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A Replenishment Policy for an Inventory Model with Price-Sensitive Demand with Linear and Quadratic Back Order in a Finite Planning Horizon

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ABSTRACT

In this research article, we proposed an inventory model for the replenishment policy. The focus of our research article is on the companies that frequently deal with backorders. An advanced inventory model considering back order has been proposed the results highlight the dynamic nature of the system, with optimal values achieved in different cycles. In this model, replenishment policy is given and to lower the economic ordering cost, we used new parameters such as price-sensitive demand, complete back ordering, and backorder is taken as a quadratic function as well as linear backorder with shortages in a finite planning horizon. The result is discussed for both backorders (linear and quadratic), to minimize the total cost obtained using the Hessian matrix to be positive definite. Software 'MATHEMATICA VERSION 12' has been used for the solution of the proposed model by using numerical iterative method. For different parameters, different tables are provided. The outcomes of the sensitivity analysis with the help of tables and graphs are depicted. Finally, we have discussed the conclusion and practical implications.

1. INTRODUCTION

While dealing with the inventory problem the basic thing to be remember the developing technologies along with the inventory models. New technologies are growing due to recent research done on inventory. Seliaman et al. [1] are developing new techniques day by day to make inventory management easier.

Backorders are for products that a firm cannot currently fill because demand exceeds supply. Backordering can refer to items that are presently in production or those that have not yet started production. For handling the backorder communication is the key. By communicating the presence of backorder, the supplier gets the information about the customer's actual demand for the product what is inbound, and when the balance items will be there. This allows both the suppliers and the customers to continue the operations uninterrupted. As the backorder may impact inventory and other holding costs.

In this article, we made an advanced model of inventory for the firms/companies who frequently deal with backorders. In this model we take the price-sensitive demand, backorder as a quadratic function with shortages in a finite planning horizon, and a case of linear backorder is discussed with lead time is zero.

2. RESEARCH GAP AND PROBLEM IDENTIFICATION

In this study, we aim to address the research gap related to price-dependent quadratic backorder. Despite the extensive research conducted in this field, there is still a lack of understanding regarding a very few research conducted in the finite planning horizon. Therefore, this study seeks to contribute to the existing literature by price-sensitive demand and price-dependent quadratic backorder. The research question addressed in this study is a replenishment policy for the quadratic backorder and linear backorder inventory models discussed in the finite planning horizon.

3. LITERATURE REVIEW

Inventory is a very interesting topic for researchers. So much work has been done from the decade and still mostly work is going to be done. Firstly, the classical EOQ formula was discovered by Harris [2] which is also known as 'Square-Root formula'. The first book on inventory management was written by Raymond [3]. Veinott [4] studied that in the real-life situations, the demand rate is dynamic. So, they developed the first dynamic economic order quantity model which is developed by modifying Harris's Square-Root model.

Ouyang et al. [5] provided a model by taking shortages and solving the total shortages as a combined form of lost sales and backorder. Scarf et al. [6] estimated a stochastic model of multi-period with shortages and given a policy (s, S) for an optimal solution with backorders. Veinott [4] think that finding the exact backorder cost was a hard task so they developed the model which calculates the backorder cost. They considered back-ordering as a constant function, with shortages. Zangwill [7] proposed a multi-period model along with shortages and backorders. Fogarty and Aucamp [8] gave the model with shortages and back-ordering. Aardal et al. [9] proposed a model by taking the random demand (q, r) model given by Hadley and Within. Backorders are not considered but they assure that the yearly backorders cannot cross the upper boundaries.

Various other models discussed in the literature are compared and contrasted to showcase the advancements made in inventory management research. Çetinkaya and Parlar [10] established a generalized model by taking two different types of backorder costs. Sarkar et al. [11] concerned with optimal inventory replenishment for a degrading item with time-quadratic demand and time-dependent partial backlogging. The analytical model yields optimum solutions, which are demonstrated numerically. Liao and Shyu [12] gave a model of predefined lot size and demand is assumed to be regularly distributed, with lead time as the variable, the estimated total cost with the backorder is minimized. Pan et al. [13] established an inventory model by taking the lead time & backorder discounts are negotiable in the way that the supplier may take into account the future & present loss & profit. The buyer may be ready to obtain the item as quickly as it can be obtained to ensure production may restart. Bayındır et al. [14] established an EPQ model taking general stock dependent backordering. San-Joséet al. [15] proposed an EOQ model for a single item with partially backlogging, shortages time-dependent, partial backordering, the demand rate is backlogged at any instant is a constant fraction with shortages & obtained an optimal policy & less inventory cost. Pan and Hsiao [16] extended the work of literature [5]. Taken an integrated inventory system with shortages and backorder as well as lead time are negotiable. A provider may provide waiting consumers with a backorder cost reduction in the first of two models they described, which had normally distributed demand, and widely dispersed demand in the second. Sazvar et al. [17] established an inventory model for deteriorating goods by taking shortages and complete backordering. Ghasemi and Afshar Nadjafi [18] proposed two models taking holding cost as increasing continuous functions. The first model with no shortages & the second model is with shortages and complete backordering. Kumar et al. [19] proposed an economic policy by taking demand as power depending on time, with shortages and complete backordering. Mishra and Ranu [20] discussed the importance of supplier-retailer coordination in managing deteriorating inventory with decreasing demand, addressing a research gap in supply chain literature. It presents a numerical solution and conducts a sensitivity analysis to illustrate the concept further.

Backordering was studied over the decade and still the work is going on. Back ordering is a major problem for the business, organization that's why researchers readily study backorder taking different types of backordering like linear, non-linear, exponential, negative exponential, constant function and quadratic function, etc.) Grubbström and Erdem [21] applied algebraic approach to develop the equations for both the EOQ (Economic Order Quantity) and the Economic Production Quantity (EPQ), while taking into account a single backordering cost that is only linear with respect to time. Cárdenas-Barrón [22] developed an algebraic method to prove the mathematical equations for EOQ and EPQ with a single cost of backordering, only linear (depending on time). Taleizadeh et al. [23] proposed an EOQ model by taking linear holding cost (depend on price), partially backlogged & backorder is a linear function. Taleizadeh et al. [24] proposed two EOQ models (a) by taking holding cost linear dependent on time, partially backlogged, backorders are linear function, lost sale cost as fixed and partially delayed payments. Taleizadeh et al. [25] by taking holding cost linear depends on time, partially backlogged, backorders are linear functions, lost sale cost is fixed & partially prepayments. Yang [26] established an EOQ model by taking non-linear stock dependent holding cost, partially backlogging, backorders are linear, a lost sale is fixed, the demand rate is stock dependent. By taking different types of backordering singly researchers were not satisfied with the output, so they started taking two types of backorders together, like linear plus fixed, linear, and quadratic Some of the literature surveys are as follows, Unwin [27] firstly took the linear plus fixed backorder and solved by calculus and solved the system of equations &they get the first-order condition. Sphicas [28] extended the study of literature [21] by taking two parameters combine i.e., Linear & fixed backorder cost for the EOQ & EPQ models. They discussed two conditions first is when fixed backorder cost is high then we can't get any optimal backorder &the second case if the back-order cost is very lesser than there should be optimally some of the backorders. The result reveals that linear backorder cost plays no role. Chung and Cárdenas-Barrón [29] given the complete solution procedure for the EOQ/EPQ models, and backorder cost is taken as fixed & linear. Most of the models are failed to give an argument & surety of the optimal situation but Chung and Lin [30] given every aspect of the approved solution procedure, we ensure the most effective possible solution. They discussed two cases in their paper for the existence of optimal solution & if the conditions are not satisfied then how to identify the condition by which optimal solution is sure. And derives four theorems & two lemmas for an optimal solution. Mishra and Namwad [31] discussed an inventory model that addresses items with minimal lead time and deterioration, utilizing cubic demand and deterioration functions. It emphasizes the advantages of employing cubic functions for practical applicability, numerical validation, and graphical representation. Additionally, it includes a numerical example and a comprehensive sensitivity analysis. Wee et al. [32] proposed an EOQ model by taking linear holding cost (depend on price), partially backlogged & backorder is linear & fixed function. Sphicas [33] proposed an EOQ model holding cost is linear & dependent on time, completely backlogged, and backorders are fixed &linear. Hu et al. [34] proposed a model of backordering as linear & quadratic function, partially backlogged. Figure 1 is the inventory model diagram. In Table 1, a literature survey is carried out.

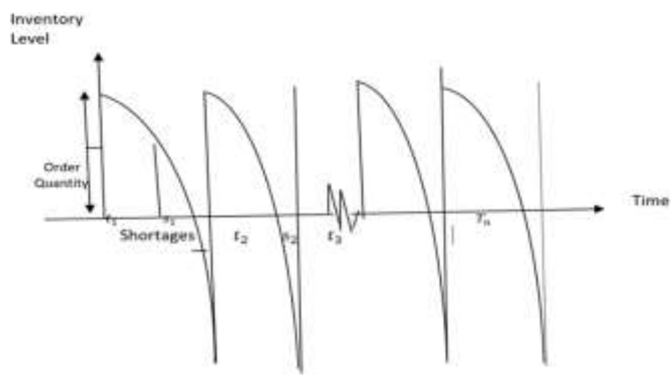

Figure 1. Inventory model diagram

Table 1. Survey of existing literature

Ref.	Demand Type	Shortages	Backorder Type	Finite Planning Horizon
[2]	Classical EOQ (Square-Root)	-	-	-
[4]	Dynamic EOQ	Yes (Dynamic)	Dynamic	-
[35]	Time-dependent	Yes	-	-
[5]	-	Lost sales & backorder	Lost sales & backorder	-
[6]	Stochastic	Optimal with backorders	Optimal	-
[7]	Multi-period	Yes	Yes	-
[12]	Normally distributed	Expected total cost with backorder	Expected total cost	-
[16]	-	Yes	Linear	-
[15]	Time-dependent	Partially backlogging	Partial	-
[17]	-	Complete backordering	Complete	-
[18]	-	No & yes with complete backordering	No & complete	-
[19]	Power depending on time	Yes, with complete backordering	Complete	Yes
[23]	Linear holding cost (on price)	Partially backlogged	Partial	-
[24]	Time-dependent holding cost	Partially backlogged	Partial	-
[25]	Time-dependent holding cost	Partially backlogged	Partial	-
[26]	Stock-dependent	Partially backlogging	Partial	-
[36]	Linear plus fixed	-	Linear & fixed	-
[32]	Linear holding cost (on price)	Partially backlogged	Partial	-
[33]	Time-dependent holding cost	Completely backlogged	Complete	-
[34]	-	Partially backlogged	Linear & quadratic	-
This paper	Price-sensitive	Yes	Complete linear and quadratic	Yes

4. ASSUMPTIONS

i. The total stock level is initially zero.
ii. The cost of storing stays constant.
iii. The lagging time is zero.
iv. The cost of ordering is predetermined.
v. Under a finite planning horizon, shortages are acceptable and a continuous one.
vi. Back ordering is complete and described as a quadratic function and linear.

5. MATHEMATICAL SOLUTION OF THE MODEL

The initial inventory equation is given by,

$$\frac{dI_{j+1}(t)}{dt} + (\theta_1)I_{j+1}(t) = -D(t) \quad t_j < t < s_{j+1} \tag{1}$$

where, j=1, 2, 3, …, n_1.

$$\frac{dI_{j+1}(t)}{dt} = -D(t) - \theta_1 I_{j+1}(t) \quad t_j < t < s_{j+1} \tag{2}$$

Considering the boundary condition $I_{i+1}(s_j) = 0$.
Solution of Eq. (2) is,

$$I_{j+1}(t) = e^{-\theta_1 * t} \int_t^{s_{j+1}} D(u)e^{u}\,du \tag{3}$$

$$I_{j+1}(t) = \int_t^{s_{j+1}} D(u)e^{\theta_1(u-t)}\,du \tag{4}$$

$$I_{j+1}(t) = \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t)} - 1\right]D(t) \tag{5}$$

During the shortage phase, the instantaneously arising shortage $I_b(t)$ is offered by,

$$I_b(t) = D_1(t_{j+1} - s_j) \tag{6}$$

where, $D_1 = a - bp - cp^2$ is the price dependent quadratic backorder.

$$I_b(t) = a - bp - cp^2(t_j - s_j) \tag{7}$$

Considering the boundary condition, $I_b(s_j) = 0$.

$$Q_{j+1} = I_{j+1}(t_j) = \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t_j)} - 1\right]D(t) \tag{8}$$

where, $D(t) = a - b * p$.

Considering the reorganization of the ordering, S_{j+1} can be given as,

$$S_{j+1} = \int_{s_j}^{t_j} I_b(t)dt = \int_{s_j}^{t_j}(a - b*p - c*p^2)(t_j - s_j)dt \tag{9}$$

The entire purchase amount for a limited time frame of planning,

$$Q_{nt} = \sum_{j=1}^{n_1} Q_{j+1} = \sum_{j=1}^{n_1}\{I_{j+1} + S_{j+1}\} \tag{10}$$

$$Q_{j+1} = \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t_j)} - 1\right]D(t) + \int_{s_j}^{t_j} (a - b*p - c*p^2)(t_j - s_j)dt \tag{11}$$

The total retailer cost over a specified time horizon is given by,

Total cost = Resupply expenses + cost of retaining stocks + purchasing cost + storage cost

$$T_R(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)dt + \sum_{j=0}^{n_1-1} W_h * Q_{j+1} + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} I_b(t)dt \tag{12}$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} I_{j+1}(t)dt + \sum_{j=0}^{n_1-1} W_h * Q_{j+1} + \sum_{j=0}^{n_1-1} s \int_{s_j}^{t_j} I_b(t)dt$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + \sum_{j=0}^{n_1-1} H \int_{t_j}^{s_{j+1}} \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t)} - 1\right] D(t)dt + \sum_{j=0}^{n_1-1} W_h * (\frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t_j)} - 1\right] D(t) + \int_{s_j}^{t_j} (a - b*p - c*p^2)(t_j - s_j)dt)$$

$$T_R(t_j, s_j, n_1) = n_1 * O_r + H \int_{t_{j-1}}^{s_j} \frac{1}{\theta_1}\left[e^{\theta_1(s_j-t)} - 1\right] D(t)dt + \int_{t_j}^{s_{j+1}} \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t)} - 1\right] D(t)dt + W_h c * (\frac{1}{\theta_1}\left[e^{\theta_1(s_j-t_{j-1})} - 1\right] D(t) + W_h * (\frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t_j)} - 1\right] D(t) + s(a - b*p - c*p^2)(t_{j-1} - s_{j-1})^2 + s(a - b*p - c*p^2)(t_j - s_i)^2 \tag{13}$$

To achieve the lowest possible total cost in the inventory system, the essential conditions for minimizing the total cost are as follows:

$$\frac{\partial TC(t_j, s_j,\ n_1)}{\partial t_j} = 0, j = 1, 2, 3, \ldots, n \tag{14}$$

$$\frac{\partial TC(t_j, s_j,\ /n)}{\partial s_j} = 0, j = 1, 2, 3, \ldots, n \tag{15}$$

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial t_j} = \sum_{j=0}^{n_1-1} -H * \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t_j)} - 1\right] D(t) - \sum_{j=0}^{n_1-1} W_h * (\left[e^{\theta_1(s_{j+1}-t_j)}\right] D(t) + \sum_{j=0}^{n_1-1} 2*s(a - b*p - c*p^2)(t_j - s_j) \tag{16}$$

$$\frac{\partial T_R(t_j, s_j, n_1)}{\partial s_j} = \sum_{j=0}^{n_1-1} H \int_{t_{i-1}}^{s_j} \left[e^{\theta_1(s_j-t_j)}\right] D(t)dt - \sum_{j=0}^{n_1-1} W_h * (\left[e^{\theta_1(s_j-t_{j-1})}\right] D(t) - \sum_{j=0}^{n_1-1} 2*s(a - b*p - c*p^2)(t_j - s_j) \tag{17}$$

The total cost's Hessian matrix must be positive definite for a fixed n in order for the total cost to be least (i.e. $\nabla^2 TC$).

$$\frac{\partial^2 T_R(t_j, s_j, n_1)}{\partial {t_j}^2} = \sum_{j=0}^{n_1-1} H * \left[e^{\theta_1(s_{j+1}-t_j)}\right] D(t) + \sum_{j=0}^{n_1-1} \theta_1 * W_h * (\left[e^{\theta_1(s_{j+1}-t_j)}\right] D(t) + \sum_{j=0}^{n_1-1} 2*s*(a - b*p - c*p^2) \tag{18}$$

$$\frac{\partial^2 T_R(t_j, s_j, n_1)}{\partial {s_i}^2} = \sum_{j=0}^{n_1-1} H * \theta_1\left(\int_{t_{j-1}}^{s_j} \left[e^{\theta_1(s_j-t_j)}\right] D(t)dt + 1\right) - \sum_{j=0}^{n_1-1} W_h * \theta_1(\left[e^{\theta_1(s_j-t_{j-1})}\right] D(t) + \sum_{j=0}^{n_1-1} 2*s(a - b*p - c*p^2) \tag{19}$$

$$\frac{\partial^2 T_R(t_j, s_j, n_1)}{\partial t_i \partial s_i} = \sum_{j=0}^{n_1-1} -2*s(a - b*p - c*p^2) \tag{20}$$

5.1 Total cost of supplier

$$T_S(t_j, s_j, n_1) = n_1 * Ss + Cs * \sum_{j=0}^{n_1-1} \frac{1}{\theta_1}\left[e^{\theta_1(s_{j+1}-t_j)} - 1\right] D(t) + \int_{s_j}^{t_j} (a - b*p - c*p^2)(t_j - s_j)dt \tag{21}$$

5.2 Numerical illustration

A numerical example to validate our model, using specific parameter values a=1.25, b=0.2, c=18.4, r=60, e=2.7, W_h=2, H=4, p=0.01, S=2, $s_1 = 0$, $\theta_1 = 0.03$ expressed in their appropriate units. For the solution of Eq. (16) and Eq. (17), Mathematica (version 12) was the computational program that we utilized. 'MATHEMATICA VERSION 12' provides efficiently handles the calculations and analysis required for the inventory model considering backorders.

5.3 Theorems

Theorem 1: If the following conditions are satisfied:

(i) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j^2} \geq 0$,

(ii) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_j^2} \geq 0$,

(iii) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j^2} - \left|\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j\, \partial s_j}\right| \geq 0$ and

(iv) $\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_j^2} - \left|\frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_j\, \partial s_j}\right| \geq 0$ for all j= 1, 2, ..., n

Then, $T_R(t_j, s_j, n_1)$ will be positive definite. This set of conditions is sufficient to ensure that $T_R(t_j, s_j, n_1)$ is at its minimum for a fixed value of n_1. The theorem establishes that $T_R(t_j, s_j, n_1)$ is indeed positive. Therefore, we can compute the optimal values of t_j and s_j for a given positive integer n_1 using iterative methods and Mathematica software based on Eq. (16) and Eq. (17).

Theorem 2: When considering a convex set S ⊆ R^n, a cost function is deemed convex across S if it satisfies the condition that, for any x_1 and x_2 belonging to S, and for any λ within the interval [0, 1], the following inequality holds: $\lambda f(x_1) + (1 - \lambda) f(x_2) \geq f(\lambda x_1 + (1 - \lambda)x_2)$. Should this inequality always be held as a strict inequality, then the function f is denoted as a strictly convex cost function on S.

Theorem 3: Consider an open convex subset S, which is non-empty, of R^n, and a cost function f: S → R that is twice differentiable on S. In this context, f is convex on S if and only if the Hessian matrix $\nabla^2 f(x)$ is positive semi-definite for all x in S.

Theorem 4: In the scenario where S is an open convex set in R^n and f: S → R is a cost function that is twice differentiable, if the Hessian matrix $\nabla^2 f(x)$ is positive definite for all x in S, then f is a strictly convex function on S.

$$\nabla T_R(t_j,s_j,n_1) = \begin{bmatrix} \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_1\, \partial s_1} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_2 \partial t_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_1 \partial t_2} & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\, \partial s_1} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_2\, \partial s_2} & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1-1} \partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1}^2} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial s_{n_1-1} \partial t_{n_1}} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}\, \partial s_{n_1-1}} & \frac{\partial^2 T_R(t_j,s_j,n_1)}{\partial t_{n_1}^2} \end{bmatrix}$$

6. SENSITIVITY ANALYSIS

The associated total cost for various resupply cycles, i.e., for n = 1, 2, ... are given in Table 2. From Table 3, Figures 2-5, we notice that for each resupply cycle, the most efficient number of replenishments time for the corresponding minimum total cost gets supplied in appropriate units. The optimal solutions for t_i and s_{i+1} for n = 4 are given in Tables 4 and 5, Figures 6 and 7 respectively. In Table 6 total cost for retailer, supplier and quantity is given for optimal value.

Table 2. Total cost for the retailer for different replenishment cycle

↓a →n	1	2	3	4	5	6
0.81675	28.8679	**28.494**	28.6444	29.2919	304364	32.078
0.9375	32.5612	31.5384	**31.1177**	31.2676	31.9882	33.2794
1.089	37.1951	35.3582	34.2208	**33.7465**	33.9352	34.7868
1.25	42.1195	39.4175	37.5185	36.3808	**36.0042**	36.3888

Table 3. The optimal solutions for t_j (replenishment time)

↓a →t_j	t_0	t_1	t_2	t_3	t_4	t_5
0.81675	**0**	3.3374	4			
0.9375	0	2.95889	3.53907	4		
1.089	0	2.58054	3.13523	3.53925	4	
1.25	0	2.20229	2.73147	3.13543	3.5393	**4**

Table 4. The optimal solutions for s_j (time of shortage)

↓a →s_j	s_0	s_1	s_2	s_3	s_4	s_5
0.81675	**0**	3.59373	4			
0.9375	0	3.19009	3.59601	4		
1.089	0	2.78658	3.19216	3.59608	4	
1.25	**0**	2.38314	2.78839	3.19226	**3.59613**	**4**

Table 5. Total cost for retailer, supplier and quantity is given for optimal value

$\downarrow a$	T_R	T_S	Q_{nt}
0.81675	28.494	12.2224	9.40793
0.9375	31.1177	16.7166	8.72184
1.089	33.7465	21.2081	8.02711
1.25	36.0042	25.6457	7.15235

Table 6. Sensitivity analysis of the parameters

Parameters	% Changes	Optimal Replenishment Cycle	Total Order Quantity Q_{nt}	Total Cost of Retailer T_R	Total Cost of Supplier T_S
a	+20	6	6.53586	38.8763	30.1608
	+10	5	7.87023	37.6106	25.8611
	0	5	7.15235	36.0042	25.6457
	−10	4	8.29353	34.3355	21.2881
	−20	4	7.36848	32.2903	21.0105
b	+20	3	8.2967	30.1857	3.69051
	+10	3	8.57701	30.8002	3.81515
	0	3	8.85732	31.4147	3.9398
	−10	4	7.23527	31.9957	4.13207
	−20	4	7.45728	32.4866	4.25878
c	+20	5	7.16731	36.0382	5.57456
	+10	5	7.16731	36.0382	5.57456
	0	5	7.16731	36.0382	5.57456
	−10	5	7.16731	36.0382	5.57456
	−20	5	7.16731	36.0382	5.57456
θ	+20	5	7.53094	36.8115	5.68158
	+10	5	7.34484	36.4147	5.62539
	0	5	7.15235	36.0042	5.56728
	−10	5	6.95334	35.5798	5.50721
	−20	5	6.74774	35.1413	5.44518
W_h	+20	5	5.87409	33.6968	5.44791
	+10	5	6.47731	34.7681	5.49681
	0	5	7.15235	36.0042	5.56728
	−10	5	7.8991	37.4047	5.65929
	−20	5	8.71747	38.9696	5.77282
r	+20	5	7.83037	37.1705	5.23008
	+10	5	7.50244	36.6031	5.37882
	0	5	7.15235	36.0042	5.56728
	−10	5	6.7769	35.3707	5.80891
	−20	5	6.37212	34.6989	6.12359
U	+20	6	6.11979	40.4676	6.73796
	+10	5	7.48108	38.3417	5.85373
	0	5	7.15235	36.0042	5.56728
	−10	5	6.78255	33.7416	5.26615
	−20	4	8.55565	31.2096	4.7597

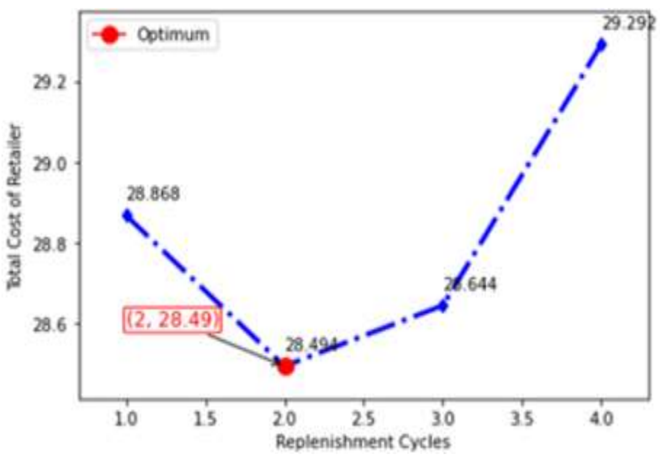

Figure 2. Convexity of total cost for retailer in 2nd replenishment cycle

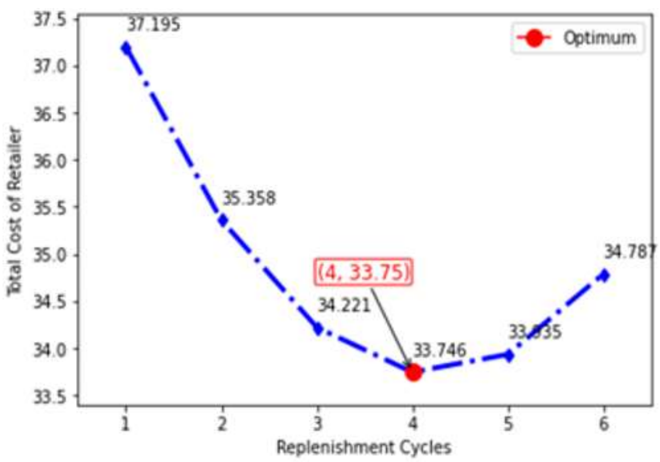

Figure 3. Convexity of total cost for retailer in 3rd replenishment cycle

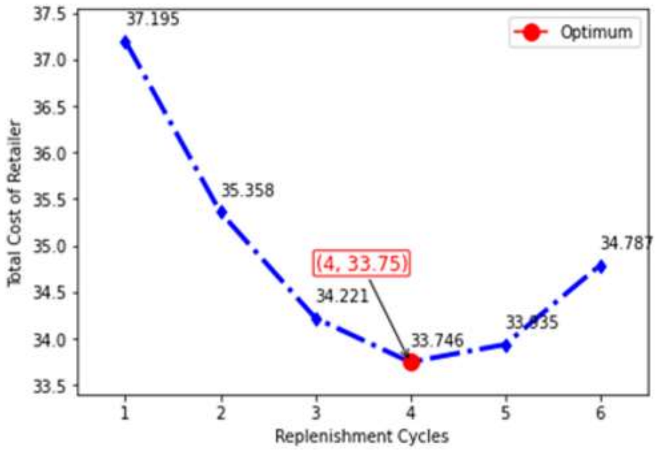

Figure 4. Convexity of total cost for retailer in 4th replenishment cycle

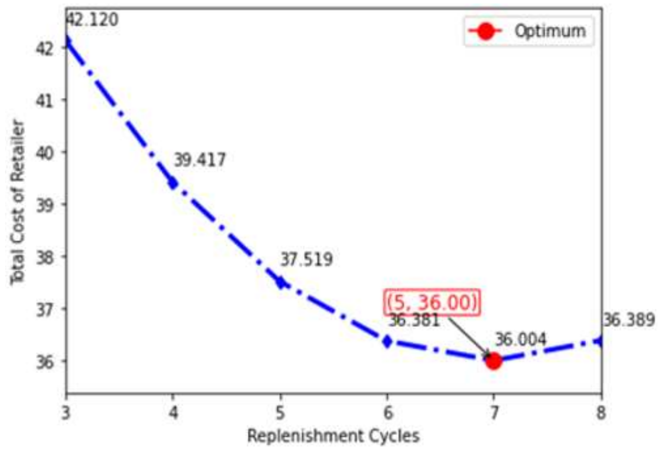

Figure 5. Convexity of total cost for retailer in 5th replenishment cycle

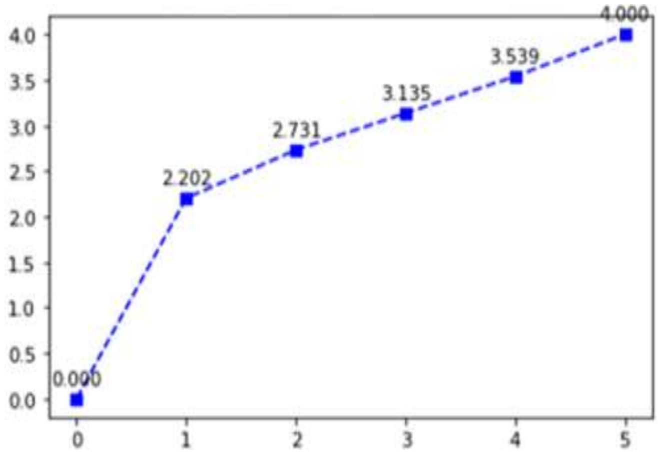

Figure 6. Increasing order of replenishment time t_j

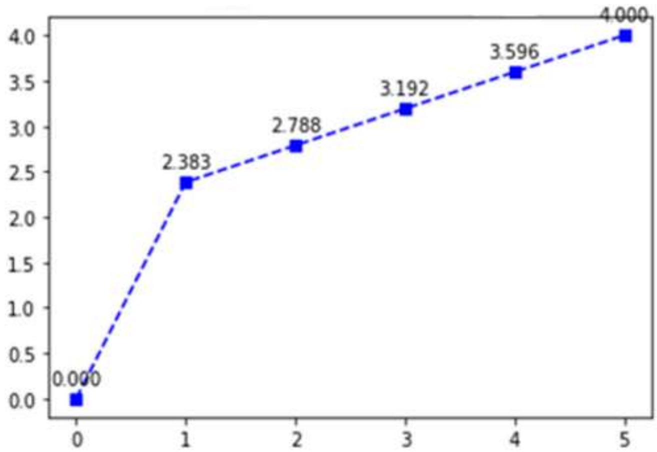

Figure 7. Increasing order of replenishment time s_j

7. ANALYSES OF SENSITIVITY FOR THE ILLUSTRATION

We will now talk about how the ideal solution responds to variations in the values of various parameters. The comparative study is carried out by altering all of the parameter's a, b, c, θ, W_h, r, and U by ±20% and ±10%, one at a time, while keeping the other parameters constant. The effect on total cost due to percentage changes in parameters a, b, c, θ, U, r, W_h and all parameters is shown in Figures 8-15. A detailed analysis of the table acknowledges the following perceptions:

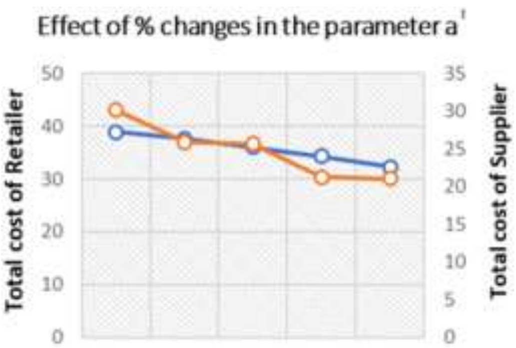

Figure 8. Effect on total cost of retailer and supplier due to parameter 'a'

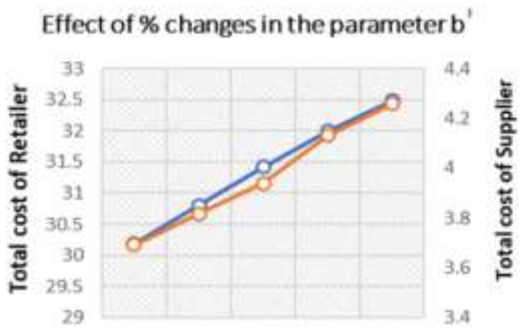

Figure 9. Effect on total cost of retailer and supplier due to parameter 'b'

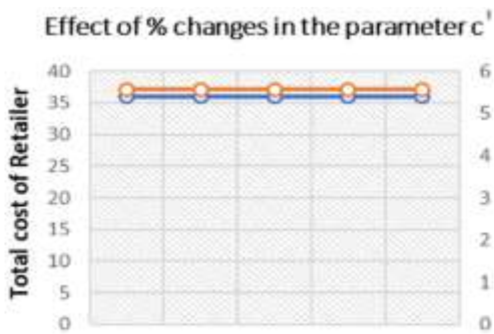

Figure 10. Effect on total cost of retailer and supplier due to parameter 'c'

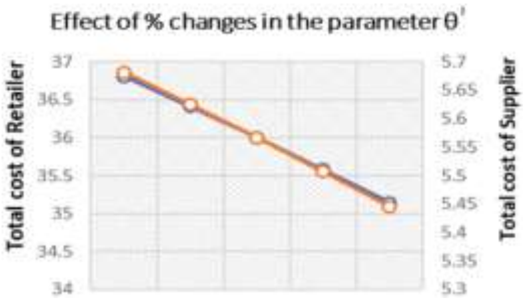

Figure 11. Effect on total cost of retailer and supplier due to parameter ‘θ’

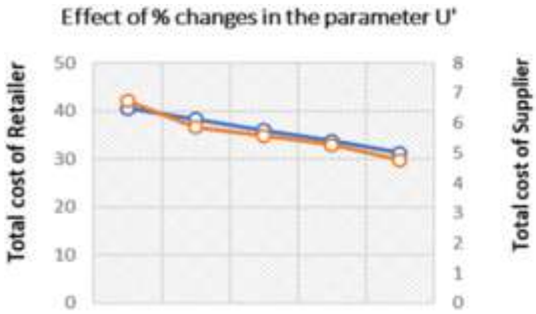

Figure 12. Effect on total cost of retailer and supplier due to parameter ‘U’

The optimal replenishment cycle, n, is sensitive to varying in most parameters. It is extremely responsive to variations in the parameter ‘a’. While decreasing 'a' by 20%, the optimal replenishment cycle, n, decreases from 5 to 4, shows a 20% decrease. On the other hand, with a 20% increase in 'a', the cycle increases to 6, a 20% increase. From this, we analyze that as 'a' expands or contracts, the optimal replenishment cycle moves in conjunction. Variations in parameters impact total cost and efficiency, providing a deeper understanding of the system's behavior under different scenarios.

Similarly, changes in the parameter 'b' also show a significant impact on the replenishment cycle. An increase of 20% in 'b' retains the cycle at 3, but a decrease of 10% in 'b' moves it to 4, a 33.33% increase. This implies that as 'b' reduces, there is an impulse to have more extended cycles.

The total cost for retailer T_R is sensitive to changes in θ and Wh. For instance, when a 20% decrease in θ shows a decrease in T_R by approximately 3.37%. Meanwhile, a 20% increase in Wh shows an increase in T_R by about 3.78%. These changes indicate the parameter's direct effect on the retailer's total costs.

The total cost for supplier, T_S, on the other hand, reacts differently to changes in parameters. An evident observation is with 'a'. A 20% increase in 'a' decreases the T_S by approximately 27.72%.

The total order quantity, Q_{nt}, shows significant changes with parameters 'b', 'θ', 'Wh', and 'U'. For 'b', a 20% increase results in an increase of approximately 7.53% in Q_{nt}. A same pattern seen for 'U'; a 20% increase in 'U' shows a decrease in Q_{nt} by approximately 14.42%.

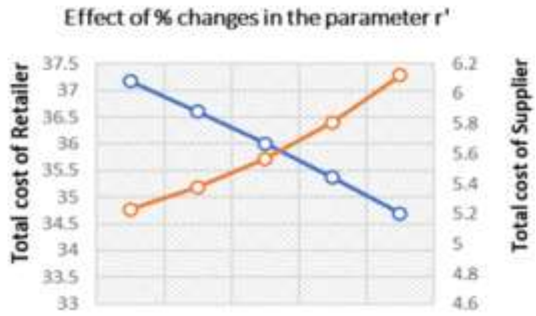

Figure 13. Effect on total cost of retailer and supplier due to parameter ‘r’

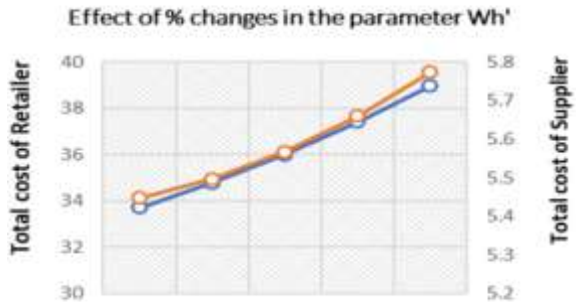

Figure 14. Effect on total cost of retailer and supplier due to parameter ‘W_h’

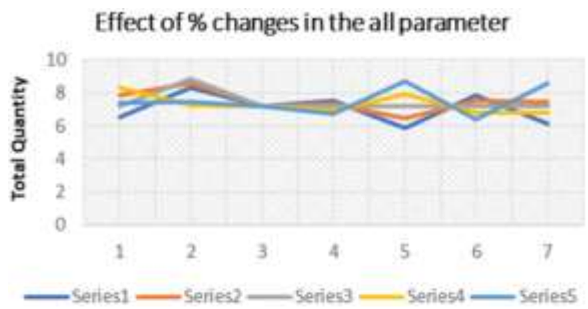

Figure 15. Effect on total cost of retailer and supplier due to all parameter

Table 7. Total cost of retailer and supplier for linear back order

Linear Back-Order Condition	Replenishment Cycle (n^*)	Q_{nt} Order Quantity	Time Intervals (Years) t_j	s_j	T_R Total Cost of Retailer	T_S Total Cost of Supplier
c=0	2	9.42888	0.174335	0.980015,	28.5422	12.2287
			1.10189	1.76513		
	3	8.74094	1.85753	2.42678	31.1614	16.7223
			2.50215	3.00577		
			3.07021	3.52538		
			3.58222	4.0000		
	4	8.04422	1.1019	1.76504	33.9692	21.2133
			1.85749	2.4267		
			2.50211	3.00571		
			3.07018	3.52535		
			3.5820	4.00000		

In the above solution, we considered quadratic back order $(a - bp - cp^2)$. If we put c=0, then we form a linear backorder case for the model. Table 7 discusses the order quantity, the total cost of retailer and supplier for the linear back order.

8. CONCLUSION

The optimization of replenishment policies outlined in this article is invaluable for businesses striving to enhance their supply chain management efficiency. By accurately modeling parameters such as price-sensitive demand and backordering, and minimizing total costs within a finite time horizon, companies can make informed decisions that lead to optimized inventory levels, reduced stockouts, and ultimately improved customer satisfaction. This approach provides a systematic framework for strategic planning, enabling businesses to allocate resources effectively, mitigate risks, and maximize profitability in a dynamic and competitive market environment.

In this article, we tackled the optimization problem associated with a replenishment policy, focusing on various parameters that influence the cost and efficiency of the system. Specifically, we considered a scenario where demand is influenced by price, modeled as (a-bp), and assumed complete backordering. Backordering was modeled both as a quadratic function and a linear function, with shortages addressed within a finite time horizon H.

Our model's primary objective was to lower the overall expense related to the replenishment procedure. We learned a lot more about how alterations to parameters like a, b, c, p, U, H, θ, W_h affect the total cost via the results of our study. Firstly, we get optimal value at 2nd cycle then in 3rd, 4th and lastly in 5th highlighting the dynamic nature of the system.

Future research can be done taking multi-item models also discussed the model in finite planning in the future this can be extended for infinite time horizon.

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NOMENCLATURE

H	Fixed time horizon.
D	The demand rate is D and D(t)= a-bp.
r	The amount that is carried per unit per order.
O_r	The cost of replenishing or purchasing per order.
S	The shortage cost per unit time.
I_j	The total inventory carried out during the interval [t_j, s_j].
S_j	The total amount of shortages in the interval [s_j, t_{j+1}]. The time at which the inventory level reaches zero in the j^{th} replenishment cycle j=1, 2, 3, …, n.
t_j	The j^{th} replenishment time j=1, 2, 3, …, n.
n	The number of orders during the time horizon H.
D_1	D_1=a-bp-cp^2 is the price dependent quadratic backordering.
Q	The total optimal order quantity during the planning horizon H.
I_b	Instantons shortage during the shortage period.
θ	An inventory dependent parameter.