, 3\}, \{1\}, \{1, 2, 3\}, \{\}\}}$$

Let ϕ be the Rough $\boldsymbol{S_{|U|}}$ submodule homomorphism then from above.

$$\boldsymbol{\phi\{\{\}\} = \{\}}$$
$$\boldsymbol{\phi\{\{3\}\} = \{1\}}$$
$$\boldsymbol{\phi\{\{1, 2\}\} = \{2, 3\}}$$
$$\boldsymbol{\phi\{\{1, 2, 3\}\} = \{1, 2, 3\}}$$

Also,

ϕ({3}△{1, 2})=ϕ({1, 2, 3})={1, 2, 3}
ϕ({3}) △ ϕ({1, `2})={1}△{2, 3}={1, 2, 3}

Definition 4.6 Kernel of Rough Homomorphism

With respect to two equivalence relations $\boldsymbol{R_1}$& $\boldsymbol{R_2}$ consider an approximation space with finite universe $\boldsymbol{U}$.

$\boldsymbol{R_{1U}}$ & $\boldsymbol{R_{2U}}$ are rough $\boldsymbol{S_{|U|}}$ submodules. Let $\boldsymbol{\phi: R_{1U} \to R_{2U}}$ be rough homomorphism then kernel is defined as $\boldsymbol{ker\ \phi = \{\eta \in R_{1U} \mid \phi(\eta) = \{\}\}}$.

Theorem 4.2

If $*: \boldsymbol{R_{1U}} \to \boldsymbol{R_{2U}}$ be rough homomorphism, then $\boldsymbol{ker} *$ is rough $\boldsymbol{S_{|U|}}$ submodule of $\boldsymbol{R_{2U}}$.

Proof:

The mapping $\cdot: \boldsymbol{S_{|U|}} \times \boldsymbol{ro_{g_1}} \to \boldsymbol{ro_{g_2}}$ defines a rough linear $\boldsymbol{S_{|U|}}$ action.

$$\text{Let } \boldsymbol{T_1}, \boldsymbol{T_2} \in \boldsymbol{ker} * \Rightarrow * (\boldsymbol{T_1}) = \{\},\ * (\boldsymbol{T_2}) = \{\}$$

$$\text{Also } * (\boldsymbol{T_1} \triangle \boldsymbol{T_2}) = * (\boldsymbol{T_1}) \triangle * (\boldsymbol{T_2}) = \{\}$$

($\because *$ is rough homomorphism) $\Rightarrow \boldsymbol{T_1} \triangle \mathbf{T_2} \in \boldsymbol{ker\phi}.$

$\therefore$ $ker\phi$ rough subgroup of $\boldsymbol{ro_{g_2}}$.

Define the map $\cdot: \boldsymbol{S_U} \times \boldsymbol{ker} * \to \boldsymbol{ker} *$ by $(\boldsymbol{\sigma}, \boldsymbol{X}) \to \boldsymbol{\sigma}(\boldsymbol{\phi}(\boldsymbol{xi}))$, $\forall \boldsymbol{xi} \in \boldsymbol{ro_{g_1}}$, satisfying the rough $\boldsymbol{S_{|U|}}$ module conditions. Hence $\boldsymbol{ker} *$ is rough $\boldsymbol{S_{|U|}}$ submodule.

5. APPLICATION OF ROUGH ACTION

A rough action example is given in this section to find missing cancer data values.

Example 5.1

Consider a finite universe with five patients and attributes Age, Gender, Treatment, and Survival Status with missing values in Age and Gender. The missing values are determined by applying a rough group action rule as follows in Table 5.

Table 5. Given information system with missing values

Patient	Age	Gender	Treatment	Survival Status
P1	45	M	Surgery	Alive
P2	?	F	Chemotherapy	Deceased
P3	60	?	Chemotherapy	Alive
P4	38	F	Surgery	Alive
P5	?	M	Surgery	Deceased

The following decision are the decision rules with respect to AGE attribute:

- If Age≤45 & Gender "M" & Treatment "Surgery" then Survival Status is "Alive."
- If Age≤60 & Treatment "Chemotherapy" then Survival Status is "Alive."
- If Age≤40 & Gender "F" & Treatment "Surgery" then Survival Status is "Alive."

Rough Group Action Rule

For each patient, if the "Age" attribute is missing, the "Treatment" attribute will be changed to "Surgery" If the "Gender" attribute is missing, the "Treatment" attribute will be changed to "Chemotherapy" as shown in Table 6.

Table 6. Rough action on information system

Patient	Age	Gender	Treatment	Survival Status
P1	45	M	Surgery	Alive
P2	?	F	Surgery	Deceased
P3	60	?	Surgery	Alive
P4	38	F	Surgery	Alive
P5	?	M	Chemotherapy	Deceased

Using the above decision rules, P2 will be of age >60 P3 Gender will be "F" & P5 Age will be >45 (Table 7).

Table 7. Information system without missing values

Patient	Age	Gender	Treatment	Survival Status
P1	45	M	Surgery	Alive
P2	>60	F	Chemotherapy	Deceased
P3	60	F	Chemotherapy	Alive
P4	38	F	Surgery	Alive
P5	>45	M	Surgery	Deceased

As a result, we are able to find the missing values by using the Rough Group Action rule.

6. RESULTS AND DISCUSSION

It is possible to apply rough sets theory to any algebraic system, in particular in this paper we incorporated rough sets into group action concepts. Using a finite universe approximation space, a rough action is introduced by considering a symmetric group that will act on all rough sets. It has been shown that every rough action of $S_{|U|}$ on rough sets induces a homomorphism on set of symmetries of rough sets. In addition, we proved that rough sets have a single orbit.

Rough $S_{|U|}$submodule is then introduced and proved that the kernel of rough homomorphism is a rough $S_{|U|}$ submodule. Also, we have shown that the intersection two of rough $S_{|U|}$ suhmodules R_1 & R_2 is a rough $S_{|U|}$ suhmodule if intersection of upper approximation of R_1 & R_2 is equal to upper approximation of their intersection. We have also provided a sample cancer data example that illustrates how rough action can be used to locate missing values.

7. CONCLUSION

In a finite universe, rough groups are defined by considering all possible rough sets. In this paper, an introduction to rough $S_{|U|}$ action on rough sets of a universe and a definition of rough σ stabilizer has been presented. We have also shown that rough sets have one $S_{|U|}$ orbit. With the help of suitable examples, rough $S_{|U|}$ submodules have been introduced with their properties. In addition, rough action was shown to be useful in finding missing values in cancer data. Rough $S_{|U|}$ submodules will be examined for their more expansive properties as future work. A rough set can also be applied to other algebraic structures in a similar way.

REFERENCES

[1] Pawlak, Z. (1982). Rough sets. International Journal of Computer & Information Sciences, 11: 341-356. https://doi.org/10.1007/BF01001956

[2] Bonikowski, Z. (1994). Algebraic Structures of Rough Sets. In: Ziarko, W.P. (eds) Rough Sets, Fuzzy Sets and Knowledge Discovery. Workshops in Computing. Springer, London. https://doi.org/10.1007/978-1-4471-3238-7_29

[3] Liu, G.L., Zhu, W. (2008). The algebraic structures of generalized rough set theory. Information Sciences, 178(21): 4105-4113. https://doi.org/10.1016/j.ins.2008.06.021

[4] Davvaz, B. (2004). Roughness in rings. Information Sciences, 164: 147-163.

https://doi.org/10.1016/j.ins.2003.10.001
[5] Kazanc, O., Davvaz, B. (2008). On the structure of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. Information Sciences, 178(5): 1343-1354. https://doi.org/10.1016/j.ins.2007.10.005
[6] Davvaz, B., Mahdavipour, M. (2006). Roughness in modules. Information Sciences, 176(24): 3658-3674. https://doi.org/10.1016/j.ins.2006.02.014
[7] Sivakumar, S., Sathish, S. (2024). Enhanced characterization of rough semigroup ideals: Extension and analysis. Mathematical Modelling of Engineering Problems, 11(3): 824-830. https://doi.org/10.18280/mmep.110328
[8] Shahzamanian, M.H., Davvaz, B. (2023). Roughness in fuzzy Cayley graphs. Bulletin of the Karaganda University. Mathematics Series, 4(112): 105-118. https://doi.org/10.31489/2023m4/105-118
[9] Pomykala, J., Pomykala, J.A. (1988). The stone algebra of rough sets. Bulletin of the Polish Academy of Sciences. Mathematics, 36: 495-508. https://api.semanticscholar.org/CorpusID:118622800.
[10] Comer, S.D. (1993). On connections between information systems, rough sets and algebraic logic. Algebraic Methods in Logic and Computer Science, 28: 117-124. https://doi.org/10.4064/-28-1-117-124
[11] Biswas, R., Nanda, S. (1994). Rough groups and rough subgroups. Bulletin of the Polish Academy of Sciences. Mathematics, 42: 251-254.
[12] Kuroki, N. (1997). Rough ideals in semigroups. Information Sciences, 100(1-4): 139-163. https://doi.org/10.1016/S0020-0255(96)00274-5
[13] Kuroki, N., Wang, P.P. (1996). The lower and upper approximations in a fuzzy group. Information Sciences, 90: 203-220. https://doi.org/10.1016/0020-0255(95)00282-0
[14] Jun, Y.B. (2003). Roughness of gamma-subsemigroups/ideals in gamma-semigroups. Bulletin of the Korean Mathematical Society, 40(3): 531-536. https://doi.org/10.4134/BKMS.2003.40.3.531
[15] Jun, Y.B. (2003). Roughness of ideals in BCK-algebras. Scientiae Mathematicae Japonicae, 57(1): 165-170.
[16] Al-Shami T.M. (2022). Topological approach to generate new rough set models. Complex & Intelligent Systems, 8: 4101-4113. https://doi.org/10.1007/s40747-022-00704-x
[17] Guler, A.C., Yildirim, E.D., Ozbakir, O.B. (2022). Rough approximations based on different topologies via ideals. Turkish Journal of Mathematics, 46: 1177-1192. https://doi.org/10.55730/1300-0098.3150
[18] Hosny, M., Al-Shami, T.M., Abdelwaheb, M. (2023). Novel approaches of generalized rough approximation spaces inspired by maximal neighbourhoods and ideals. Alexandria Engineering Journal, 69: 497-520. https://doi.org/10.1016/j.aej.2023.02.008
[19] Mordeson, J.N. (2001). Rough set theory applied to (fuzzy) ideal theory. Fuzzy Sets and Systems, 121: 315-324. https://doi.org/10.1016/S0165-0114(00)00023-3
[20] Davvaz, B. (2006). Roughness based on fuzzy ideals. Information Sciences, 176(16): 2417-2437. https://doi.org/10.1016/j.ins.2005.10.001
[21] Xiao, Q.M., Zhang, Z.L. (2006). Rough prime ideals and rough fuzzy prime ideals in semigroups. Information Sciences, 176: 725-733. https://doi.org/10.1016/j.ins.2004.12.010
[22] Sangeetha, S., Sathish, S. (2023). Aspects of algebraic structure of rough sets. Mathematics and Statistics, 11(5): 820-826. http://doi.org/10.13189/ms.2023.110508
[23] Bağırmaz, N., İçen, İ., Özcan, A.F. (2016). Topological rough groups. Topological Algebra and Its Applications, 4(1): 31-38. https://doi.org/10.1515/taa-2016-0004
[24] Altassan, A., Alharbi, N., Aydi, H., Özel, C. (2020). Rough action on topological rough groups. Applied General Topology, 21(2): 295-304. https://doi.org/10.4995/agt.2020.13156
[25] Artin, M. (1991). Algebra. Prentice Hall, Inc. https://search.worldcat.org/en/title/24364036.